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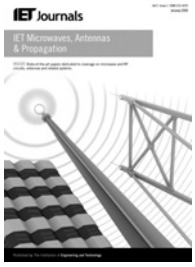

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Robust technique based on transition matrix method to electromagnetic characterisation of anisotropic material

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Abstract: This study presents a parameter retrieval technique based on the state space approach for the electromagnetic characterisation of biaxial anisotropic structures. First, the formulation of a 4 × 4 transition matrix method for the analysis of forward scattering problems is reviewed; then a procedure for the extraction of the constitutive tensor parameters of a biaxial anisotropic medium from the knowledge of reflection and transmission coefficients is implemented. In the proposed retrieval method, scattering parameters are employed for a plane wave incident normally and obliquely on a biaxial anisotropic slab. This characterisation algorithm is based on the state transition matrix and its properties in the biaxial anisotropic layers are presented as two theorems. In this method, it is not necessary to solve directly the wave equation in complex media and then apply the boundary conditions. To demonstrate the validity of the proposed method, the constitutive parameters of two non-dispersive and dispersive biaxial anisotropic slabs at microwave frequencies are retrieved. From the numerical results, one can find out that when the scattering parameters are combined with the properties of a state transition matrix, a robust technique is provided for the parameter retrieval of the anisotropic structures.

1 Introduction

Complex artificial electromagnetic structures with interesting electromagnetic properties have generated an enormous research interest over the years. Many natural or artificial electromagnetic materials exhibit unusual properties that would be beneficial to engineers if they were artificially designed to suit our needs. Therefore the study and the electromagnetic characterisation of such complex structures are recognised subjects which date back to the last decade.

Recently, with the increased interest in electromagnetic metamaterials (MTMs) and their wide applications in different microwave devices [1–5], various methods have been proposed for retrieving the effective electromagnetic parameters of such artificial structures. Some of these methods have been based on the electromagnetic fields inside the MTM structures, and so they are not practical for application in the experimental setups measurements [6–8]. In another well-known method, parameter extraction of the electromagnetic MTM structures is achieved by using analytical dispersion models [9, 10]. As a manifest disadvantage, the application of this method for complex structures is difficult. A more commonly used scattering parameter method is generally based on the inversion of the reflection and the transmission parameters of a plane wave incident on the MTM structure to give effective electromagnetic parameters [11–13]. Many papers have discussed the application of this method in different media and situations [14–19]. Most attempts at measuring the MTMs electromagnetic parameters at an oblique incidence or accounting for anisotropy or bianisotropy have relied on fully numerical optimisation and curve fitting techniques [20, 21].

The objective of the present paper is to characterise the biaxial anisotropic media using the state space approach. Although the application of the state transition matrix method in the forward scattering problems has been well studied over the years [22–27], its application in the formulation for the inverse scattering problems of the biaxial anisotropic mediums has not been reported. In the present paper, an electromagnetic characterisation procedure is presented for retrieving the constitutive tensor parameters of a biaxial anisotropic slab by using the scattering parameters based on the state transition matrix and its characteristics expressed as two theorems. The proposed technique allows for a characterisation at oblique incidences.

This paper is organised as follows. In Section 2, the analysis of the forward problem through the state space approach is reviewed. Two interesting properties of the state transition matrix of the biaxial anisotropic layers are presented in Section 3. Section 4 deals with the formulation of the inverse scattering of a biaxial anisotropic medium based on the 4 × 4 state transition matrix method. In Section 5, some numerical results are provided to validate the proposed formulation, from which it is found that the proposed scheme works well.

2 Forward problem analysis

In this section, we review the analysis of the problem of plane wave scattering from a biaxial anisotropic layer which has diagonal constitutive parameters such as

$$\bar{\bar{\varepsilon}} = \varepsilon_0 \mathrm{diag}\left(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}\right),$$
$$\bar{\bar{\mu}} = \mu_0 \mathrm{diag}\left(\mu_{xx}, \mu_{yy}, \mu_{zz}\right) \tag{1}$$

where ε_0 and μ_0 are the free space permittivity and permeability, respectively. It is assumed that the transverse electric (TE) or transverse magnetic (TM) polarised plane waves are obliquely incident at the angle θ_0 from the free space to the anisotropic slab, as shown in Fig. 1. The planar structure is of infinite extent along the y-direction, and hence the derivative of the fields with respect to the y variable vanishes. In addition, the derivative of the fields with respect to the x variable in the slab must take on the same value as in the free space in order to satisfy the boundary conditions on the tangential fields at the boundaries, and hence $\partial/\partial x = -jk_0 \sin\theta_0$ (assuming a time harmonic field with $e^{j\omega t}$) where k_0 is the free space wave number. By substituting the constitutive relations (1) into Maxwell's curl equations and by eliminating the z-components of the electric and the magnetic fields one can write

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} \bar{E}_\mathrm{T} \\ \bar{H}_\mathrm{T} \end{pmatrix} = \boldsymbol{\Gamma}_\omega \begin{pmatrix} \bar{E}_\mathrm{T} \\ \bar{H}_\mathrm{T} \end{pmatrix} \tag{2}$$

where $\bar{E}_\mathrm{T} = (E_x, E_y)$ and $\bar{H}_\mathrm{T} = (H_x, H_y)$ are the transverse components of the electric and the magnetic fields, respectively, and the $\boldsymbol{\Gamma}_\omega$-matrix is given by

$$\boldsymbol{\Gamma}_\omega = \frac{\omega}{c}\boldsymbol{\Gamma} = \frac{\omega}{c}\begin{pmatrix} 0 & 0 & 0 & \Gamma_{14} \\ 0 & 0 & \Gamma_{23} & 0 \\ 0 & \Gamma_{32} & 0 & 0 \\ \Gamma_{41} & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

where

$$\Gamma_{14} = \mathrm{j}\eta_0\left(-\mu_{yy} + \frac{1}{\varepsilon_{zz}}\sin^2\theta_0\right) \tag{4a}$$

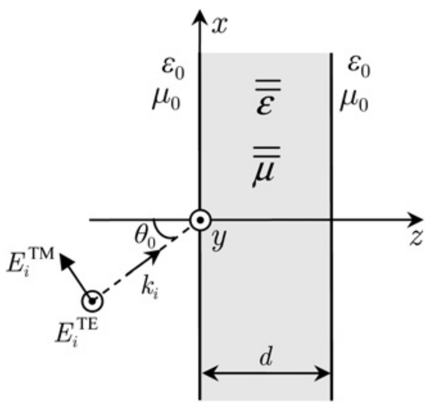

Fig. 1 *Biaxial anisotropic slab exposed to a linearly polarised plane wave*

$$\Gamma_{23} = \mathrm{j}\eta_0\mu_{xx} \tag{4b}$$

$$\Gamma_{32} = \mathrm{j}\eta_0^{-1}\left(\varepsilon_{yy} - \frac{1}{\mu_{zz}}\sin^2\theta_0\right) \tag{4c}$$

$$\Gamma_{41} = -\mathrm{j}\eta_0^{-1}\varepsilon_{xx} \tag{4d}$$

where ω is the angular frequency and $\eta_0 = \sqrt{\mu_0 / \varepsilon_0}$ is the intrinsic impedance of the free space.

We define a 4×4 state transition matrix $\boldsymbol{\Phi}$ that relates the transverse components of the electric and the magnetic fields at the two boundaries of a chiral slab

$$\begin{pmatrix} \bar{E}_\mathrm{T}(0) \\ \bar{H}_\mathrm{T}(0) \end{pmatrix} = \boldsymbol{\Phi}\begin{pmatrix} \bar{E}_\mathrm{T}(d) \\ \bar{H}_\mathrm{T}(d) \end{pmatrix} = \begin{pmatrix} (\boldsymbol{\Phi}_1)_{2\times2} & (\boldsymbol{\Phi}_2)_{2\times2} \\ (\boldsymbol{\Phi}_3)_{2\times2} & (\boldsymbol{\Phi}_4)_{2\times2} \end{pmatrix}\begin{pmatrix} \bar{E}_\mathrm{T}(d) \\ \bar{H}_\mathrm{T}(d) \end{pmatrix} \tag{5}$$

where the state transition matrix $\boldsymbol{\Phi}$ is given by

$$\boldsymbol{\Phi} = \mathrm{e}^{-\boldsymbol{\Gamma}_\omega d} \tag{6}$$

Several methods have been proposed for the computation of the exponential of a square matrix [28, 29], such as the expansion in the power series, the Laplace transform, the Jordan normal form and the Cayley–Hamilton theorem discussed in Appendix 2.

Generally, we can define the reflection and the transmission matrices as

$$\bar{E}^\mathrm{r}_\mathrm{T}(z=0) = \begin{pmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{pmatrix}\bar{E}^\mathrm{i}_\mathrm{T}(z=0) \tag{7}$$

$$\bar{E}^\mathrm{t}_\mathrm{T}(z=d) = \begin{pmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{pmatrix}\bar{E}^\mathrm{i}_\mathrm{T}(z=0) \tag{8}$$

where the superscripts i, r and t denote the incident, the reflected and the transmitted fields, respectively. However, in the case of the biaxial anisotropic slab, the crosspolarised components of the reflection and the transmission coefficients are zero. After some simple manipulations on (5), we may write

$$\bar{E}^\mathrm{i}_\mathrm{T}(0) + \bar{E}^\mathrm{r}_\mathrm{T}(0) = \boldsymbol{\Phi}_1\bar{E}^\mathrm{t}_\mathrm{T}(d) + \boldsymbol{\Phi}_2\bar{H}^\mathrm{t}_\mathrm{T}(d) \tag{9}$$

$$\mathbf{Z}_0^{-1}\bar{E}^\mathrm{i}_\mathrm{T}(0) - \mathbf{Z}_0^{-1}\bar{E}^\mathrm{r}_\mathrm{T}(0) = \boldsymbol{\Phi}_3\bar{E}^\mathrm{t}_\mathrm{T}(d) + \boldsymbol{\Phi}_4 Z_0^{-1}\bar{\boldsymbol{E}}^\mathrm{t}_\mathrm{T}(d) \tag{10}$$

where the wave impedance matrix $\boldsymbol{Z}_0$ is defined as

$$\boldsymbol{Z}_0 = \begin{pmatrix} 0 & \eta_0\cos\theta_0 \\ -\eta_0/\cos\theta_0 & 0 \end{pmatrix} \tag{11}$$

By using (9) and (10) and by considering (7) and (8), after some simple matrix manipulations one obtains [22]

$$\boldsymbol{R} = \left[\boldsymbol{\Phi}_1\boldsymbol{Z}_0 + \boldsymbol{\Phi}_2 - \boldsymbol{Z}_0\left(\boldsymbol{\Phi}_3\boldsymbol{Z}_0 + \boldsymbol{\Phi}_4\right)\right] \times \left[\boldsymbol{\Phi}_1\boldsymbol{Z}_0 + \boldsymbol{\Phi}_2 + \boldsymbol{Z}_0\left(\boldsymbol{\Phi}_3\boldsymbol{Z}_0 + \boldsymbol{\Phi}_4\right)\right]^{-1} \tag{12}$$

$$\boldsymbol{T} = 2\boldsymbol{Z}_0\left[\boldsymbol{\Phi}_1\boldsymbol{Z}_0 + \boldsymbol{\Phi}_2 + \boldsymbol{Z}_0\left(\boldsymbol{\Phi}_3\boldsymbol{Z}_0 + \boldsymbol{\Phi}_4\right)\right]^{-1} \tag{13}$$

3 Properties of the state transition matrix of a biaxial anisotropic slab

In this section, we introduce and prove two interesting properties of the state transition matrix $\mathbf{\Phi}$ of the biaxial anisotropic slabs which can be used for the proposed parameter retrieval algorithm which will be discussed in the next section.

Theorem 1: The determinant of the state transition matrix $\mathbf{\Phi}$ of a biaxial anisotropic slab is equal to unity. Its proof is presented in Appendix 1.

Theorem 2: The absolute values of the entries of the state transition matrix $(\mathbf{\Phi})$ and its inverse (Φ^{-1}) are equal. As proved in Appendix 2, the state transition matrix of a biaxial anisotropic slab and its inverse are

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_{11} & 0 & 0 & \Phi_{14} \\ 0 & \Phi_{22} & \Phi_{23} & 0 \\ 0 & \Phi_{32} & \Phi_{22} & 0 \\ \Phi_{41} & 0 & 0 & \Phi_{11} \end{pmatrix} \tag{14a}$$

$$\Phi^{-1} = \begin{pmatrix} \Phi_{11} & 0 & 0 & -\Phi_{14} \\ 0 & \Phi_{22} & -\Phi_{23} & 0 \\ 0 & -\Phi_{32} & \Phi_{22} & 0 \\ -\Phi_{41} & 0 & 0 & \Phi_{11} \end{pmatrix} \tag{14b}$$

Observe that the 4×4 matrix $\mathbf{\Phi}$ has only six distinct non-zero entries, namely $\Phi_{11}, \Phi_{14}, \Phi_{22}, \Phi_{23}, \Phi_{32}$ and Φ_{41}, respectively, which are given in Appendix 2.

4 Formulation of the parameter retrieval technique

This section deals with the formulation of the electromagnetic characterisation method for the retrieval of the constitutive tensor parameters of a biaxial anisotropic slab. The first step in the inverse problem is to find the state transition matrix $\mathbf{\Phi}$ by using the scattering parameters.

It is clear that four tensor parameters $\varepsilon_{xx}, \varepsilon_{yy}, \mu_{xx}$ and μ_{yy} are active when the slab is normally illuminated by the TE and the TM plane waves, while two other tensor parameters ε_{zz} and μ_{zz} are also active when the slab is obliquely illuminated. Hence, for retrieving all the constitutive tensor parameters of a biaxial anisotropic slab, the scattering parameters corresponding to the illuminations at two different angles of incidence are required.

4.1 Determination of the state transition matrices at oblique and normal incidences

First, let us consider the perpendicular polarisation incidence, where the TE incident wave is assumed to be incident from the free space to the slab at the angle θ_0. Hence, (9) and (10) may be rewritten as

$$1 + E_y^{\mathrm{r}} = \left(\Phi_{22}^{\mathrm{oblique}} - \frac{\Phi_{23}^{\mathrm{oblique}}}{\eta_0}\cos\theta_0\right)E_y^{\mathrm{t}} \tag{15a}$$

$$-\frac{1}{\eta_0}\cos\theta_0\left(1 - E_y^{\mathrm{r}}\right) = \left(\Phi_{32}^{\mathrm{oblique}} - \frac{\Phi_{22}^{\mathrm{oblique}}}{\eta_0}\cos\theta_0\right)E_y^{\mathrm{t}} \tag{15b}$$

where the oblique superscript denotes an oblique incidence. Then, consider the TM incident wave, for which (9) and (10) can be rewritten as

$$1 + E_x^{\mathrm{r}} = \left(\Phi_{11}^{\mathrm{oblique}} + \frac{\Phi_{14}^{\mathrm{oblique}}}{\eta_0\cos\theta_0}\right)E_x^{\mathrm{t}} \tag{16a}$$

$$\frac{1}{\eta_0\cos\theta_0}\left(1 - E_x^{\mathrm{r}}\right) = \left(\Phi_{41}^{\mathrm{oblique}} + \frac{\Phi_{11}^{\mathrm{oblique}}}{\eta_0\cos\theta_0}\right)E_x^{\mathrm{t}} \tag{16b}$$

Observe that by using the measured reflection and transmission coefficients, a simple set of four equations is derived, whereas the number of the distinct and the non-zero elements of the state transition matrix $\mathbf{\Phi}^{\mathrm{oblique}}$ in an oblique incidence (considered as unknown parameters of the problem) is six. Here, our proposed method for equating the number of the unknowns and the equations is to use the properties of the state transition matrix of the biaxial anisotropic slab discussed in the previous section. In fact, by considering the presented theorems, the necessary and sufficient equations are provided to uniquely determine the unknown entries of an $\mathbf{\Phi}^{\mathrm{oblique}}$-matrix.

It is clear that in the case of a normal incidence, we can consider (15) and (16) assuming $\theta_0 = 0°$ and the state transition matrix $\mathbf{\Phi}^{\mathrm{normal}}$, where a normal superscript denotes a normal incidence. Similar to the previous case, $\mathbf{\Phi}^{\mathrm{normal}}$ can be determined by solving the obtained equations.

4.2 Determination of the constitutive tensor parameters

Once the state transition matrix $\mathbf{\Phi}$ ($\mathbf{\Phi}^{\mathrm{normal}}$ and $\mathbf{\Phi}^{\mathrm{oblique}}$) is determined, the $\mathbf{\Gamma}$-matrices ($\mathbf{\Gamma}^{\mathrm{normal}}$ and $\mathbf{\Gamma}^{\mathrm{oblique}}$) can be subsequently identified as

$$\mathbf{\Gamma} = -\frac{c}{\omega d}\ln(\Phi) = -\frac{\lambda_0}{2\pi d}\ln(\Phi) \tag{17}$$

where c and λ_0 are the speed of light and the wavelength in free space, respectively. Once the $\mathbf{\Gamma}^{\mathrm{normal}}$ and $\mathbf{\Gamma}^{\mathrm{oblique}}$ matrices are obtained, the constitutive tensor parameters can be determined by

$$\varepsilon_{xx} = \mathrm{j}\eta_0\Gamma_{41}^{\mathrm{normal}} \tag{18a}$$

$$\varepsilon_{yy} = -\mathrm{j}\eta_0\Gamma_{32}^{\mathrm{normal}} \tag{18b}$$

$$\varepsilon_{zz} = \mathrm{j}\eta_0\sin^2\theta_0\left(\Gamma_{14}^{\mathrm{oblique}} - \Gamma_{14}^{\mathrm{normal}}\right) \tag{18c}$$

$$\mu_{xx} = -\mathrm{j}\eta_0^{-1}\Gamma_{23}^{\mathrm{normal}} \tag{19a}$$

$$\mu_{yy} = \mathrm{j}\eta_0^{-1}\Gamma_{14}^{\mathrm{normal}} \tag{19b}$$

$$\mu_{zz} = -\mathrm{j}\eta_0^{-1}\sin^2\theta_0\left(\Gamma_{32}^{\mathrm{oblique}} - \Gamma_{32}^{\mathrm{normal}}\right) \tag{19c}$$

Briefly, the procedure of the electromagnetic characterisation of the biaxial anisotropic materials from the knowledge of

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the normal and the oblique scattering parameters can be summarised as follows:

Solve (15a)–(16b) by considering the oblique scattering parameters along with Theorems 1 and 2 to find $\boldsymbol{\Phi}^{\text{oblique}}$.
Solve (15a)–(16b) by considering $\theta_0 = 0$ and the normal S-parameters along with Theorems 1 and 2 to find $\boldsymbol{\Phi}^{\text{normal}}$.
Find the $\boldsymbol{\Gamma}$-matrices ($\boldsymbol{\Gamma}^{\text{nor}}$ and $\boldsymbol{\Gamma}^{\text{obl}}$) by using (17).
Determine the constitutive tensor parameters by using (18a)–(19c).

4.3 Investigation of a probable ambiguity in the results

Notice that for the $\boldsymbol{\Gamma}$-matrix with the eigenvalues γ_1, γ_2, γ_3 and γ_4, and the eigenvectors $\bar{V}_1$, $\bar{V}_2$, $\bar{V}_3$ and $\bar{V}_4$, respectively, there exists a matrix $\boldsymbol{M}$ such that

$$\boldsymbol{\Gamma} = \boldsymbol{M}\boldsymbol{\Lambda}\boldsymbol{M}^{-1} \tag{20}$$

where

$$\boldsymbol{M} = \left[\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4\right] \tag{21}$$

and $\boldsymbol{\Lambda}$ is a diagonal matrix as diag($\gamma_1, \gamma_2, \gamma_3, \gamma_4$). As proved in Appendix 1, one can write

$$\boldsymbol{\Phi} = \boldsymbol{M}\text{diag}\left(\text{e}^{-\gamma_1\omega d/c}, \text{e}^{-\gamma_2\omega d/c}, \text{e}^{-\gamma_3\omega d/c}, \text{e}^{-\gamma_4\omega d/c}\right)M^{-1} \tag{22}$$

On considering (20) and (22) it is clear that $\boldsymbol{\Gamma}$ and $\boldsymbol{\Phi}$ have an identical set of eigenvectors $\bar{V}_n$. Note that in the proposed characterisation method, once the state transition matrix $\boldsymbol{\Phi}$ is determined, its eigenvalues λ_n ($n = 1$, 2, 3 and 4, respectively) and the eigenvectors are readily obtained. Therefore, for the computation of the $\boldsymbol{\Gamma}$-matrix through (20), the eigenvalues γ_n are

$$\gamma_n = -\frac{c}{\omega d}\ln\lambda_n = -\frac{c}{\omega d}\left[\ln\left|\lambda_n\right| + \text{j}\left(\arg\lambda_n + 2m\pi\right)\right] \tag{23}$$

where m can be an arbitrary integer number at any frequency. The resulting uncertainty because of the existence of m in (23) is referred to as a branching problem, which is due to the multibranch form of a complex logarithmic function. Clearly, if the thickness d is much smaller than the free space wavelength, the eigenvalues γ_n and consequently the $\boldsymbol{\Gamma}$-matrix are unambiguously identified.

Here, we can use an advanced and well-known technique based on the Kramers–Kronig (K–K) relations for resolving the branch selection problem which has been well discussed in ordinary and chiral MTMs [13, 30, 31]. On considering (3), one can easily see that the eigenvalues of $\boldsymbol{\Gamma}^{\text{normal}}$ are $\gamma_1 = -\gamma_2 = -\text{j}\sqrt{\mu_{yy}\varepsilon_{xx}} = -\text{j}n_1$ and $\gamma_3 = -\gamma_4 = \text{j}\sqrt{\mu_{xx}\varepsilon_{yy}} = \text{j}n_2$, respectively. Note that the real and the imaginary parts of n_1 and n_2 satisfy the K–K relations

$$\text{Re}\left\{n_{1(2)}(\omega)\right\} = \text{Re}\left\{n_{1(2)}(\infty)\right\} - \frac{2}{\pi}\text{P.V.}\int_0^\infty \frac{u\text{Im}\left\{n_{1(2)}(u)\right\}}{u^2 - \omega^2}\text{d}u \tag{24}$$

$$\text{Im}\left\{n_{1(2)}(\omega)\right\} = \frac{2\omega}{\pi}\text{P.V.}\int_0^\infty \frac{\text{Re}\left\{n_{1(2)}(u)\right\} - \text{Re}\left\{n_{1(2)}(\infty)\right\}}{u^2 - \omega^2}\text{d}u \tag{25}$$

where P.V. refers to the principal value of the integral defined as follows

$$\text{P.V.}\int_{-\infty}^{\infty} f(u)\text{d}u = \int_{-\infty}^{\infty}\text{P.V.}\big(f(u)\big)\text{d}u = \lim_{\text{eps}\to 0}\left[\int_{-\infty}^{\omega-\text{eps}} f(u)\text{d}u + \int_{\omega+\text{eps}}^{\infty} f(u)\text{d}u\right] \tag{26}$$

Therefore we can rewrite (24) for γ_1 and γ_4 as

$$\text{Im}\left\{\gamma_{1(4)}(\omega)\right\} = \text{Im}\left\{\gamma_{1(4)}(\infty)\right\} + \frac{2}{\pi}\text{P.V.}\int_0^\infty \frac{u\text{Re}\left\{\gamma_{1(4)}(u)\right\}}{u^2 - \omega^2}\text{d}u \tag{27}$$

Note that the parameter retrieval procedure takes advantage of the fact that the real parts of the eigenvalues are not affected by the branches of the logarithmic function and can be unambiguously determined by using (23). Therefore the imaginary parts of the eigenvalues can be calculated from (27) without any ambiguity. However, note that in the simulations and the experiments, the S-parameters are measured in limited frequency ranges, and hence the integration of (27) should be truncated. In addition, in the inverse problem the values of $\gamma_n(\infty)$ are unknown, and hence (27) yields an approximated solution determining the general behaviour of the exact solution properly.

Note that although it is shown that the constitutive tensor parameters of the biaxial anisotropic materials can be successfully retrieved without a multibranch ambiguity even for a thick material, which is one of the main concerns for the material characterisation, the phase ambiguity in materials with large-valued parameters cannot be completely solved by using the K–K relationships [32].

4.4 Discussions

In the standard full wave methods [11–20], we should directly solve the set of the obtained non-linear equations from the boundary conditions to derive the analytical formulae for the constitutive tensor parameters of the unknown slab in terms of the reflection and the transmission coefficients. In such methods, the identification of the multibranch ambiguity which is one of the main concerns for the material characterisation requires having these analytical formulae where their derivations in complex structures such as a biaxial anisotropic medium are not simple. In the proposed characterisation method, it is not necessary to obtain the eigenwaves of the biaxial anisotropic layer through a direct solution of the wave equation. In addition, a derivation of the analytical formulae for the unknown constitutive parameters in terms of the reflection and the transmission coefficients is not required and only some simple matrix operations should be performed.

5 Numerical examples and results

Two examples are provided to validate the proposed formulation. Hitherto, we used the Cayley–Hamilton theorem for computing the exponential and the logarithm of a square matrix, while in the following numerical examples for simplicity, the *expm* and the *logm* commands in MATLAB are used.

5.1 Non-dispersive biaxial anisotropic slab

As the first example, consider a non-dispersive biaxial anisotropic slab with the following constitutive tensor parameters

$$\bar{\bar{\varepsilon}} = \varepsilon_0 \mathrm{diag}(4 - \mathrm{j}0.5, 7 - \mathrm{j}0.1, 2 - \mathrm{j}2) \quad (28a)$$

$$\bar{\bar{\mu}} = \mu_0 \mathrm{diag}(1 - \mathrm{j}0.3, 2, 5 - \mathrm{j}2) \quad (28b)$$

Assume that the thickness of the slab and the excitation frequency are 5 mm and 2 GHz, respectively. The forward problem analysis shows that the reflection and the transmission coefficients are given by $R_{xx} = -0.143 - \mathrm{j}0.126$, $R_{yy} = -0.322 - \mathrm{j}0.371$, $T_{xx} = 0.767 - \mathrm{j}0.538$ and $T_{yy} = 0.593 - \mathrm{j}0.569$ for a normal incidence, and $R_{xx} = -0.091 - \mathrm{j}0.077$, $R_{yy} = -0.393 - \mathrm{j}0.396$, $T_{xx} = 0.784 - \mathrm{j}0.525$ and $T_{yy} = 0.536 - \mathrm{j}0.569$ for an oblique incidence at $\theta_0 = 30°$, respectively. The proposed reconstruction technique has been applied to the obtained reflection and transmission coefficients and the discussed set of equations in Section 4 has been solved analytically by using MATLAB. The elements of the computed $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ matrices corresponding to the normal and the oblique incidences and the retrieved constitutive parameters of the slab are presented in Table 1. The comparison among the parameters generated by the proposed technique and the exact ones illustrates the good behaviour of the technique.

5.2 Dispersive biaxial anisotropic slab

As the second example, consider a biaxial anisotropic slab with thickness $d = 0.5$ mm whose constitutive parameters have the following dispersion relations [16]

$$\varepsilon = \varepsilon_\mathrm{b} - \frac{A_\varepsilon f_{\mathrm{e}0}^2}{f^2 - f_{\mathrm{e}0}^2 - \mathrm{j}f f_{\mathrm{e}0} \delta_\varepsilon} \quad (29a)$$

$$\mu = \mu_\mathrm{b} - \frac{A_\mu f^2}{f^2 - f_{\mathrm{m}0}^2 - \mathrm{j}f f_{\mathrm{m}0} \delta_\mu} \quad (29b)$$

where $f_{\mathrm{e}0}$, $f_{\mathrm{m}0}$, ε_b, μ_b, δ_ε and δ_μ are the electric and magnetic resonant frequencies, the background relative permittivity and permeability and the damping factors, respectively. Assume that $\varepsilon_\mathrm{b} = 4.2$, $\mu_\mathrm{b} = 1$, $f_{\mathrm{e}0} = 4$ GHz and $f_{\mathrm{m}0} = 6$ GHz, for all the tensors elements, $A_\varepsilon = 1$, 0.5, 1.2 and $\delta_\varepsilon = 0.02$, 0.1, 0.05 for ε_{xx}, ε_{yy} and ε_{zz}, respectively. Also, assume $A_\mu = 0.1$, 0.01, 0.05 and $\delta_\mu = 0.005$, 0.01, 0.05 for μ_{xx}, μ_{yy} and μ_{zz}, respectively. The amplitudes and the phases of the reflection and the transmission coefficients of this slab against the frequency obtained from the forward problem analysis at the normal and the oblique incidences at $\theta_0 = 60°$ are illustrated in Fig. 2.

The proposed retrieval method based on the state transition matrix method is applied to the reflection and the transmission coefficients, and the $\mathbf{\Phi}$ and the $\mathbf{\Gamma}$ matrices at the normal and the oblique incidences are determined. For instance, the real and the imaginary parts of the eigenvalues of the $\mathbf{\Phi}$ and the $\mathbf{\Gamma}$ matrices at a normal incidence are shown in Figs. 3 and 4, respectively. Observe that the γ_n curves are continuous and all of them are determined without any ambiguity. The approximate curves for the imaginary parts of γ_n obtained by the K–K relation, that is, (27), are also reported in Fig. 4*b*. As stated previously, in the inverse problem, the values of $\gamma_n(\infty)$ are unknown, and hence (27) yields an approximate solution determining the general behaviour of the exact solution, which causes the difference between the exact and the approximate solutions. Once the $\mathbf{\Gamma}$-matrices at the normal and the oblique incidences are determined, the constitutive tensor parameters are computed by (18) and (19) as shown in Fig. 5. Observe that there is an excellent agreement between the true and the retrieved values of the constitutive parameters of the anisotropic slab. These results show that in the electromagnetic characterisation of a 0.5-mm-thick anisotropic slab, no branching problem occurs.

Now, consider a further study, a 5-mm-thick biaxial anisotropic slab ten times thicker than the former one. The amplitudes and the phases of the reflection and the transmission coefficients at the normal and the oblique incidences are shown in Fig. 6. The real and the imaginary parts of the eigenvalues of the obtained $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ matrices are shown in Figs. 7 and 8, respectively. Note that, unlike the prior case, there are considerable discontinuities in the γ_n curves which may be attributed to the branching problem in (23). However, no discontinuity is seen in the approximate solutions by the K–K relations. Therefore, considering the K–K solutions, the discontinuities which have an allowable value $2\pi c/\omega d$ about 6 GHz should be removed and the nearest solution to the K–K one is achieved. The modified curve of $\gamma_3 = -\gamma_4$ is also shown in

Table 1 Computed $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ matrices and constitutive tensor parameters of a biaxial anisotropic slab

Computed $\mathbf{\Phi}^{\mathrm{normal}}$	Computed $\mathbf{\Phi}^{\mathrm{oblique}}$	Computed $\mathbf{\Gamma}^{\mathrm{normal}}$	Computed $\mathbf{\Gamma}^{\mathrm{oblique}}$	Computed ε	Computed μ
$\Phi_{11} = 0.8295 + \mathrm{j}0.0207$	$\Phi_{11} = 0.8353 + \mathrm{j}0.0252$	$\Gamma_{14} = 0.0003 - \mathrm{j}753.98$	$\Gamma_{14} = -23.562 - \mathrm{j}730.42$	$\varepsilon_{xx} = 4.000 - \mathrm{j}0.5000$	$\mu_{xx} = 1.0000 - \mathrm{j}0.3000$
$\Phi_{14} = -1.1145 + \mathrm{j}148.84$	$\Phi_{14} = 3.3434 + \mathrm{j}144.53$	$\Gamma_{23} = 113.10 + \mathrm{j}376.99$	$\Gamma_{23} = 113.10 + \mathrm{j}376.99$		
$\Phi_{22} = 0.8506 + \mathrm{j}0.0458$	$\Phi_{22} = 0.8516 + \mathrm{j}0.0459$	$\Gamma_{32} = 0.0003 + \mathrm{j}0.0186$	$\Gamma_{32} = 0.0003 + \mathrm{j}0.0185$	$\varepsilon_{yy} = 7.0000 - \mathrm{j}0.1000$	$\mu_{yy} = 2.0000 + \mathrm{j}0.0000$
$\Phi_{23} = -21.2649 - \mathrm{j}73.3579$	$\Phi_{23} = -21.2707 - \mathrm{j}73.3860$	$\Gamma_{41} = -0.0013 - \mathrm{j}0.0106$	$\Gamma_{41} = -0.0013 - \mathrm{j}0.0106$		
$\Phi_{32} = 0.0000 - \mathrm{j}0.0037$	$\Phi_{32} = 0.0000 - \mathrm{j}0.0037$	other elements are in the order of 10^{-16}	other elements are in the order of 10^{-16}	$\varepsilon_{zz} = 2.0000 - \mathrm{j}2.0000$	$\mu_{zz} = 4.9996 - \mathrm{j}2.0006$
$\Phi_{41} = 0.0002 + \mathrm{j}0.0021$	$\Phi_{41} = 0.0002 + \mathrm{j}0.0021$				

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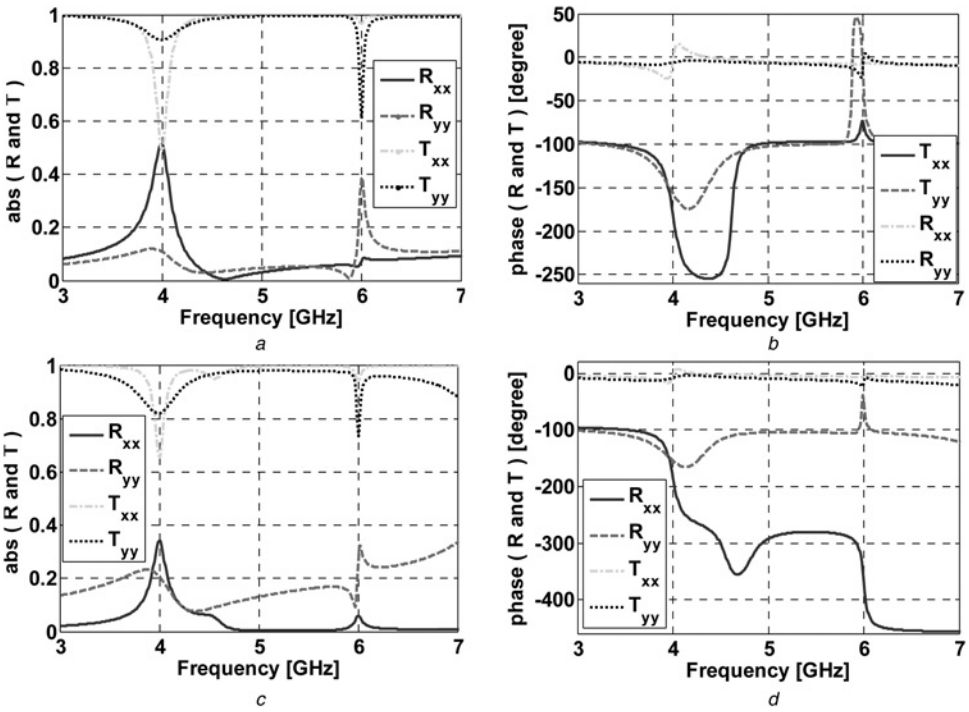

Fig. 2 *Amplitudes and phases of the reflection and the transmission coefficients of a dispersive biaxial anisotropic slab with thickness d = 0.5 mm at*

a and *b* Normal
c and *d* Oblique incidence with $\theta_0 = 60°$

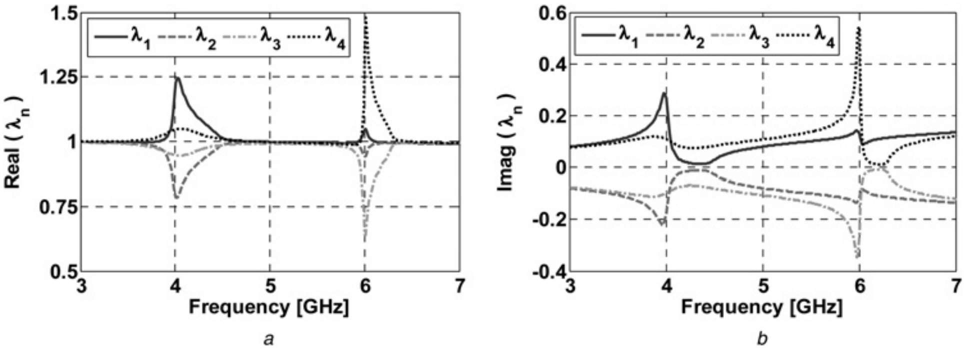

Fig. 3 *For instance, the real and the imaginary parts of the eigenvalues of the* $\boldsymbol{\Phi}$ *and* $\boldsymbol{\Gamma}$ *at a normal incidence*

a Real
b Imaginary parts of the eigenvalues of the $\boldsymbol{\Phi}$-matrix for a biaxial anisotropic slab with thickness $d = 0.5$ mm at a normal incidence

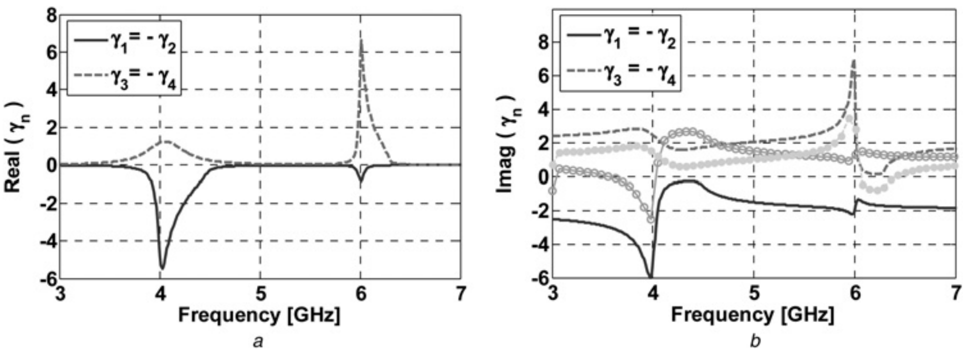

Fig. 4 *For instance, the real and the imaginary parts of the eigenvalues of the* $\boldsymbol{\Phi}$ *and* $\boldsymbol{\Gamma}$ *at a normal incidence*

a Real
b Imaginary parts of the eigenvalues of the $\boldsymbol{\Gamma}$-matrix for a biaxial anisotropic slab with thickness $d = 1.6$ mm at a normal incidence
Obtained approximate results by the K–K relations for $\gamma_1 = -\gamma_2$ and $\gamma_3 = -\gamma_4$ are shown with hollow and solid circles, respectively

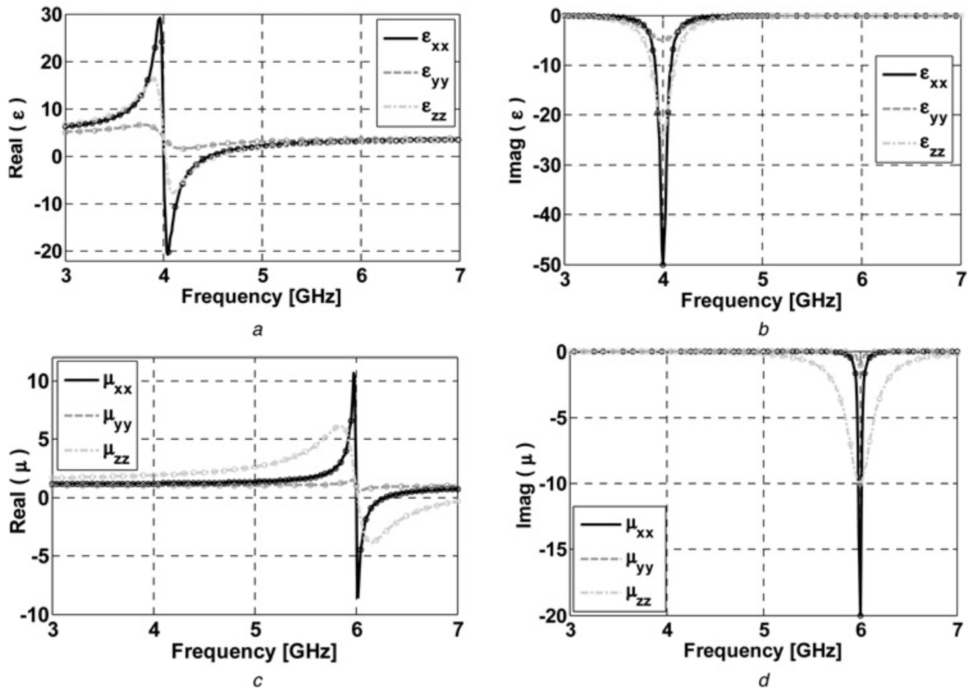

Fig. 5 *Real and imaginary parts of the retrieved constitutive parameters of a biaxial anisotropic slab*

a and *b* Relative permittivity
c and *d* Relative permeability
True values of the parameters are shown with circles

Fig. 8*a*. Observe that as expected, the obtained eigenvalues of the **Γ**-matrix are independent of the slab thickness and are the same as the previous values. After a unique determination of the **Γ** matrices at the normal and the oblique incidences, the constitutive tensor parameters of the slab are computed. The purpose of the current paper was to demonstrate the applicability and the robustness of the K–K relations for removing the branching problem.

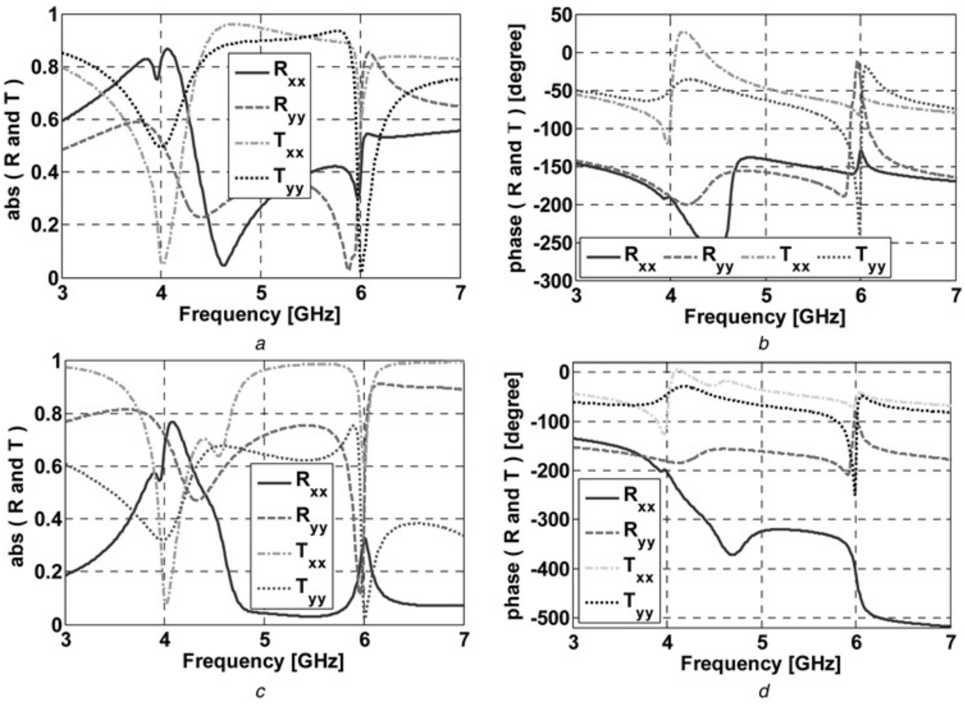

Fig. 6 *Amplitudes and phases of the reflection and the transmission coefficients of a dispersive biaxial anisotropic slab with thickness d = 5 mm at*

a and *b* Normal
c and *d* Oblique incidence with $\theta_0 = 60°$

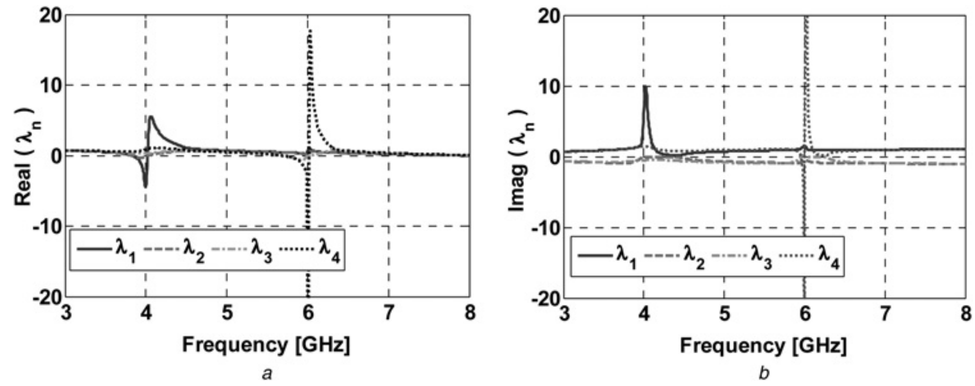

Fig. 7 *Real and imaginary parts of the eigenvalues of the obtained **Φ** and **Γ***

a Real
b Imaginary parts of the eigenvalues of the **Φ**-matrix for a biaxial anisotropic slab with thickness $d = 5$ mm at a normal incidence

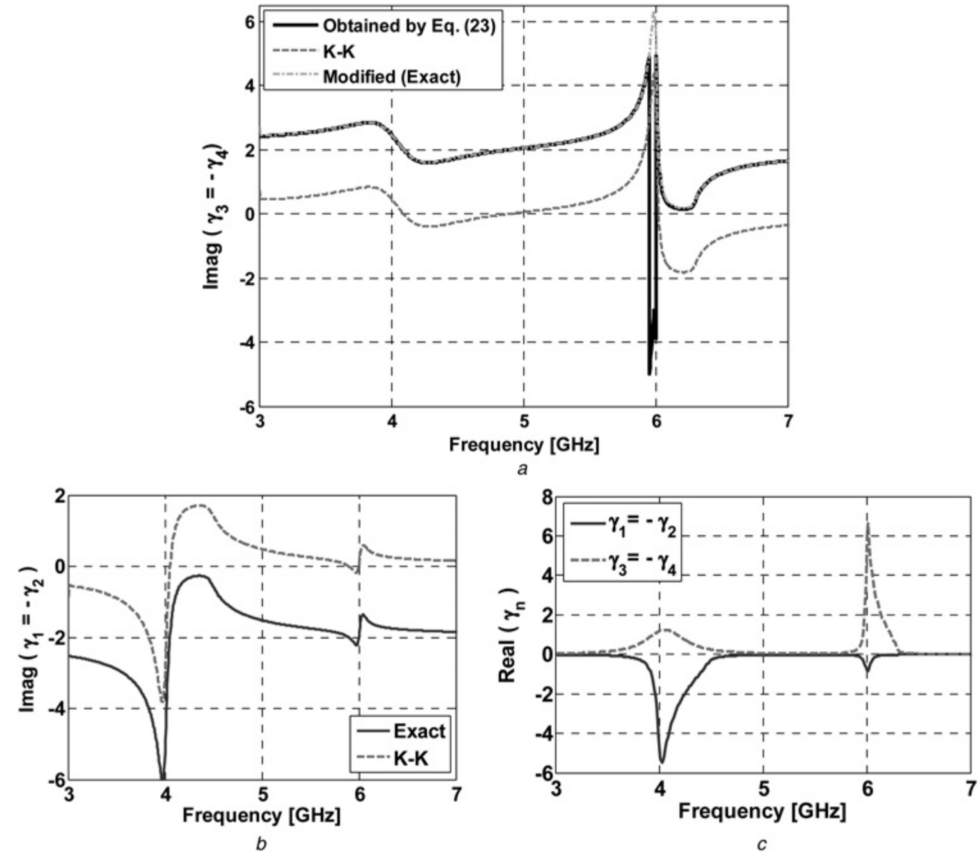

Fig. 8 *Real and imaginary parts of the eigenvalues of the obtained **Φ** and **Γ***

a Imaginary parts of $\gamma_3 = -\gamma_4$ for a biaxial anisotropic slab with thickness $d = 5$ mm at a normal incidence
b Imaginary parts $\gamma_1 = -\gamma_2$. The obtained approximate results by the K–K are also shown
c Real parts of $\gamma_1 = -\gamma_2$ and $\gamma_3 = -\gamma_4$

6 Summary and conclusions

In this paper, an analytical frequency-domain formulation is presented for the inverse problem of a biaxial anisotropic medium based on the state space approach. The proposed characterisation algorithm is mainly based on the state transition matrix and its interesting properties, presented as two proven theorems. In addition, the probable cause of ambiguity in the proposed method is fully investigated and a robust technique based on the K–K relations is presented to solve this challenge. Some numerical examples are given to validate the performance of the proposed formulations. From the numerical results, one can find that they work well for the reconstruction of the constitutive tensor parameters of an unknown biaxial anisotropic slab. In the future, the proposed electromagnetic characterisation method is expected to be used for retrieving the constitutive parameters of generalised anisotropic materials.

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8 Appendix 1

For the $\boldsymbol{\Gamma}$-matrix with the distinct eigenvalues γ_1, γ_2, γ_3 and γ_4, respectively, there exists a matrix $\boldsymbol{M}$ such that $\boldsymbol{\Gamma}$ is equal to $\boldsymbol{M\Lambda M}^{-1}$ where $\boldsymbol{\Lambda}$ is a diagonal matrix in the form of $\mathrm{diag}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. The exponential function of the square matrix $-\Gamma d$ is defined in terms of an infinite Taylor series as

$$\mathrm{e}^{-\boldsymbol{\Gamma} d} = \boldsymbol{I} - \boldsymbol{\Gamma} d + \frac{1}{2!}\boldsymbol{\Gamma}^2 d^2 - \frac{1}{3!}\boldsymbol{\Gamma}^3 d^3 + \cdots \tag{30}$$

where $\boldsymbol{I}$ is a 4×4 identity matrix and the above series is convergent for all the square matrices. By substituting $\boldsymbol{M\Lambda M}^{-1}$ in $\boldsymbol{\Gamma}^2$, we have

$$\begin{aligned}\Gamma^2 &= (\boldsymbol{M\Lambda M}^{-1})(\boldsymbol{M\Lambda M}^{-1}) = \boldsymbol{M\Lambda}(\boldsymbol{M}^{-1}\boldsymbol{M})\boldsymbol{\Lambda M}^{-1} \\ &= \boldsymbol{M}\Lambda^2\boldsymbol{M}^{-1}\end{aligned} \tag{31}$$

and all the powers of $\boldsymbol{\Gamma}$ are similarly reduced. Thus

$$\begin{aligned}\mathrm{e}^{-\Gamma d} &= \boldsymbol{M}\left[\boldsymbol{I} - \Lambda d + \frac{1}{2!}\Lambda^2 d^2 - \frac{1}{3!}\Lambda^3 d^3 + \cdots\right]\boldsymbol{M}^{-1} \\ &= \boldsymbol{M}\mathrm{e}^{-\Lambda d}\boldsymbol{M}^{-1}\end{aligned} \tag{32}$$

Since the determinant of a product of the square matrices is the product of their determinants, the determinant of $\mathrm{e}^{-\Gamma d}$ is equal to that of $\mathrm{e}^{-\Lambda d}$ which due to the diagonal nature of $\boldsymbol{\Lambda}$ is $\exp[-(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)d]$. Beside this, we know that the sum of the eigenvalues of a square matrix is equal to the sum of its diagonal elements [29]. Thus, according to (6), the determinant of the state transition matrix $\boldsymbol{\Phi} = \mathrm{e}^{-\boldsymbol{\Gamma}_\omega d}$ of a biaxial anisotropic slab is equal to unity. The proof is complete.

9 Appendix 2

The Cayley–Hamilton theorem relates a matrix to its characteristic polynomial [28, 29]. By using this theorem, the exponential of the $\boldsymbol{W}$-matrix can be simply calculated. Assuming a 4×4 $\boldsymbol{W}$-matrix with the distinct eigenvalues w_1, w_2, w_3 and w_4, respectively, we can write the equations as

$$\mathrm{e}^{w_p} = \sum_{q=0}^{3} c_q w_p^q \tag{33}$$

where $p = 0$, 1, 2 and 3, respectively. By solving this set of four linear equations, the unknown coefficients c_q for $q = 0, 1, 2$ and 3, respectively, are determined. Finally, the exponential of $\boldsymbol{W}$ can be written as

$$e^{\boldsymbol{W}} = c_0 I + c_1 \boldsymbol{W} + c_2 \boldsymbol{W}^2 + c_3 \boldsymbol{W}^3 \tag{34}$$

Assuming $\boldsymbol{W} = -\boldsymbol{\Gamma}_{\omega} d$, $g_1 = \mathrm{j}\omega\mu_0 d(-\mu_{yy} + \sin^2\theta_0/\varepsilon_{zz})$, $g_2 = \mathrm{j}\omega\mu_0\mu_{yy} d$, $g_3 = \mathrm{j}\omega\varepsilon_0 d(\varepsilon_{yy} - \sin^2\theta_0/\mu_{zz})$ and $g_4 = -\mathrm{j}\omega\varepsilon_0\varepsilon_{xx} d$, respectively, the eigenvalues of $\boldsymbol{W}$ are given by

$$w_{\frac{1}{2}} = \pm\sqrt{g_1 g_4}, \quad w_{\frac{3}{4}} = \pm\sqrt{g_2 g_3} \tag{35}$$

By substituting these eigenvalues in (33) and by solving the equations, the unknown coefficients c_q for $q = 0, 1, 2$ and 3, respectively, are fully determined

$$c_0 = \frac{w_3^2 \cosh(w_1) - w_1^2 \cosh(w_3)}{w_3^2 - w_1^2} \tag{36}$$

$$c_1 = \frac{w_3^3 \sinh(w_1) - w_1^3 \sinh(w_3)}{w_1 w_3 \left(w_3^2 - w_1^2\right)} \tag{37}$$

$$c_2 = -\frac{\cosh(w_1) - \cosh(w_3)}{w_3^2 - w_1^2} \tag{38}$$

$$c_3 = -\frac{w_3 \sinh(w_1) - w_1 \sinh(w_3)}{w_1 w_3 \left(w_3^2 - w_1^2\right)} \tag{39}$$

Then, after some matrix manipulations in (34), the matrix exponential $\boldsymbol{\Phi} = \mathrm{e}^{-\boldsymbol{\Gamma}_{\omega} d\, c}$ may be written as (14a) where

$$\begin{cases} \Phi_{11} = c_0 + c_2 g_1 g_4, & \Phi_{14} = c_1 g_1 + c_3 g_1^2 g_4 \\ \Phi_{22} = c_0 + c_2 g_2 g_3, & \Phi_{23} = c_1 g_2 + c_3 g_2^2 g_3 \\ \Phi_{32} = c_1 g_3 + c_3 g_3^2 g_2, & \Phi_{41} = c_1 g_4 + c_3 g_4^2 g_1 \end{cases} \tag{40}$$

For the computation of the inverse matrix $\boldsymbol{\Phi}^{-1}$, we can use the identity $(\mathrm{e}^{-\boldsymbol{\Gamma}\omega d/c})^{-1} = \mathrm{e}^{+\boldsymbol{\Gamma}\omega d/c}$ and compute $\mathrm{e}^{+\boldsymbol{\Gamma}\omega d/c}$. It is evident that the eigenvalues remain unchanged rather than the prior case. In this case, we should consider

$$\mathrm{e}^{-\boldsymbol{W}} = c_0' \mathrm{I} + c_1' \boldsymbol{W} + c_2' \boldsymbol{W}^2 + c_3' \boldsymbol{W}^3 \tag{41}$$

and solve the four equations

$$\mathrm{e}^{-w_p} = \sum_{q=0}^{3} c_q' w_p^q \tag{42}$$

where $p = 0, 1, 2$ and 3, respectively. It can be easily seen that the coefficients of c_q' for $q = 0, 1, 2$ and 3, respectively, are given by

$$c_0' = c_0, \quad c_1' = -c_1, \quad c_2' = c_2, \quad c_3' = -c_3 \tag{43}$$

After some simple matrix manipulations, one can obtain the inverse matrix $\boldsymbol{\Phi}^{-1}$ as presented in (14b).