Analysis of Reflection and Transmission from Biaxial Chiral Slabs Using the State Space Approach

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Abstract— **In this paper, the propagation of electromagnetic waves through an infinite slab of biaxial chiral medium is analytically formulated for an arbitrary incidence using the 4×4 transition matrix method. In this powerful method, a state vector differential equation is extracted whose solution is given in terms of a transition matrix relating the tangential field components at the input and output planes of the biaxial chiral layer. The validity of the method is verified using two typical examples at the microwave frequencies.**

Keywords-component: Reflection and Transmission; Biaxial Chiral media; State Space Approach

I. INTRODUCTION

The study of artificial composite structures as an important topic in electromagnetic research has been the subject of many endeavors over the years. Applications of such structures in microwave and millimeter wave regimes have prompted a renewed interest over the last decade leading to the design of different microwave devices, such as cavities, resonators, waveguides, and lenses [1]-[4] and has allowed for the realization of negative refraction indices [5]. Studying artificial bianisotropic media is a recognized subject which dates back to the last decades [6]. Unlike the ordinary materials, described by electric permittivity and magnetic permeability, bianisotropic media include a magneto-electric coupling yielding to interesting properties of the electromagnetic fields. The biaxial bianisotropic chiral medium is a special type of bianisotropic medum, where the chirality appears in two directions. A biaxial bianisotropic chiral slab can be realized by placing miniature wire spirals immersed in a host dielectric slab [7, 8]. One of the well-known applications of uni- and bi-axial chiral layers is in polarization transformers, whereby any polarization can be transformed into any other polarization [9].

The reflection and transmission properties of a plane wave incident normally or obliquely from free space to a uni- and biaxial bianisotropic chiral slab have been well studied over the years [10-12]. This paper presents the powerful 4×4 transition matrix method, where Maxwell's equations in the biaxial chiral region are only cast into a 4×4 matrix formulation, where it is not necessary to specify the unintelligible eigenpolarizations of this layer. Then, the complete solution is derived by combining the boundary conditions at the interfaces with the transition matrix.

The paper is organized as follows: In Sec. I, the differential equations describing the biaxial bianisotropic chiral slab are extracted, and the formulas of the reflection and transmission are then derived. Section III deals with the validation of the presented method based on the state space approach along with some example calculations.

II. FORMULATION OF PROBLEM

To obtain the solution for the reflection and transmission coefficients of a biaxial chiral layer, consider the problem illustrated in Fig. 1. The constitutive relations in such a biaxial chiral medium can be written as

$$\bar{D} = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} \bar{E} - j\sqrt{\varepsilon_0\mu_0} \begin{pmatrix} \kappa & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \kappa \end{pmatrix} \bar{H} \tag{1}$$

$$\bar{B} = \begin{pmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{pmatrix} \bar{H} + j\sqrt{\varepsilon_0\mu_0} \begin{pmatrix} \kappa & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \kappa \end{pmatrix} \bar{E} \tag{2}$$

where κ is the chirality parameter, and ε_0 and μ_0 are permittivity and permeability of free space. It is assumed that an arbitrarily polarized plane wave is incident from free space to the biaxial chiral slab at an oblique angle θ_0. Substituting the constitutive equations into Maxwell's equations, considering $\partial/\partial y = 0$ and $\partial/\partial x = -jk_0\sin\theta_0$ where k_0 is the wave number in vacuum, the differential equations describing biaxial chiral layer are given by

$$\frac{d}{dz}\begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix} = \Gamma \begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix} \tag{3}$$

where the elements of the Γ-matrix are given by:

$$
\Gamma=\begin{pmatrix}
0 & \frac{\omega\kappa}{c\left(\varepsilon_z^r\mu_z^r-\kappa^2\right)}\sin^2\theta_0 & 0 & -j\omega\mu_0\left(\mu_y^r-\frac{\mu_z^r}{\varepsilon_z^r\mu_z^r-\kappa^2}\sin^2\theta_0\right)\\
-\frac{\omega}{c}\kappa & 0 & j\omega\mu_0\mu_x^r & 0\\
0 & j\omega\varepsilon_0\left(\varepsilon_y^r-\frac{\varepsilon_z^r}{\varepsilon_z^r\mu_z^r-\kappa^2}\sin^2\theta_0\right) & 0 & \frac{\omega\kappa}{c\left(\varepsilon_z^r\mu_z^r-\kappa^2\right)}\sin^2\theta_0\\
-j\omega\varepsilon_0\varepsilon_x^r & 0 & -\frac{\omega}{c}\kappa & 0
\end{pmatrix} \tag{4}
$$

where superscript r on ε and μ indicates relative permittivity and permeability, respectively; and c and ω are the speed of light in vacuum and the angular frequency, respectively.

We may define a 4×4 transition matrix Φ that relates the transverse components of electric and magnetic fields at the two boundaries of the slab

$$
\begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix}_{z=0} = \Phi_{4\times4}\begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix}_{z=t} \tag{5}
$$

Similar to state-space equations in linear control systems [13, 14], it can be easily seen that the transition matrix Φ is given by

$$
\Phi = e^{-\Gamma t}. \tag{6}
$$

For the computation of the matrix, many methods have been proposed [14] such as expansion of Φ in a power series, methods based on diagonalization of matrix Γ, Cayley-Hamilton theorem, *expm* command in MATLAB, etc.

By introducing the reflection and transmission matrices, T and R, we can write:

$$
\begin{pmatrix} E_x^r \\ E_y^r \end{pmatrix}_{z=0} = \begin{pmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{pmatrix}\begin{pmatrix} E_x^i \\ E_y^i \end{pmatrix}_{z=0} \tag{7}
$$

$$
\begin{pmatrix} E_x^t \\ E_y^t \end{pmatrix}_{z=t} = \begin{pmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{pmatrix}\begin{pmatrix} E_x^i \\ E_y^i \end{pmatrix}_{z=0} \tag{8}
$$

where the superscripts i, r, and t denote the incident, reflected, and transmitted fields, respectively. If the transition matrix Φ is partitioned into four 2×2 submatrices, such that

$$
\Phi = \begin{pmatrix} (\Phi_1)_{2\times2} & (\Phi_2)_{2\times2} \\ (\Phi_3)_{2\times2} & (\Phi_4)_{2\times2} \end{pmatrix}, \tag{9}
$$

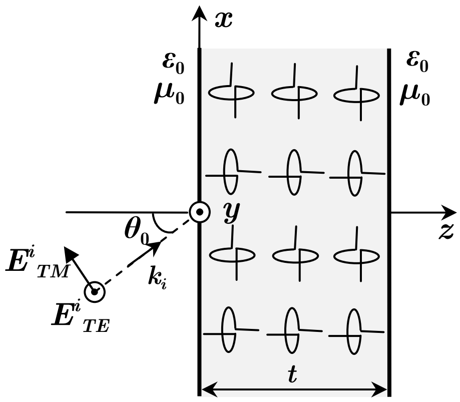

Fig. 1. A typical biaxial chiral slab. The chirality appears in x and z directions.

then equation (5) may be rewritten as

$$
\begin{pmatrix} E_x^i \\ E_y^i \end{pmatrix}_{z=0} + \begin{pmatrix} E_x^r \\ E_y^r \end{pmatrix}_{z=0} = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}\begin{pmatrix} E_x^t \\ E_y^t \end{pmatrix}_{z=t} + \begin{pmatrix} \Phi_{13} & \Phi_{14} \\ \Phi_{23} & \Phi_{24} \end{pmatrix}\begin{pmatrix} H_x^t \\ H_y^t \end{pmatrix}_{z=t} \tag{10}
$$

$$
Z_0^{-1}\begin{pmatrix} E_x^i \\ E_y^i \end{pmatrix}_{z=0} - Z_0^{-1}\begin{pmatrix} E_x^r \\ E_y^r \end{pmatrix}_{z=0} = \begin{pmatrix} \Phi_{31} & \Phi_{32} \\ \Phi_{41} & \Phi_{42} \end{pmatrix}\begin{pmatrix} E_x^t \\ E_y^t \end{pmatrix}_{z=t} + \begin{pmatrix} \Phi_{33} & \Phi_{34} \\ \Phi_{43} & \Phi_{44} \end{pmatrix} Z_0^{-1}\begin{pmatrix} E_x^t \\ E_y^t \end{pmatrix}_{z=t} \tag{11}
$$

wherein Z_0 is the wave impedance matrix given by:

$$
Z_0 = \begin{pmatrix} 0 & \sqrt{\mu_0/\varepsilon_0}\,\cos\theta_0 \\ -\sqrt{\mu_0/\varepsilon_0}\,/\cos\theta_0 & 0 \end{pmatrix}. \tag{12}
$$

Using Equations (10) and (11) and by considering Equations (7) and (8), after some matrix computations one can write

$$
R = \left[\Phi_1 Z_0 + \Phi_2 - Z_0\left(\Phi_3 Z_0 + \Phi_4\right)\right]\left[\Phi_1 Z_0 + \Phi_2 + Z_0\left(\Phi_3 Z_0 + \Phi_4\right)\right]^{-1}, \tag{13}
$$

$$
T = 2\,Z_0\left[\Phi_1 Z_0 + \Phi_2 + Z_0\left(\Phi_3 Z_0 + \Phi_4\right)\right]^{-1}. \tag{14}
$$

Once R and T matrices are determined, co- and cross-reflection and transmission coefficients could be identified as

$$
\begin{pmatrix} R_{TM-TM} & R_{TM-TE} \\ R_{TE-TM} & R_{TE-TE} \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy}/\cos\theta_0 \\ R_{yx}\cos\theta_0 & R_{yy} \end{pmatrix}. \tag{15}
$$

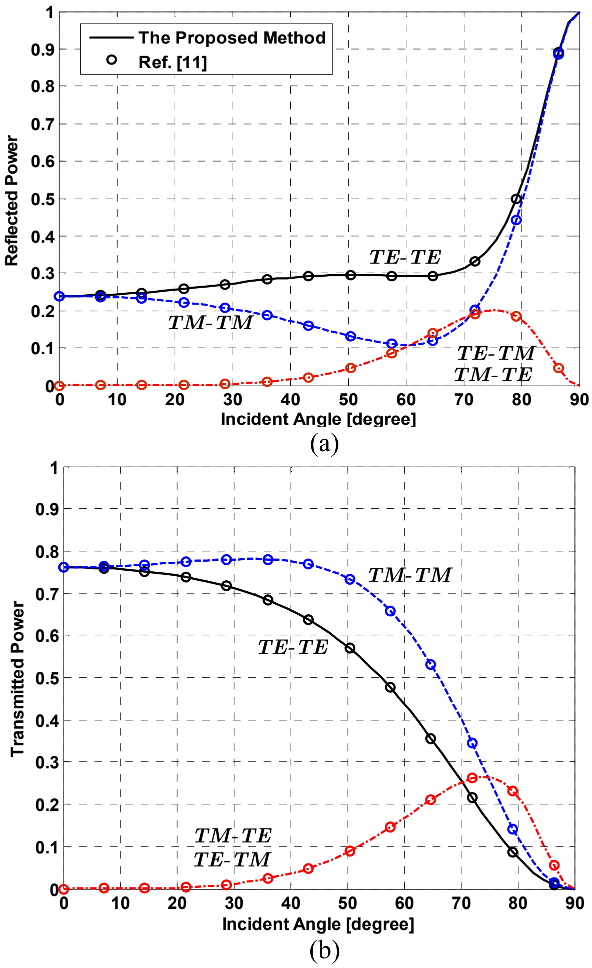

Fig. 2. The reflected power (a), and transmitted power (b) as a function of incident angle for uniaxial chiral slab discussed in example 1.

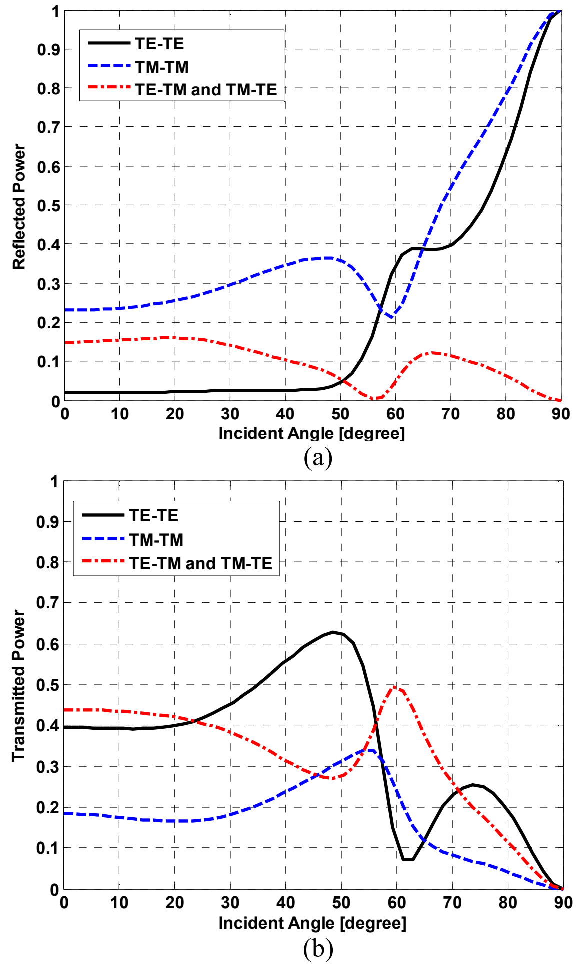

Fig. 3. The reflected power (a), and transmitted power (b) as a function of incident angle for biaxial chiral slab discussed in example 2.

III. Numerical Examples

In this section, in order to verify the validity of the method, two uni- and b-iaxial chiral slabs are considered and analyzed using the presented method.

A. Example 1

Consider a uniaxial chiral slab with thickness of $t = 5$ mm, and the electromagnetic parameters $\varepsilon_x = \varepsilon_y = \varepsilon_z = 3\varepsilon_0$, $\mu_x = \mu_y = \mu_z = \mu_0$, and $\kappa = 1.5$. Assume a plane wave with unity amplitude and excitation frequency 10 GHz obliquely illuminating the uniaxial chiral slab. The reflected and transmitted powers versus the angle of incidence obtained by the proposed method and the exact results presented in [11] are shown in Fig. 2. Comparison between the two results illustrates the good behavior of the proposed method.

B. Example 2

As the second example, consider a biaxial chiral slab with thickness of $t = 20$ cm, and the electromagnetic parameters $\varepsilon_x = \varepsilon_0$, $\varepsilon_y = 2\varepsilon_0$, $\varepsilon_z = 4\varepsilon_0$, $\mu_x = 3\mu_0$, $\mu_y = 1.5\mu_0$, $\mu_z = \mu_0$, and $\kappa = 1.5$. Assume a plane wave with unity amplitude and excitation frequency 1 GHz obliquely illuminating the biaxial chiral slab. Figure 3 shows the reflected and transmitted power obtained from the proposed method versus the angle of incidence.

IV. Conclusions

This paper presents an analytic formulation for reflection and transmission problems involving biaxial chiral layers. In the presented method, a 4×4 transition matrix that relates the transverse components of electric and magnetic fields at the two boundaries of the slab is employed and is combined with the boundary conditions. The validity of the presented method is achieved by providing some numerical examples and comparing the obtained results with those of other available methods for two special cases. As an interesting, significant, and applicable property of the presented method, it is not necessary to specify the eigenpolarizations of the biaxial chiral layers. In addition, the presented method can be used to analysis of the reflection and transmission problems involving more complex bianisotropic layers. In fact, the Γ-matrix should be only calculated for more complex layers and the next steps to obtain reflection and transmission matrices are the same procedure describing in this paper.

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