State-transition-matrix method for inverse scattering in one-dimensional inhomogeneous media

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This study presents an analytical approach for the electromagnetic characterization of one-dimensional inhomogeneous media. The proposed approach provides the permittivity profile of the medium in terms of the reflection and transmission coefficients. The inverse solution of the permittivity profile is obtained with the help of the state-transition matrix (STM) and its properties, which are presented and proved. The advantage of using this analytic reconstruction technique is its ability to remove complexity and nonlinearity of the inverse problem. Several examples have been considered for validation of the proposed technique and, in each case, quite good agreement has been found between the original and reconstructed profiles. It has been established from the obtained results that when the scattering parameters are combined with the properties of STM, a robust and reliable technique is provided for the electromagnetic characterization of one-dimensional inhomogeneous media.

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I. INTRODUCTION

The study of electromagnetic inverse scattering is a widely encountered problem and has been a subject of extensive research. The inverse scattering of one-dimensional inhomogeneous media is of great interest due to its potential functional benefits for many applications, such as remote sensing, biomedical diagnosis, industrial tomography, nondestructive testing, military surveillance, and many others. In general, the reconstruction process involves the measurement of scattering data due to an illuminating wave. Information about the unknown permittivity profile of an object in terms of measured scattering data is obtained by using some inverse techniques. Various methods have been used to reconstruct one-dimensional permittivity profiles from electromagnetic scattering data which can be categorized into time domain and frequency domain methods [1–12]. The time domain methods always require a very narrow pulse which is difficult to radiate and to be received in practice.

A survey of the literature on inverse scattering indicates that the conventional methods basically depend on source reconstruction which leads to nonlinear equations that can only be solved using iterative and optimization algorithms [13–15]. In fact, the problem with most of these numerical and quasinumerical techniques is that they are usually computationally intensive, and sometimes it is difficult to obtain a unique and stable solution for the corresponding inverse problem. Although a number of analytical techniques have been proposed to reconstruct the permittivity profiles in terms of the inverse Fourier transform of the scattering data, most of these approaches usually assume that the scattering data are available over the whole frequency band which is difficult in practice. In addition, a large number of investigations have been carried out using the Gel'fand-Levitan-Marchenko theory [16–18]. Unfortunately, this exact approach is actually very difficult to implement due to considerable mathematical complexity. It is mainly due to these reasons that a unique analytical approach is proposed in this paper to obtain the inverse solution of the one-dimensional inhomogeneous medium.

Briefly, for homogeneous material, one can find numerous studies in which the analytical expression between material parameters and scattering parameters was solved either directly or iteratively [19–21]. A more complicated problem compared to that of homogeneous materials is the measurement of inhomogeneous structures, which has been mainly treated by the aid of numerical and optimization techniques. In this contribution, the aim of the study is to present an analytic methodology based on the state-transition-matrix (STM) method to the characterization of inhomogeneous media. The STM method has been well described in forward scattering problems, including isotropic, anisotropic, and bianisotropic media, over the years [22–26]. Recently, its application in the formulation for inverse scattering problems, including homogeneous media, has been proposed [27,28].

Organization of this paper is as follows. The paper starts with a brief discussion of application of the STM method in inhomogeneous media. Section III deals with the proof of some useful properties of the STM of an inhomogeneous layer. The inverse algorithm based on the transition-matrix method is then explained in detail in Sec. IV. The proposed approach is validated along with example computations in Sec. V. Finally, a summary and conclusions are made in Sec. VI.

II. FORMULATION OF FORWARD SCATTERING PROBLEM

The geometry of the medium under investigation is shown in Fig. 1. A time-harmonic electromagnetic wave is normally incident from the left upon an inhomogeneous dielectric slab with permittivity ε which is a function of the geometric distance z. In the spectral domain approach, the time dependence is $e^{j\omega t}$. The planar structure is of infinite extent along the y direction, and so the derivative of the fields with respect to the y variable vanishes. In addition, the derivative of the fields with respect to the x variable in the slab must take

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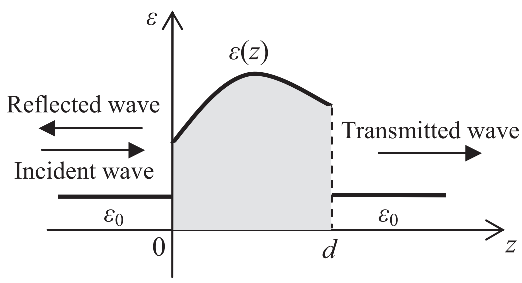


FIG. 1. An inhomogeneous slab exposed to an electromagnetic wave.

on the same value as in free space in order to satisfy the boundary conditions on tangential fields at the boundaries. So, due to the normal incidence of plane wave, $\partial/\partial x = 0$. By substituting the constitutive equations of inhomogeneous media into curl Maxwell's equations and by eliminating z components of electric and magnetic fields one can write

$$\frac{d}{dz}\begin{pmatrix}\bar{E}_T\\ \bar{H}_T\end{pmatrix} = \Gamma\begin{pmatrix}\bar{E}_T\\ \bar{H}_T\end{pmatrix}, \tag{1}$$

where $\bar{E}_T = (E_x, E_y)$ and $\bar{H}_T = (H_x, H_y)$ are the transverse components of the electric and magnetic fields, respectively, and the elements of a 4×4 Γ matrix are given by

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 & -j\omega\mu_0\mu_r \\ 0 & 0 & j\omega\mu_0\mu_r & 0 \\ 0 & j\omega\varepsilon_0\varepsilon_r(z) & 0 & 0 \\ -j\omega\varepsilon_0\varepsilon_r(z) & 0 & 0 & 0 \end{pmatrix}, \tag{2}$$

where ω is the angular frequency, and ε_0 and μ_0 are the permittivity and the permeability of vacuum.

Defining a 4×4 STM (Φ) that relates the transverse components of electric and magnetic fields at the two boundaries of the inhomogeneous slab,

$$\begin{pmatrix}\bar{E}_T(0)\\ \bar{H}_T(0)\end{pmatrix} = \Phi\begin{pmatrix}\bar{E}_T(d)\\ \bar{H}_T(d)\end{pmatrix} = \begin{pmatrix}(\Phi_1)_{2\times2} & (\Phi_2)_{2\times2}\\ (\Phi_3)_{2\times2} & (\Phi_4)_{2\times2}\end{pmatrix}\begin{pmatrix}\bar{E}_T(d)\\ \bar{H}_T(d)\end{pmatrix}, \tag{3}$$

one can write

$$\bar{E}_T^i(0) + \bar{E}_T^r(0) = \Phi_1\bar{E}_T^t(d) + \Phi_2\bar{H}_T^t(d), \tag{4}$$

$$Z_0^{-1}\bar{E}_T^i(0) - Z_0^{-1}\bar{E}_T^r(0) = \Phi_3\bar{E}_T^t(d) + \Phi_4 Z_0^{-1}\bar{E}_T^t(d), \tag{5}$$

where the superscripts i, r, and t denote the incident, reflected, and transmitted field, respectively, and the wave impedance matrix Z_0 is defined as

$$Z_0 = \begin{pmatrix} 0 & \eta_0 \\ -\eta_0 & 0\end{pmatrix}, \tag{6}$$

where η_0 is the intrinsic impedance of free space. After some matrix manipulations, reflection and transmission coefficients can be obtained which have been presented in [27].

III. STM OF AN INHOMOGENEOUS LAYER

A. Computation of STM of inhomogeneous layers

The computation of the STM of inhomogeneous layers is more complicated than that of homogeneous layers. If the slab is homogeneous, similar to state-space analysis in linear systems, the STM is $\exp(-\Gamma d)$.

The most straightforward method is subdividing the inhomogeneous slab into N homogeneous electrically thin layers. Using the Cayley-Hamilton theorem discussed in [29,30], the STM of the nth homogeneous layer with constitutive parameters ε_n and μ_n is given by

$$\Phi^{\text{layer}} = \begin{pmatrix} \Phi_{11}^{\text{layer}} & 0 & 0 & \Phi_{14}^{\text{layer}} \\ 0 & \Phi_{11}^{\text{layer}} & -\Phi_{14}^{\text{layer}} & 0 \\ 0 & -\Phi_{41}^{\text{layer}} & \Phi_{11}^{\text{layer}} & 0 \\ \Phi_{41}^{\text{layer}} & 0 & 0 & \Phi_{11}^{\text{layer}} \end{pmatrix}, \tag{7}$$

where its entries are given in Appendix A. The STM of an inhomogeneous slab is obtained from those of thin layers as $\Phi = \Phi^{\text{layer}(1)}\Phi^{\text{layer}(2)}\Phi^{\text{layer}(3)}\dots\Phi^{\text{layer}(N)}$. It can be seen that the STM of the inhomogeneous slab has the following form:

$$\Phi = \begin{pmatrix} \Phi_{11} & 0 & 0 & \Phi_{14} \\ 0 & \Phi_{11} & -\Phi_{14} & 0 \\ 0 & -\Phi_{41} & \Phi_{44} & 0 \\ \Phi_{41} & 0 & 0 & \Phi_{44} \end{pmatrix}. \tag{8}$$

As an interesting point, observe that unlike the transition matrix of a homogeneous layer, this matrix has four distinct nonzero entries and its diagonal elements are generally different.

It is convenient for some purposes to compute analytically the STM in the transition-matrix method. In addition to the above method for calculating the matrix, we can use an analytic technique based on the Peano-Baker series [31]. In this method, the Ψ matrix of an inhomogeneous layer defined as $\exp(\Gamma z)$ is given by

$$\Psi(z) = \mathrm{I} + \int_0^z \Gamma(z_0)dz_0 + \int_0^z \Gamma(z_0)\int_0^{z_0}\Gamma(z_1)dz_1dz_0 + \int_0^z \Gamma(z)\int_0^{z_0}\Gamma(z_1)\int_0^{z_1}\Gamma(z_2)dz_2dz_1dz_0 + \cdots, \tag{9}$$

where the STM of the inhomogeneous slab is $\Phi = \Psi^{-1}(z = d)$. It has been shown that the Peano-Baker series is unique and converges absolutely and uniformly [31]. In this study, due to theoretical purposes, we use this analytical method for the computation of the STM of inhomogeneous media.

B. Properties of STM of an inhomogeneous layer

Prior to the discussion of the characterization methodology, it is required to discuss some useful properties of the state-transition matrix of a one-dimensional inhomogeneous layer. The following two theorems are introduced here and then proved in the Appendices.

Theorem 1. The determinant of the STM of an inhomogeneous layer is equal to unity.

Theorem 2. The inverse of the STM of an inhomogeneous layer could be written as follows:

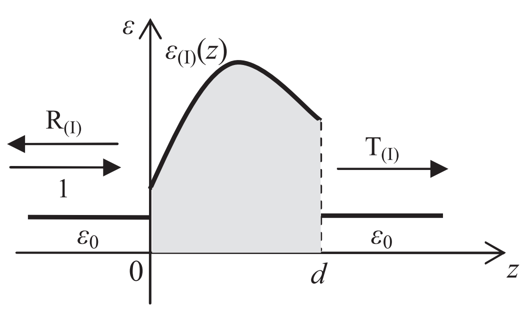
$$\Phi^{-1} = \begin{pmatrix} \Phi_{44} & 0 & 0 & -\Phi_{14} \\ 0 & \Phi_{44} & \Phi_{14} & 0 \\ 0 & \Phi_{41} & \Phi_{11} & 0 \\ -\Phi_{41} & 0 & 0 & \Phi_{11} \end{pmatrix}. \quad (10)$$

Comparing this matrix with (8), observe that the positions of the major diagonal elements are swapped and the signs of the minor diagonal elements are changed.

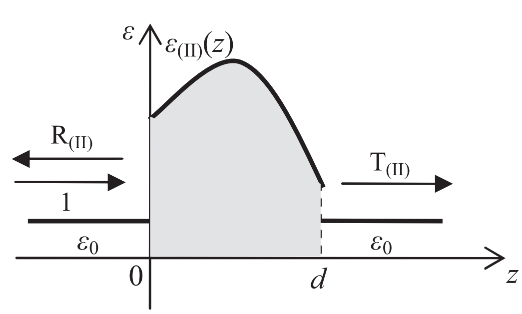
C. Nonreciprocity of an inhomogeneous slab

The plane wave transmission coefficients for an inhomogeneous slab are the same when the wave incidences on the slab are from the left or from the right, but the reflection coefficients usually differ [32]. Figure 2 shows these two cases. When a time-harmonic electromagnetic plane wave is incident from the left, free space, upon the inhomogeneous layer, the STM Φ is given by (8). In the other case, where an electromagnetic wave is incident from the right, using $\Phi_{(\mathrm{II})} = \Phi^{\mathrm{layer}(N)}\Phi^{\mathrm{layer}(N-1)}\Phi^{\mathrm{layer}(N-2)}\ldots\Phi^{\mathrm{layer}(1)}$, one can see that

$$\Phi_{(\mathrm{II})} = \begin{pmatrix} \Phi_{44} & 0 & 0 & \Phi_{14} \\ 0 & \Phi_{44} & -\Phi_{14} & 0 \\ 0 & -\Phi_{41} & \Phi_{11} & 0 \\ \Phi_{41} & 0 & 0 & \Phi_{11} \end{pmatrix}. \quad (11)$$



(a) Case (I)



(b) Case (II)

FIG. 2. (a) The wave incidences on the inhomogeneous slab from the left. (b) The wave incidences on the inhomogeneous slab from the right. This case is as if the slab is rotated.

IV. FORMULATION OF RECONSTRUCTION ALGORITHM

In this section, a general method is proposed to reconstruct an electric permittivity profile using scattering parameters, based on the STM and its properties. In the first step, from the knowledge of reflection and transmission coefficients and the properties of STM, the STM of an inhomogeneous slab is determined. Then, considering the permittivity profile by a polynomial function and with the help of a Peano-Baker series, the unknown profile is determined.

A. Determination of STM

Without loss of generality, we suppose that the incident wave has the form of $\bar{E}_i = e^{-jk_0 z}\hat{a}_y$ where $k_0 = \omega\sqrt{\mu_0\varepsilon_0}$ is the free space wave number. When this electromagnetic wave is incident from the left upon the inhomogeneous layer, considering the STM in (8) along (4) and (5), one can write the following equations:

$$1 + E^r_{y(\mathrm{I})} - \left(\Phi_{11} + \frac{\Phi_{14}}{\eta_0}\right)E^t_{y(\mathrm{I})} = 0,$$
$$1 - E^r_{y(\mathrm{I})} - (\eta_0\Phi_{41} + \Phi_{44})E^t_{y(\mathrm{I})} = 0. \quad (12)$$

Similarly, in case (II) shown in Fig. 2(b), considering the STM in (11), one can rewrite (4) and (5) as follows:

$$1 + E^r_{y(\mathrm{II})} - \left(\Phi_{44} + \frac{\Phi_{14}}{\eta_0}\right)E^t_{y(\mathrm{II})} = 0,$$
$$1 - E^r_{y(\mathrm{II})} - (\eta_0\Phi_{41} + \Phi_{11})E^t_{y(\mathrm{II})} = 0. \quad (13)$$

In addition, the set of equations obtained by the discussed theorems in the prior section are considered as follows:

$$\det(\Phi) = 1, \quad \Phi(1,1) - \Phi^{-1}(3,3) = 0,$$
$$\Phi(1,4) + \Phi^{-1}(1,4) = 0, \quad \Phi(4,1) + \Phi^{-1}(4,1) = 0. \quad (14)$$

Thus, the STMs are found by solving the above system of equations.

B. Permittivity profile reconstruction

Once the STM of the inhomogeneous layer is determined, the dielectric profile can be identified. The second step of the reconstruction algorithm is based on the consideration of the permittivity profile by a continuous function defined by a small number of coefficients. In order to reconstruct the dielectric profile in z space, we expand the profile in a polynomial series as

$$\varepsilon_r(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_N z^N. \quad (15)$$

According to this expansion, we can compute the STM of an inhomogeneous slab using (9). For instance, consider a linear profile for dielectric permittivity and $j\omega\mu_0\mu_r = u$ and $j\omega\varepsilon_0\varepsilon_r(z) = v + wz$ wherein u, v, and w are arbitrary constants. Using the Peano-Baker series, the STM of an inhomogeneous slab is given by

$$\Phi = \left.\begin{pmatrix} \Psi_{11} & 0 & 0 & \Psi_{14} \\ 0 & \Psi_{11} & -\Psi_{14} & 0 \\ 0 & -\Psi_{41} & \Psi_{44} & 0 \\ \Psi_{41} & 0 & 0 & \Psi_{44} \end{pmatrix}^{-1}\right|_{z=d}, \quad (16)$$

wherein

$$\Psi_{11} = 1 + \left(\tfrac{1}{2}uv\right)z^2 + \left(\tfrac{1}{6}uw\right)z^3 + \cdots,$$
$$\Psi_{14} = -uz - \left(\tfrac{1}{6}u^2v\right)z^3 - \left(\tfrac{1}{12}u^2w\right)z^4 + \cdots,$$
$$\Psi_{41} = -vz - \left(\tfrac{1}{2}w\right)z^2 - \left(\tfrac{1}{6}uv^2\right)z^3 + \cdots,$$
$$\Psi_{44} = 1 + \left(\tfrac{1}{2}uv\right)z^2 + \left(\tfrac{1}{3}uw\right)z^3 + \cdots. \quad (17)$$

Observe that only the first four terms in the series are written. By comparing these terms with the STM obtained in the previous section, unknown coefficients u, v, and w and subsequently unknown profiles are determined. For high order profiles such as quadratic, cubic, etc., one can obtain the STM of an inhomogeneous slab using the Peano-Baker series.

V. NUMERICAL EXAMPLES, RESULTS, AND DISCUSSION

To test the validity of the presented reconstruction algorithm, this section is devoted to some computation examples. The reflection and transmission coefficient data are simulated by the forward STM method discussed in Sec. II and also the wave splitting method. Here, we use MATLAB for the computation procedures. The computation times of the inversion algorithm were less than a few minutes for all examples.

A. Inhomogeneous slab with quadratic permittivity profile

Firstly, a nonmagnetic inhomogeneous layer with thickness of 2 cm and quadratic permittivity profile of

$$\varepsilon_r(z) = 3 - 6\left(\frac{z}{d}\right) + 8\left(\frac{z}{d}\right)^2 \quad (18)$$

is considered. Using the forward STM method discussed in Sec. II, transmission and reflection coefficients at a frequency of 1 GHz are $T_{(II)} = T_{(I)} = 0.7031 - j0.6427$, $R_{(II)} = -0.1847 - j0.2417$, and $R_{(I)} = -0.2242 - j0.2056$. These calculated S-parameter data are fed into the reconstruction algorithm to determine the STM of an inhomogeneous layer. By solving the discussed system of equations in Sec. IV A, the STM is obtained as follows:

$$\Phi = \begin{pmatrix} 0.7467 + j0.0000 & 0.0000 + j0.0000 & 0.0000 + j0.0000 & 0.0046 + j147.09 \\ 0.0000 + j0.0000 & 0.7467 + j0.0000 & -0.0046 - j147.09 & 0.0000 + j0.0000 \\ 0.0000 + j0.0000 & -0.0000 - j0.0027 & 0.8029 + j0.0000 & 0.0000 + j0.0000 \\ 0.0000 + j0.0027 & 0.0000 + j0.0000 & 0.0000 + j0.0000 & 0.8029 + j0.0000 \end{pmatrix}. \quad (19)$$

In the next step, we can determine the permittivity profile using the discussed approach in Sec. IV B. Notice that when using (9) in numerical calculations, we can only consider a limited number of terms of the series. Here, the computation is performed considering the first ten terms in (9). In order to reconstruct the permittivity profile of an inhomogeneous slab, three linear, quadratic, and cubic profiles with unknown coefficients are considered. In each of these cases, using (9), (16), and (17), the STM of an inhomogeneous layer is computed and by comparing it with (19), the unknown coefficients are determined. The computation results are presented in Table I. The reconstructed linear, quadratic, and cubic profiles along the true profile are displayed in Fig. 3. The horizontal axis is normalized by the thickness of the inhomogeneous slab d. Observe that the linear approximation is not accurate, but the reconstructed quadratic and cubic profiles are in very good agreement with the original profile.

TABLE I. Reconstructed linear, quadratic, cubic, and fourth degree polynomial profiles.

Linear
$\varepsilon_r(z) = c_0 + c_1\left(\frac{z}{d}\right)$
$c_0 = 0.6011$, $c_1 = 3.5837$
Quadratic
$\varepsilon_r(z) = c_0 + c_1\left(\frac{z}{d}\right) + c_2\left(\frac{z}{d}\right)^2$
$c_0 = 3.0367$, $c_1 = -6.2377$, $c_2 = 8.2649$
Cubic
$\varepsilon_r(z) = c_0 + c_1\left(\frac{z}{d}\right) + c_2\left(\frac{z}{d}\right)^2 + c_3\left(\frac{z}{d}\right)^3$
$c_0 = 3.1458$, $c_1 = -7.7032$, $c_2 = 12.2757$, $c_3 = -2.8997$

B. Inhomogeneous layer with exponential permittivity profile

As the second illustrative example, the exponential permittivity profile,

$$\varepsilon_r(z) = 2 + 0.05\exp\left(\frac{5z}{d}\right), \quad (20)$$

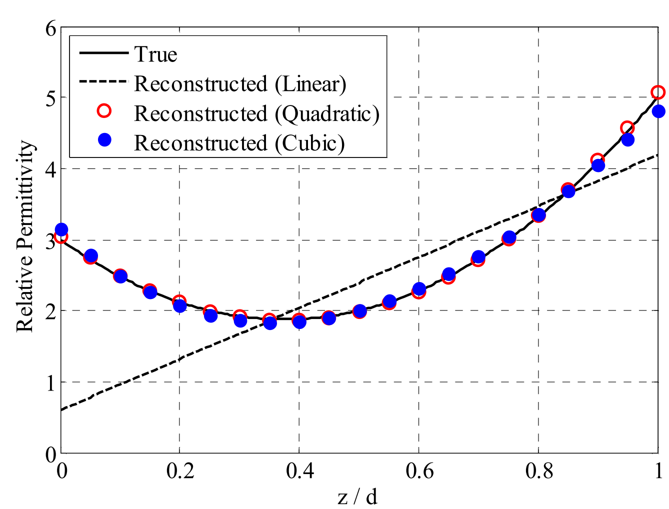


FIG. 3. (Color online) True and reconstructed permittivity profiles of an inhomogeneous slab with quadratic permittivity profile.

TABLE II. Reconstructed linear, quadratic, cubic, and fourth degree polynomial profiles.

Linear
$\varepsilon_r(z) = c_0 + c_1 \left(\frac{z}{d}\right)$
$c_0 = 0.7569,\ c_1 = 5.4465$
Quadratic
$\varepsilon_r(z) = c_0 + c_1 \left(\frac{z}{d}\right) + c_2 \left(\frac{z}{d}\right)^2$
$c_0 = 2.3010,\ c_1 = -4.6051,\ c_2 = 10.2921$
Cubic
$\varepsilon_r(z) = c_0 + c_1 \left(\frac{z}{d}\right) + c_2 \left(\frac{z}{d}\right)^2 + c_3 \left(\frac{z}{d}\right)^3$
$c_0 = 2.4087,\ c_1 = -2.6490,\ c_2 = 2.7346,\ c_3 = 5.9094$
Fourth degree polynomial: $\varepsilon_r(z) = c_0 + c_1 \left(\frac{z}{d}\right) + c_2 \left(\frac{z}{d}\right)^2 + c_3 \left(\frac{z}{d}\right)^3 + c_4 \left(\frac{z}{d}\right)^4$
$c_0 = 2.1102,\ c_1 = -1.5314,\ c_2 = 11.9241,\ c_3 = -24.1788,\ c_4 = 21.01$

of an inhomogeneous layer with thickness of 2 cm is to be reconstructed. Using the forward STM method discussed in Sec. II, reflection and transmission coefficients at a frequency of 1 GHz are $T_{(II)} = T_{(I)} = 0.5805 - j0.6935$, $R_{(II)} = -0.2795 - j0.3223$, and $R_{(I)} = -0.3665 - j0.2184$. In order to verify the accuracy of these results, the scattering parameters by the wave splitting method based on cascading thin linear layers discussed in [33] are also computed as $T_{(II)} = T_{(I)} = 0.5801 - j0.6938$, $R_{(II)} = -0.2797 - j0.3223$, and $R_{(I)} = -0.3668 - j0.2183$.

The calculated S-parameter data are fed into the extraction algorithm to obtain the STM as

$$\Phi = \begin{pmatrix} 0.6348 + j0.0000 & 0.0000 + j0.0000 & 0.0000 + j0.0000 & 0.0046 + j144.05 \\ 0.0000 + j0.0000 & 0.6348 + j0.0000 & -0.0046 - j144.05 & 0.0000 + j0.0000 \\ 0.0000 + j0.0000 & -0.0000 - j0.0035 & 0.7846 + j0.0000 & 0.0000 + j0.0000 \\ 0.0000 + j0.0035 & 0.0000 + j0.0000 & 0.0000 + j0.0000 & 0.7846 + j0.0000 \end{pmatrix}. \tag{21}$$

These obtained values are then used to reconstruct the permittivity profile. In this example, linear, quadratic, cubic, and fourth degree polynomial profiles with unknown coefficients are considered. In each of these case, using (9), (16), and (17), the STM of an inhomogeneous layer is computed and by comparing it with (19), the unknown coefficients are determined. Table II indicates the unknown coefficients of reconstructed profiles. Also, Fig. 4 shows the original and reconstructed profiles using our proposed method. As may be seen from the different curves, there is a much better agreement between the true and reconstructed profiles using a higher degree polynomial as compared to lower ones. Observe that an excellent agreement between the original and reconstructed profile using a fourth degree polynomial has been achieved.

C. Inhomogeneous layer with discontinuous permittivity profile

The last example is devoted to reconstructing a profile composed of two different materials. A profile composed of two different materials with electric permittivity of 3 and 6 and thicknesses of 2.5 cm is considered. Using the forward STM method, reflection and transmission coefficients at a frequency of 1 GHz are simply computed. The reconstructed STM of the slab is

$$\Phi = \begin{pmatrix} -0.8927 + j0.0000 & 0.0000 + j0.0000 & 0.0000 + j0.0000 & 0.0000 + j139.65 \\ 0.0000 + j0.0000 & -0.8927 + j0.0000 & 0.0000 - j139.65 & 0.0000 + j0.0000 \\ 0.0000 + j0.0000 & 0.0000 - j0.0049 & -0.3588 + j0.0000 & 0.0000 + j0.0000 \\ 0.0000 + j0.0049 & 0.0000 + j0.0000 & 0.0000 + j0.0000 & -0.3588 + j0.0000 \end{pmatrix}. \tag{22}$$

In Fig. 5, the true and reconstructed profiles which have been found assuming linear, cubic, and eighth degree polynomial profiles are plotted. Observe that due to the discontinuity of the permittivity profile, the desired agreement between the original and reconstructed profiles using a high degree polynomial profiles is achieved.

D. Discussion

A more complicated problem compared to inverse scattering from homogeneous materials is the characterization of multilayered or inhomogeneous structures, which has been mainly treated by the aid of bio-inspired techniques such as sequential quadratic programming and genetic algorithm.

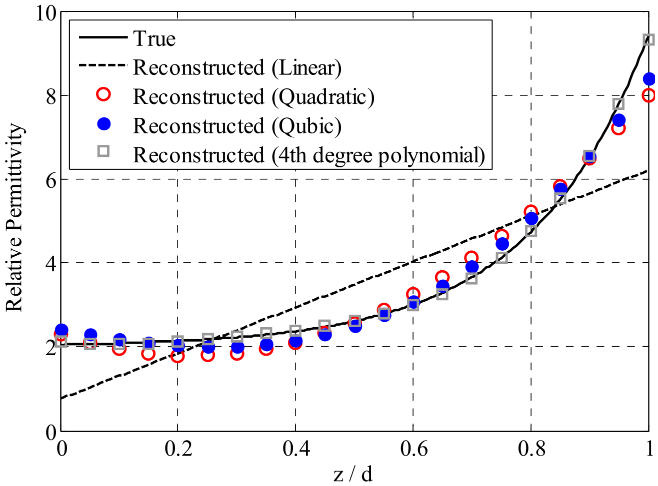


FIG. 4. (Color online) True and reconstructed permittivity profiles of an inhomogeneous slab with exponential permittivity profile.

The conventional methods basically lead to nonlinear differential or integral equations that can only be solved using iterative, numerical, and optimization algorithms wherein obtaining a unique and stable solution is difficult. The idea of the state-transition-matrix method for the electromagnetic characterization is to exploit the state-transition matrix and then go back to the inhomogeneous slab parameters. This allows avoiding the nonlinearity of the problem but requires getting enough equations to fulfill the task. From a scientific point of view, the main difference with respect to other well established retrieval procedures based on the use of the scattering parameters relies on the direct computation of the transfer matrix of the slab as opposed to the conventional calculation of the wave equation in the inhomogeneous medium. This reconstruction method allows one to simply implement it in a programming language supporting matrix and symbolic manipulations, such as MATLAB and MATHEMATICA.

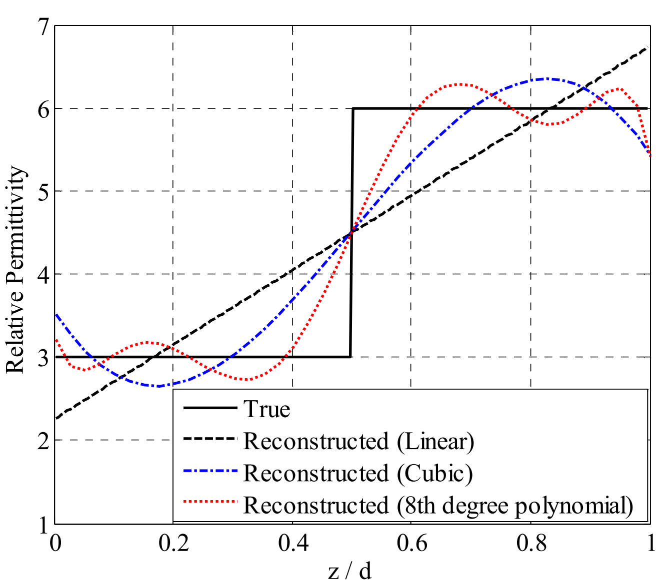


FIG. 5. (Color online) True and reconstructed permittivity profiles of an inhomogeneous slab with discontinuous permittivity profile.

VI. CONCLUSIONS

Some alternative ideas for reconstructing permittivity profiles in planar inhomogeneous structures illuminated by electromagnetic waves are presented in this paper. The proposed method for reconstructing the permittivity profile of one-dimensional inhomogeneous materials is an analytic approach based on the STM method. In fact, it utilizes the measured scattering parameters along properties of the STM. Reconstructions using different materials have been carried out to validate the quality of the proposed technique. Some known permittivity profile has been taken to generate synthetic reflection and transmission coefficient data by the forward STM method. These data have been used in conjunction with the proposed technique to reconstruct the permittivity profile. The results demonstrate excellent agreement between originally assumed and reconstructed dielectric profiles. From the numerical examples, one can conclude that the presented method is able to reconstruct both smooth and step profiles. Generally, the presented method can represent a basis for a wide class of inverse problems.

APPENDIX A

An inhomogeneous slab can be discretized into a number of homogeneous electrically thin layers. The Γ matrix for a homogeneous layer with constitutive parameters ε_n and μ_n is given by

$$\Gamma^{\text{layer}} = \begin{pmatrix} 0 & 0 & 0 & -j\omega\mu_n \\ 0 & 0 & j\omega\mu_n & 0 \\ 0 & j\omega\varepsilon_n & 0 & 0 \\ -j\omega\varepsilon_n & 0 & 0 & 0 \end{pmatrix}, \tag{A1}$$

and its STM is $\Phi^{\text{layer}} = e^{-\Gamma^{\text{layer}} d}$. For the computation of the exponential function of a square matrix, we can use the Cayley-Hamilton theorem [29,30]. Assuming $\mathrm{W} = -\Gamma^{\text{layer}} d$, $\Gamma_{23}^{\text{layer}} = j\omega\mu_n d$, and $\Gamma_{32}^{\text{layer}} = j\omega\varepsilon_n d$, the eigenvalues of the W matrix are given by

$$w_{\substack{1\\3}} = w_{\substack{2\\4}} = \pm\sqrt{\Gamma_{23}^{\text{layer}}\Gamma_{32}^{\text{layer}}}. \tag{A2}$$

Using the Cayley-Hamilton theorem, the exponential of W can be written as

$$e^{\mathrm{W}} = x_0\mathrm{I} + x_1\mathrm{W} + x_2\mathrm{W}^2 + x_3\mathrm{W}^3, \tag{A3}$$

where x_0, x_1, x_2, and x_3 are unknown coefficients which are determined by solving the following set of equations:

$$e^{w_1} = x_0 + x_1 w_1 + x_2 w_1^2 + x_3 w_1^3, \quad w_1 e^{w_1} = x_1 + 2x_2 w_1 + 3x_3 w_1^2,$$
$$e^{w_3} = x_0 + x_1 w_3 + x_2 w_3^2 + x_3 w_3^3, \quad w_3 e^{w_3} = x_1 + 2x_2 w_3 + 3x_3 w_3^2. \tag{A4}$$

By solving these equations, one can write

$$x_0 = \left(1 - \frac{w_1^2}{2}\right)\cosh(w_1), \quad x_1 = -\frac{w_1^2 - 3}{2w_1}\sinh(w_1), \quad x_2 = \frac{1}{2}\cosh(w_1), \quad x_3 = \frac{w_1^2 - 1}{2w_1^3}\sinh(w_1). \tag{A5}$$

Finally, by substituting these coefficients in (A3) and after some simple matrix manipulations, the STM of the homogeneous layer can be written as

$$\Phi^{\text{layer}} = \begin{bmatrix} \cosh\sqrt{\Gamma_{23}^{\text{layer}}\Gamma_{32}^{\text{layer}}} & 0 & 0 & \sqrt{\frac{\Gamma_{23}^{\text{layer}}}{\Gamma_{32}^{\text{layer}}}}\sinh\sqrt{\Gamma_{23}^{\text{layer}}\Gamma_{32}^{\text{layer}}} \\ 0 & \cosh\sqrt{\Gamma_{23}^{\text{layer}}\Gamma_{32}^{\text{layer}}} & -\sqrt{\frac{\Gamma_{23}^{\text{layer}}}{\Gamma_{32}^{\text{layer}}}}\sinh\sqrt{\Gamma_{23}^{\text{layer}}\Gamma_{32}^{\text{layer}}} & 0 \\ 0 & -\sqrt{\frac{\Gamma_{32}^{\text{layer}}}{\Gamma_{23}^{\text{layer}}}}\sinh\sqrt{\Gamma_{23}^{\text{layer}}\Gamma_{32}^{\text{layer}}} & \cosh\sqrt{\Gamma_{23}^{\text{layer}}\Gamma_{32}^{\text{layer}}} & 0 \\ \sqrt{\frac{\Gamma_{32}^{\text{layer}}}{\Gamma_{23}^{\text{layer}}}}\sinh\sqrt{\Gamma_{23}^{\text{layer}}\Gamma_{32}^{\text{layer}}} & 0 & 0 & \cosh\sqrt{\Gamma_{23}^{\text{layer}}\Gamma_{32}^{\text{layer}}} \end{bmatrix}. \tag{A6}$$

APPENDIX B

In [27], it has been shown that the determinant of the STM of a homogeneous layer with constitutive parameters ε_n and μ_n is

$$\det(\Phi_{\text{layer}}) = \exp\left[\sum_{\alpha=1}^{4}\Gamma(\alpha,\alpha)\right]. \tag{B1}$$

According to (A1), the sum of the diagonal elements of the Γ matrix of a homogeneous slab is zero, and so the determinant of its STM is equal to unity. By subdividing an inhomogeneous slab into N homogeneous electrically thin layers, its STM is given by $\Phi = \Phi^{\text{layer}(1)}\Phi^{\text{layer}(2)}\Phi^{\text{layer}(3)}\ldots\Phi^{\text{layer}(N)}$. Since the determinant of a product of square matrices is the product of their determinants, the determinant of the STM of an inhomogeneous slab (Φ) is also equal to unity.

APPENDIX C

For the computation of the inverse matrix $(\Phi^{\text{layer}})^{-1}$, $e^{+\Gamma d}$ can be computed. Using a similar procedure based on the Cayley-Hamilton theorem discussed in Appendix B, one can easily see that the inverse matrix $(\Phi^{\text{layer}})^{-1}$ is given by

$$(\Phi^{\text{layer}})^{-1} = \begin{pmatrix} \Phi_{11}^{\text{layer}} & 0 & 0 & -\Phi_{14}^{\text{layer}} \\ 0 & \Phi_{11}^{\text{layer}} & \Phi_{14}^{\text{layer}} & 0 \\ 0 & \Phi_{41}^{\text{layer}} & \Phi_{11}^{\text{layer}} & 0 \\ -\Phi_{41}^{\text{layer}} & 0 & 0 & \Phi_{11}^{\text{layer}} \end{pmatrix}. \tag{C1}$$

The inverse of the STM of an inhomogeneous slab is given by $\Phi^{-1} = (\Phi^{\text{layer}(N)})^{-1}(\Phi^{\text{layer}(N-1)})^{-1}(\Phi^{\text{layer}(N-2)})^{-1}\ldots(\Phi^{\text{layer}(1)})^{-1}$. Using (C1) it can be seen that the inverse of the STM of an inhomogeneous slab has the form presented in (10).

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