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State transition matrix of inhomogeneous planar layers

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Abstract: In this study, the computation and properties of the state transition matrix of inhomogeneous planar layered media are investigated. Furthermore, non-reciprocity of inhomogeneous planar layered media is shown using the transition matrix method. This theorem says that the plane wave transmission coefficients for a slab with inhomogeneity along the direction perpendicular to the interfaces are the same when the wave incidences on the slab from left or from right, but that the reflection coefficients usually differ. The validation of the results is studied finally through some typical examples.

1 Introduction

Features of electromagnetic wave propagation in inhomogeneous media have been intensively studied during the last decades. Inhomogeneous media are widely used in microwave and antenna engineering as shields, filters, absorbers and radomes [1–7]. In addition, inhomogeneous media provide less scattering, larger bandwidth and better coupling effects than homogeneous media.

Although analysis of scattering from stratified inhomogeneous media is more complicated than that from homogeneous media, several approaches have been presented for the analysis of the scattering from inhomogeneous media such as Richmond method [8], Riccati equation [9], Taylor's series expansion [10] and Fourier series expansion [11]. The transition matrix method is a commonly used to deal with the problems of plane wave scattering from planar layered inhomogeneous media [12–15]. The most important feature of this method is that no matter how complex the medium is under study, the transverse components of electric and magnetic fields with some algebraic manipulating become four coupled first-order differential equations.

This paper deals with the computation of state transition matrix of inhomogeneous planar layered media using an analytic method based on the Peano–Baker series. In addition, we show non-reciprocity of one-dimensional (1D) inhomogeneous planar layered media using the transition matrix method. In the other word, it is shown that the plane wave transmission coefficients for a slab with inhomogeneity along the direction perpendicular to the interfaces are the same when the wave incidences on the slab from left or from right, but that the reflection coefficients usually differ. Notice that, here, the medium is reciprocal and non-reciprocity is used for the interaction of electromagnetic waves with the inhomogeneous slab from left and right.

2 Review on the transition matrix method

Consider an inhomogeneous slab characterised by a set of constitutive relations

$$\bar{D} = \varepsilon_0\, \varepsilon(z)\, \bar{E} \tag{1a}$$

$$\bar{B} = \mu_0\, \mu(z)\, \bar{H} \tag{1b}$$

where $\varepsilon(z)$ and $\mu(z)$ are relative permittivity and permeability, respectively. As shown in Fig. 1, the planar structure is of infinite extent along the y-direction so, derivatives of the fields vanish with respect to y and z variables, that is, $\partial/\partial y = 0$ and $\partial/\partial x = -jk_0 \sin\theta_0$. By eliminating z-components of electric and magnetic fields from curl Maxwell's equations one can write

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix} \bar{E}_\mathrm{T} \\ \bar{H}_\mathrm{T} \end{bmatrix} = \boldsymbol{\Gamma}\begin{bmatrix} \bar{E}_\mathrm{T} \\ \bar{H}_\mathrm{T} \end{bmatrix} \tag{2}$$

where $\bar{E}_\mathrm{T} = (E_x, E_y)$ and $\bar{H}_\mathrm{T} = (H_x, H_y)$ and are transverse components of electric and magnetic fields, respectively, and $\boldsymbol{\Gamma}$-matrix is given by (see equation (3) on the bottom of the next page)

where ω, c and $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ are the angular frequency, speed of light and the intrinsic impedance of free space, respectively.

Defining a 4×4 state transition matrix $\boldsymbol{\Phi}$ with 2×2 sub-matrices $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$, $\boldsymbol{\Phi}_3$ and $\boldsymbol{\Phi}_4$ that relates the transverse components of electric and magnetic fields at two

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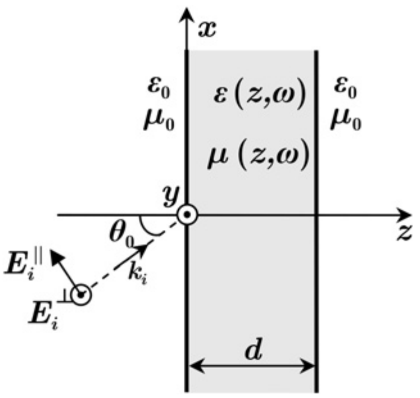

Fig. 1 *Inhomogeneous slab exposed to a linearly polarised plane wave*

boundaries of the inhomogeneous slab, we can write

$$\begin{bmatrix} \bar{E}_{\mathrm{T}}(0) \\ \bar{H}_{\mathrm{T}}(0) \end{bmatrix} = \boldsymbol{\Phi} \begin{bmatrix} \bar{E}_{\mathrm{T}}(d) \\ \bar{H}_{\mathrm{T}}(d) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 \\ \boldsymbol{\Phi}_3 & \boldsymbol{\Phi}_4 \end{bmatrix} \begin{bmatrix} \bar{E}_{\mathrm{T}}(d) \\ \bar{H}_{\mathrm{T}}(d) \end{bmatrix} \tag{4}$$

By introducing the reflection and transmission matrices of [T] and [R] as

$$\bar{E}^r_{\mathrm{T}}(0) = \boldsymbol{R}\bar{E}^i_{\mathrm{T}}(0) = \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \bar{E}^i_{\mathrm{T}}(0) \tag{5}$$

$$\bar{E}^t_{\mathrm{T}}(d) = \boldsymbol{T}\bar{E}^i_{\mathrm{T}}(0) = \begin{bmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{bmatrix} \bar{E}^i_{\mathrm{T}}(0) \tag{6}$$

where superscripts i, r and t denote incident, reflected and transmitted field, respectively, it can be shown that [12]

$$\begin{aligned} \boldsymbol{R} = {} & [\boldsymbol{\Phi}_1\boldsymbol{Z}_0 + \boldsymbol{\Phi}_2 - \boldsymbol{Z}_0(\boldsymbol{\Phi}_3\boldsymbol{Z}_0 + \boldsymbol{\Phi}_4)] \\ & \times [\boldsymbol{\Phi}_1\boldsymbol{Z}_0 + \boldsymbol{\Phi}_2 + \boldsymbol{Z}_0(\boldsymbol{\Phi}_3\boldsymbol{Z}_0 + \boldsymbol{\Phi}_4)]^{-1} \end{aligned} \tag{7}$$

$$\boldsymbol{T} = 2\boldsymbol{Z}_0[\boldsymbol{\Phi}_1\boldsymbol{Z}_0 + \boldsymbol{\Phi}_2 + \boldsymbol{Z}_0(\boldsymbol{\Phi}_3\boldsymbol{Z}_0 + \boldsymbol{\Phi}_4)]^{-1} \tag{8}$$

where in

$$\boldsymbol{Z}_0 = \begin{bmatrix} 0 & \eta_0 \cos\theta_0 \\ -\eta_0/\cos\theta_0 & 0 \end{bmatrix} \tag{9}$$

is the wave impedance matrix.

3 Computation of the state transition matrix of inhomogeneous planar layers

The computation of the state transition matrix of inhomogeneous layers is more complicated than that of homogeneous layers. Notice that if the slab is homogeneous, similar to state-space analysis in linear time-invarient systems, the state transition matrix $\boldsymbol{\Phi}$ is exp $(-\boldsymbol{\Gamma}d)$. For the computation of such a matrix, many methods have been proposed such as expansion of $\boldsymbol{\Phi}$ in a power series, Cayley–Hamilton theorem and Leverrier's algorithm [16]. For inhomogeneous layers, $\boldsymbol{\Gamma}$-matrix is dependent to z and so the state transition matrix cannot be computed using $\exp(-\boldsymbol{\Gamma}d)$. In this section, two numerical and analytic methods for the computation of the state transition matrix of inhomogeneous layers based on the cascading thin homogeneous layers and Peano-Baker series are proposed.

3.1 Cascading thin homogeneous layers

As shown in Fig. 2, an inhomogeneous slab can be considered as laterally N homogenous stratified medium. The state transition matrix of a homogeneous layer with constitutive parameters $\varepsilon_0\varepsilon_n$ and $\mu_0\mu_n$ is computed using the Cayley–Hamilton theorem [16]. As proved in the Appendix, the state transition matrix of a homogeneous isotropic layer is given by

$$\boldsymbol{\Phi}^{(\text{layer}n)} = \begin{bmatrix} \Phi^{(n)}_{11} & 0 & 0 & \Phi^{(n)}_{14} \\ 0 & \Phi^{(n)}_{22} & \Phi^{(n)}_{23} & 0 \\ 0 & \Phi^{(n)}_{32} & \Phi^{(n)}_{22} & 0 \\ \Phi^{(n)}_{41} & 0 & 0 & \Phi^{(n)}_{11} \end{bmatrix} \tag{10}$$

has only six distinct and non-zero elements ($\Phi^{(n)}_{11}$, $\Phi^{(n)}_{14}$, $\Phi^{(n)}_{22}$, $\Phi^{(n)}_{23}$, $\Phi^{(n)}_{32}$ and $\Phi^{(n)}_{41}$) that are

$$\begin{cases} \Phi^{(n)}_{11} = \cosh\sqrt{g_1 g_4} \\ \Phi^{(n)}_{14} = \sqrt{g_1/g_4}\sinh\sqrt{g_1 g_4} \\ \Phi^{(n)}_{22} = \cosh\sqrt{g_2 g_3} \\ \Phi^{(n)}_{23} = -\sqrt{g_2/g_3}\sinh\sqrt{g_2 g_3} \\ \Phi^{(n)}_{32} = -\sqrt{g_3/g_2}\sinh\sqrt{g_2 g_3} \\ \Phi^{(n)}_{41} = \sqrt{g_4/g_1}\sinh\sqrt{g_1 g_4} \end{cases} \tag{11}$$

where $g_1 = j\omega\mu_0 d(-\mu_n + (1/\varepsilon_n)\sin^2\theta_0)$, $g_2 = j\omega\mu_0\mu_n d$, $g_3 = j\omega\varepsilon_0 d(\varepsilon_n - (1/\mu_n)\sin^2\theta_0)$ and $g_4 = j\omega\varepsilon_0\varepsilon_n d$. Clearly, according to (4), the state transition matrix of inhomogeneous slab can be written as

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}^{(\text{layer}1)}\boldsymbol{\Phi}^{(\text{layer}2)}, \ldots, \boldsymbol{\Phi}^{(\text{layer}N)} \tag{12}$$

Considering (10) and (12), it can be seen that the state transition matrix of inhomogeneous slab has the following form

$$\boldsymbol{\Phi} = \begin{bmatrix} \Phi_{11} & 0 & 0 & \Phi_{14} \\ 0 & \Phi_{22} & \Phi_{23} & 0 \\ 0 & \Phi_{32} & \Phi_{33} & 0 \\ \Phi_{41} & 0 & 0 & \Phi_{44} \end{bmatrix} \tag{13}$$

$$\boldsymbol{\Gamma} = \frac{j\omega}{c} \begin{bmatrix} 0 & 0 & 0 & \eta_0\left(-\mu(z) + \dfrac{1}{\varepsilon(z)}\sin^2\theta_0\right) \\ 0 & 0 & \eta_0\mu(z) & 0 \\ 0 & \eta_0^{-1}\left(\varepsilon(z) - \dfrac{1}{\varepsilon(z)}\sin^2\theta_0\right) & 0 & 0 \\ -\eta_0^{-1}\varepsilon(z) & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

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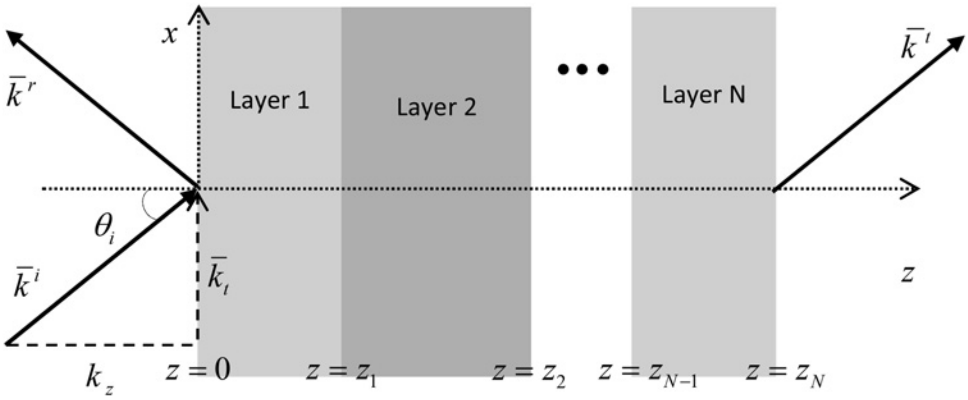

Fig. 2 *Plane wave incidence upon a stratified structure of N layers*

Notice that unlike the transition matrix of a homogeneous layer, the diagonal elements of this matrix are generally different and there are four distinct and non-zero elements.

3.2 Peano-Bakers series

It is convenient for some purposes to compute analytically the state transition matrix in the transition matrix method. In general there is no closed form solution for $\mathbf{\Phi}$-matrix. Assuming $\mathbf{\Psi}$-matrix as $\mathbf{\Psi} = \exp\left(\int_0^z \mathbf{\Gamma}(z_0)\mathrm{d}z_0\right)$, using the Peano-Baker series we can write [17] (see (14))

where $\boldsymbol{I}$ is the 4×4 identity matrix. The $\mathbf{\Psi}$-matrix is given by the Peano-Baker series is unique and converges absolutely and uniformly. For the simple computation of (14) using programming language supporting matrix and symbolic manipulations such as MATLAB, we can use a successive procedure as the follows

$$
\begin{aligned}
\mathbf{\Psi}_{(0)} &= \int_0^z \mathbf{\Gamma}(z)\mathrm{d}z \\
\mathbf{\Psi}_{(1)} &= \mathbf{\Psi}_{(0)} + \int_0^z \mathbf{\Psi}_{(0)}\mathbf{\Gamma}(z)\mathrm{d}z \\
\mathbf{\Psi}_{(2)} &= \mathbf{\Psi}_{(1)} + \int_0^z \left(\mathbf{\Psi}_{(1)} - \mathbf{\Psi}_{(0)}\right)\mathbf{\Gamma}(z)\mathrm{d}z \\
&\vdots \\
\mathbf{\Psi}_{(M)} &= \mathbf{\Psi}_{(M-1)} + \int_0^z \left(\mathbf{\Psi}_{(M-1)} - \mathbf{\Psi}_{(M-2)}\right)\mathbf{\Gamma}(z)\mathrm{d}z \quad \text{for } M \geq 2
\end{aligned} \tag{15}
$$

where $\mathbf{\Psi} = \boldsymbol{I} + \mathbf{\Psi}_{(M)}\big|_{z=d}$ and the state transition matrix given by as $\mathbf{\Phi} = \mathbf{\Psi}^{-1}$. Notice that truncating the above series after a finite number of terms (M) gives an approximation for the state transition matrix. Clearly, for homogeneous layers, that is, when $\mathbf{\Gamma}$ is a 'constant' matrix, the state transition matrix is given as $\mathbf{\Phi} = \exp(-\mathbf{\Gamma}d)$.

In the end of this section, in order to compare the results of these two methods, two inhomogeneous layers with linear and exponential inhomogeneity are considered. Notice that using the Peano-Baker series, with fewer terms more accurate answer can be found. Also, the studies show that as the thickness of the inhomogeneous layer with respect to the wavelength increases, the necessary number of homogeneous sub-layers (N) and number of terms (M) increases (see Tables 1 and 2).

Table 1 State transition matrix of an inhomogeneous layer with linear inhomogeneity

$\varepsilon(z) = 4 + 5(z/d)$, $d = 2$ cm, $f = 1$ GHz, $\theta_0 = 0°$			
Cascading thin homogeneous layers		Peano-Bakers Series	
$N = 10$	$N > 80$	$M = 5$	$M > 10$
$\Phi_{11} = 0.4169$	$\Phi_{11} = 0.4163$	$\Phi_{11} = 0.4186$	$\Phi_{11} = 0.4163$
$\Phi_{14} = j129.55$	$\Phi_{14} = j129.55$	$\Phi_{14} = j129.60$	$\Phi_{14} = j129.55$
$\Phi_{22} = 0.4169$	$\Phi_{22} = 0.4163$	$\Phi_{22} = 0.4186$	$\Phi_{22} = 0.4163$
$\Phi_{23} = -j129.55$	$\Phi_{23} = -j129.55$	$\Phi_{23} = -j129.60$	$\Phi_{23} = -j129.55$
$\Phi_{32} = -j0.0006$	$\Phi_{32} = -j0.0060$	$\Phi_{32} = -j0.0006$	$\Phi_{32} = -j0.0060$
$\Phi_{33} = 0.5459$	$\Phi_{33} = 0.5465$	$\Phi_{33} = 0.5482$	$\Phi_{33} = 0.5465$
$\Phi_{41} = j0.0060$	$\Phi_{41} = j0.0060$	$\Phi_{41} = j0.0060$	$\Phi_{41} = j0.0060$
$\Phi_{44} = 0.5459$	$\Phi_{44} = 0.5465$	$\Phi_{41} = j0.0060$	$\Phi_{44} = 0.5465$

Table 2 State transition matrix of an inhomogeneous layer with exponential inhomogeneity

$\varepsilon(z) = 2\exp(2z/d)$, $d = 10$ cm, $f = 1$ GHz, $\theta_0 = 45°$			
Cascading thin homogeneous layers		Peano-Bakers series	
$N = 10$	$N > 150$	$M = 15$	$M > 20$
$\Phi_{11} = 0.3136$	$\Phi_{11} = 0.3265$	$\Phi_{11} = 0.3199$	$\Phi_{11} = 0.3265$
$\Phi_{14} = -j146.93$	$\Phi_{14} = -j147.14$	$\Phi_{14} = -j147.11$	$\Phi_{14} = -j147.14$
$\Phi_{22} = 0.4038$	$\Phi_{22} = 0.4201$	$\Phi_{22} = 0.4125$	$\Phi_{22} = 0.4201$
$\Phi_{23} = j168.56$	$\Phi_{23} = j168.84$	$\Phi_{23} = j168.79$	$\Phi_{23} = j168.84$
$\Phi_{32} = j0.0060$	$\Phi_{32} = j0.0060$	$\Phi_{32} = j0.0060$	$\Phi_{32} = j0.0060$
$\Phi_{33} = -0.0177$	$\Phi_{33} = -0.0238$	$\Phi_{33} = -0.0263$	$\Phi_{33} = -0.0238$
$\Phi_{41} = -j0.0068$	$\Phi_{41} = -j0.0068$	$\Phi_{41} = -j0.0068$	$\Phi_{41} = -j0.0068$
$\Phi_{44} = 0.0163$	$\Phi_{44} = 0.0112$	$\Phi_{44} = 0.0084$	$\Phi_{44} = 0.0112$

$$
\mathbf{\Psi} = \boldsymbol{I} + \int_0^z \mathbf{\Gamma}(z_0)\mathrm{d}z_0 + \int_0^z \mathbf{\Gamma}(z_0)\int_0^{z_0} \mathbf{\Gamma}(z_1)\mathrm{d}z_1\mathrm{d}z_0 + \int_0^z \mathbf{\Gamma}(z)\int_0^{z_0} \mathbf{\Gamma}(z_1)\int_0^{z_1} \mathbf{\Gamma}(z_2)\mathrm{d}z_2\mathrm{d}z_1\mathrm{d}z_0 + \cdots \tag{14}
$$

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4 Non-reciprocity of inhomogeneous planar layers

The geometry of the problem is shown in Fig. 3. As shown in this figure, both sides of the inhomogeneous slab having thickness d are free space and a plane wave impinges normally from free space onto it. As the first case (case I), consider Fig. 3*a* wherein the slab is illuminated at $z=0$. In this case, the state transition matrix is given by

$$\boldsymbol{\Phi}^{(\mathrm{I})}=\boldsymbol{\Phi}^{(\mathrm{layer1})}\boldsymbol{\Phi}^{(\mathrm{layer2})}\cdots\boldsymbol{\Phi}^{(\mathrm{layer}N)}=\begin{bmatrix}\boldsymbol{\Phi}_1^{(\mathrm{I})} & \boldsymbol{\Phi}_2^{(\mathrm{I})}\\ \boldsymbol{\Phi}_3^{(\mathrm{I})} & \boldsymbol{\Phi}_4^{(\mathrm{I})}\end{bmatrix} = \begin{bmatrix}\Phi_{11} & 0 & 0 & \Phi_{14}\\ 0 & \Phi_{22} & \Phi_{23} & 0\\ 0 & \Phi_{32} & \Phi_{33} & 0\\ \Phi_{41} & 0 & 0 & \Phi_{44}\end{bmatrix} \tag{16}$$

Therefore using (8) the transmission matrix is given by

$$\boldsymbol{T}^{(\mathrm{I})}=2\boldsymbol{Z}_0\left[\boldsymbol{\Phi}_1^{(\mathrm{I})}\boldsymbol{Z}_0+\boldsymbol{\Phi}_2^{(\mathrm{I})}+\boldsymbol{Z}_0\left(\boldsymbol{\Phi}_3^{(\mathrm{I})}\boldsymbol{Z}_0+\boldsymbol{\Phi}_4^{(\mathrm{I})}\right)\right]^{-1} \tag{17}$$

In the second case (case II), wherein the slab is illuminated at $z=d$, the state transition matrix is given by

$$\boldsymbol{\Phi}^{(\mathrm{II})}=\boldsymbol{\Phi}^{(\mathrm{layer}N)}\boldsymbol{\Phi}^{(\mathrm{layer}N-1)}\cdots\boldsymbol{\Phi}^{(\mathrm{layer1})}=\begin{bmatrix}\boldsymbol{\Phi}_1^{(\mathrm{II})} & \boldsymbol{\Phi}_2^{(\mathrm{II})}\\ \boldsymbol{\Phi}_3^{(\mathrm{II})} & \boldsymbol{\Phi}_4^{(\mathrm{II})}\end{bmatrix} = \begin{bmatrix}\Phi_{44} & 0 & 0 & \Phi_{14}\\ 0 & \Phi_{33} & \Phi_{23} & 0\\ 0 & \Phi_{32} & \Phi_{22} & 0\\ \Phi_{41} & 0 & 0 & \Phi_{11}\end{bmatrix} \tag{18}$$

and the transmission matrix is

$$\boldsymbol{T}^{(\mathrm{II})}=2\boldsymbol{Z}_0\left[\boldsymbol{\Phi}_1^{(\mathrm{II})}\boldsymbol{Z}_0+\boldsymbol{\Phi}_2^{(\mathrm{II})}+\boldsymbol{Z}_0\left(\boldsymbol{\Phi}_3^{(\mathrm{II})}\boldsymbol{Z}_0+\boldsymbol{\Phi}_4^{(\mathrm{II})}\right)\right]^{-1} \tag{19}$$

By considering (16) and (18), we can write

$$\begin{cases}\boldsymbol{\Phi}_1^{(\mathrm{II})}=\boldsymbol{\Phi}_4^{(\mathrm{I})}\\ \boldsymbol{\Phi}_2^{(\mathrm{II})}=\boldsymbol{\Phi}_2^{(\mathrm{I})}\\ \boldsymbol{\Phi}_3^{(\mathrm{II})}=\boldsymbol{\Phi}_3^{(\mathrm{I})}\\ \boldsymbol{\Phi}_4^{(\mathrm{II})}=\boldsymbol{\Phi}_1^{(\mathrm{I})}\end{cases} \tag{20}$$

Considering (20) one can rewrite (16) as

$$\boldsymbol{T}^{(\mathrm{II})}=2\boldsymbol{Z}_0\left[\boldsymbol{\Phi}_4^{(\mathrm{I})}\boldsymbol{Z}_0+\boldsymbol{\Phi}_2^{(\mathrm{I})}+\boldsymbol{Z}_0\left(\boldsymbol{\Phi}_3^{(\mathrm{I})}\boldsymbol{Z}_0+\boldsymbol{\Phi}_1^{(\mathrm{I})}\right)\right]^{-1} \tag{21}$$

Now notice that because of $\boldsymbol{\Phi}_4^{(\mathrm{I})}\boldsymbol{Z}_0=\boldsymbol{Z}_0\boldsymbol{\Phi}_4^{(\mathrm{I})}$ and $\boldsymbol{Z}_0\boldsymbol{\Phi}_1^{(\mathrm{I})}=\boldsymbol{\Phi}_1^{(\mathrm{I})}\boldsymbol{Z}_0$ we conclude that $\boldsymbol{T}^{(\mathrm{II})}=\boldsymbol{T}^{(\mathrm{I})}$. Thus, the transmission of electromagnetic waves through an inhomogeneous layer is independent of the direction of transmission.

Similarly, for the reflection matrices, we can write

$$\boldsymbol{R}^{(\mathrm{I})}=\left[\boldsymbol{\Phi}_1^{(\mathrm{I})}\boldsymbol{Z}_0+\boldsymbol{\Phi}_2^{(\mathrm{I})}-\boldsymbol{Z}_0\left(\boldsymbol{\Phi}_3^{(\mathrm{I})}\boldsymbol{Z}_0+\boldsymbol{\Phi}_4^{(\mathrm{I})}\right)\right]\left[\boldsymbol{\Phi}_1^{(\mathrm{I})}\boldsymbol{Z}_0+\boldsymbol{\Phi}_2^{(\mathrm{I})}+\boldsymbol{Z}_0\left(\boldsymbol{\Phi}_3^{(\mathrm{I})}\boldsymbol{Z}_0+\boldsymbol{\Phi}_4^{(\mathrm{I})}\right)\right]^{-1} \tag{22}$$

$$\boldsymbol{R}^{(\mathrm{II})}=\left[\boldsymbol{\Phi}_4^{(\mathrm{I})}\boldsymbol{Z}_0+\boldsymbol{\Phi}_2^{(\mathrm{I})}-\boldsymbol{Z}_0\left(\boldsymbol{\Phi}_3^{(\mathrm{I})}\boldsymbol{Z}_0+\boldsymbol{\Phi}_1^{(\mathrm{I})}\right)\right]\left[\boldsymbol{\Phi}_4^{(\mathrm{I})}\boldsymbol{Z}_0+\boldsymbol{\Phi}_2^{(\mathrm{I})}+\boldsymbol{Z}_0\left(\boldsymbol{\Phi}_3^{(\mathrm{I})}\boldsymbol{Z}_0+\boldsymbol{\Phi}_1^{(\mathrm{I})}\right)\right]^{-1} \tag{23}$$

Notice that because of $\boldsymbol{\Phi}_4^{(\mathrm{I})}\boldsymbol{Z}_0=\boldsymbol{Z}_0\boldsymbol{\Phi}_4^{(\mathrm{I})}$ and $\boldsymbol{Z}_0\boldsymbol{\Phi}_1^{(\mathrm{I})}=\boldsymbol{\Phi}_1^{(\mathrm{I})}\boldsymbol{Z}_0$, terms inside the second brackets in (22) and (23) are the same; but equality of $\boldsymbol{\Phi}_1^{(\mathrm{I})}$ and $\boldsymbol{\Phi}_4^{(\mathrm{I})}$ is necessary to establish the equality of terms inside the first brackets in these equations. Owing to the discussed properties of the state transition matrix of homogeneous and inhomogeneous layers, equality of $\boldsymbol{\Phi}_1^{(\mathrm{I})}$ and $\boldsymbol{\Phi}_4^{(\mathrm{I})}$ is possible in two special cases, so that the layer is homogeneous, or inhomogeneous with symmetric inhomogeneity around the centre of the layer.

To sum up, the transmission of electromagnetic waves through an inhomogeneous layer is independent of the direction of transmission and transmission coefficients for such a slab are the same when the wave impinges on the

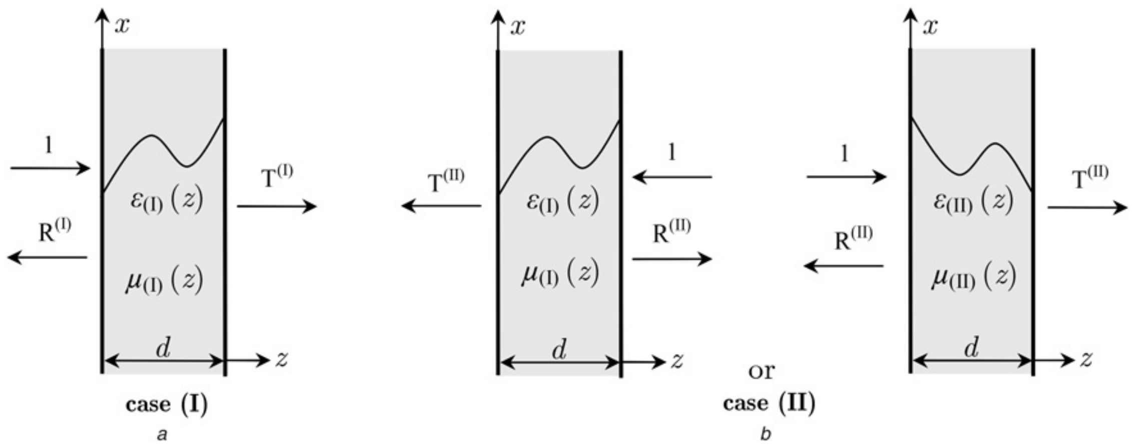

Fig. 3 *An inhomogeneous slab exposed to a linearly polarised plane wave*

a Wave incidences on the slab from left
b Wave incidences on the slab from right
Relation between $\varepsilon_{(\mathrm{I})}(z)$ and $\varepsilon_{(\mathrm{II})}(z)$ is shown in this figure

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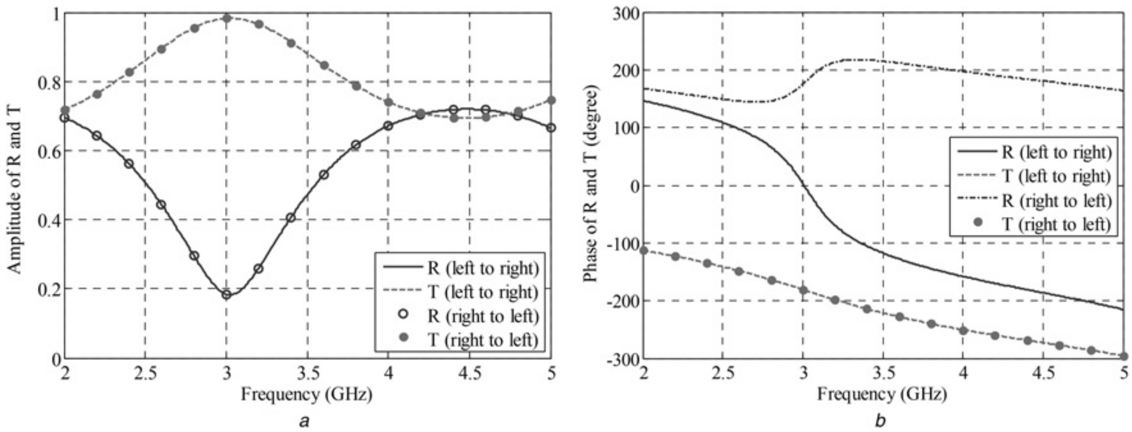

Fig. 4 *Amplitudes and phases of reflection and transmission coefficients of inhomogeneous slab when the wave impinges normally on the slab*
a From left
b From right

slab at a given angle from left or from right, but that the reflection coefficients usually differ under the same circumstances. Although notice that if the slab is lossless, because of the energy conservation principle, the amplitudes of reflection coefficients are the same and have different phases. Here, non-reciprocity is used for the interaction of electromagnetic waves with the inhomogeneous slab while the medium is reciprocal.

In the end of this section, an inhomogeneous layer is considered to verify the non-reciprocity of one-dimensional inhomogeneous layers. Consider a non-magnetic inhomogeneous slab with a thickness of $d = 0.2$ m and relative permittivity $\varepsilon(z) = 4 + 5(z/d)$ that varies linearly from 4 to 9. The reflection and transmission coefficients of inhomogeneous slab when the wave impinges normally on the slab from left or from right are shown in Fig. 4. Observe that as we expected the transmission coefficients are identical, while the amplitudes of the reflection coefficients are the same and have different phases. Owing to the lossless dielectric layer, this is justified by the energy conservation principle.

5 Conclusions

This paper deals with the computation and properties of the state transition matrix of inhomogeneous planar layered media. First, application of the transition matrix method to analysis of the problems of plane wave scattering from planar layered inhomogeneous media is reviewed. Then, the computation of the state transition matrix using two different methods as cascading thin sub-layers and Peano-Baker series are investigated. Finally, non-reciprocity of inhomogeneous planar layered media is shown using the transition matrix method. It is shown that the plane wave transmission coefficients for an inhomogeneous layer are the same when the wave incidences on the slab from left or from right, but that the reflection coefficients usually differ. The validation of the introduced methods and properties of the state transition matrix of inhomogeneous layers was studied using some typical examples. In future, the results of this paper are expected to be used for the implementation of a retrieval method for the electromagnetic characterisation of inhomogeneous materials.

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7 Appendix

Assuming $\boldsymbol{X} = -\boldsymbol{\Gamma} d$, $g_1 = j\omega\mu_0 d\big(-\mu_n + (1/\varepsilon_n)\sin^2\theta_0\big)$, $g_2 = j\omega\mu_0\mu_n d$, $g_3 = j\omega\varepsilon_0 d\big(\varepsilon_n - (1/\mu_n)\sin^2\theta_0\big)$ and $g_4 = j\omega\varepsilon_0\varepsilon_n d$.

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The eigenvalues of $\boldsymbol{X}$-matrix are given by

$$\lambda_1 = \lambda_2 = \sqrt{g_1 g_4}, \quad \lambda_3 = \lambda_4 = -\sqrt{g_2 g_3} \tag{24}$$

Using the Cayley-Hamilton theorem [16], the exponential of $\boldsymbol{X}$ can be written as

$$e^{\boldsymbol{X}} = a_0 \boldsymbol{I} + a_1 \boldsymbol{X} + a_2 \boldsymbol{X}^2 + a_3 \boldsymbol{X}^3 \tag{25}$$

where a_0, a_1, a_2 and a_3 are unknown coefficients that are determined by solving the following set of equations

$$\begin{cases} e^{\lambda_1} = a_0 + a_1\lambda_1 + a_2\lambda_1^2 + a_3\lambda_1^3 \\ \lambda_1 e^{\lambda_1} = a_1 + 2a_2\lambda_1 + 3a_3\lambda_1^2 \\ e^{\lambda_3} = a_0 + a_1\lambda_3 + a_2\lambda_3^2 + a_3\lambda_3^3 \\ \lambda_3 e^{\lambda_3} = a_1 + 2a_2\lambda_3 + 3a_3\lambda_3^2 \end{cases} \tag{26}$$

By solving these set of equations, we can write

$$\begin{cases} a_0 = \left(1 - \dfrac{\lambda_1^2}{2}\right)\cosh(\lambda_1) \\ a_1 = -\dfrac{\lambda_1^2 - 3}{2\lambda_1}\sinh(\lambda_1) \\ a_2 = \dfrac{1}{2}\cosh(\lambda_1) \\ a_3 = \dfrac{\lambda_1^2 - 1}{2\lambda_1^3}\sinh(\lambda_1) \end{cases} \tag{27}$$

Finally, by substituting these coefficients in (25) and after some simple matrix manipulations, the state transition matrix of layer can be written as (see (28))

$$\boldsymbol{\Phi}^{(layern)} = \begin{bmatrix} \cosh\sqrt{g_1 g_4} & 0 & 0 & \sqrt{\dfrac{g_1}{g_2}}\sinh\sqrt{g_1 g_4} \\ 0 & \cosh\sqrt{g_2 g_3} & -\sqrt{\dfrac{g_2}{g_3}}\sinh\sqrt{g_2 g_3} & 0 \\ 0 & -\sqrt{\dfrac{g_3}{g_2}}\sinh\sqrt{g_2 g_3} & \cosh\sqrt{g_2 g_3} & 0 \\ \sqrt{\dfrac{g_4}{g_1}}\sinh\sqrt{g_1 g_4} & 0 & 0 & \cosh\sqrt{g_1 g_4} \end{bmatrix} \tag{28}$$

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