Chapter 4

NETWORK FORMULATION OF STRUCTURAL ANALYSIS

4.1 INTRODUCTION

Graph theoretical concepts have been widely employed for the analysis of networks in the field of electrical engineering. Kirchhoff was the first to establish the basis for the analysis of networks, as early as 1874. In introducing the concepts of circulating loop currents, independent loops (cycles, circuits) and methods for finding branch currents in a network, Kirchhoff [101] employed a number of topological ideas long before graph theory became an established branch of mathematics.

Maxwell developed both the node method and the mesh method which are described in his famous book on Electricity and Magnetism of 1892. Ku [109] studied the rules of Maxwell and Kirchhoff in the light of the principle of duality. Synge [178] investigated the matrix theory of electrical networks. Further applications of graph theory to the conceptual analysis of electrical networks may be found in Trent [185], Roth [160] and Kron [105] among others.

The first application of topology and graph theory to the analysis of elastic networks (structures) seems to be due to Kron [106]. In this paper an analogy was made between electrical networks and elastic structures. Worked examples using equivalent circuits for elastic structures can be found in Carter and Kron [14].

In a series of papers Kron [105] presented the method of piecewise solutions of large scale systems, so called "*diakoptics*". In this method a physical system is torn into an appropriate number of small subdivisions, each of which is analyzed and solved separately. The partial solutions are then interconnected step by step until the solution of the entire system is obtained. Further consideration of topological concepts for elastic structures was given by Kron [106].

Langefors [112-117] established the system of equations for the analysis of indeterminate continuous frames, and presented a complete topological approach to structural analysis. Samuelsson [165] used algebraic topology for the analysis of skeletal structures, and Wiberg [196] employed a mixed method for the analysis of more general structures including continuum problems.

Henderson and Bickley [60] related the degree of statical indeterminacy of a rigid-jointed frame to the first Betti number of its graph model. Henderson [59], Henderson and Maunder [61] and Kaveh [77] used various embedding techniques in order to overcome the difficulties involved in the force method of structural analysis.

The analysis for the cases when the structures are viewed as general systems or networks was formulated by Lind [119], Dimaggio [29], Fenves [36], Fenves and Branin [37], Spillers [170,171], Mauch and Fenves [126]. Oden and Neighbors [137] employed graph theory to the non-linear analysis of structures, and Fenves and Gonzalez-Caro [38] and Munro [133] used it in plastic analysis and design. Cassell [15] and Shinghui and Guohua [167] employed graph theory in finite element analysis.

Different formulations of structural analysis have been made using various mathematical tools. Perhaps Russopoulos [163] was the first to achieve this goal by using tensorial approaches. Argyris and Kelsey [4] employed matrix algebra, Langefors [112-117] and Samuelsson [165] used algebraic topology, Maunder [127] employed vector spaces, and Fenves and Branin [37] used graphs and networks for the formulations of structural analysis.

In this chapter a network-topological formulation of structural analysis, which in the main follows that of Fenves and Branin [37] is presented. A complete matrix formulation of displacement and force methods will be provided in subsequent chapters.

4.2 THEORY OF NETWORKS

The common properties of various networks such as electrical, hydraulic and mechanical networks make a unified treatment of their properties feasible. As an example, the theory developed for the analysis of each type should be modifiable to apply to the analysis of the other types. Since the history of the application of graph theory to the analysis of electrical networks is much older than that of non-electrical networks, a formulation of the theory of structures based on network theory is justifiable. This unified treatment opens the door for the extension of advances made in network theory to structural theory and *vice versa*.

For defining a structure, three types of property should be studied separately. These properties, as already been mentioned in Chapter 2, consist of

1. Topological properties: which contain information about the interconnection of the members, provided by the the graph model of a structure.

2. Geometrical properties: which specify the spatial position of each member, implied by the dimensions of the structural drawing.

3. Mechanical properties: which consist of the load-deflection relationships of the members forming the entire structure.

Each of the above-mentioned properties is independent of the other two and each plays its own important role in characterizing the structure. However, it is easy to lose sight of these roles in the usual treatment of the structures. One of the main objectives of this chapter is to separate these properties, in order to find the common aspects of structural and network theories. It is shown that the force and displacement methods of structural analysis for rigid-jointed skeletal structures can easily be developed as the counterparts of mesh and node methods of network analysis.

4.3 BASIC CONCEPTS OF NETWORK THEORY

A system which is made up from several individual members interconnected in some particular way can be defined as a network. As a result of this interconnection, the variables specifying the behaviour of the individual members are inter-related. This is true regardless of whether the network is electrical, hydraulic or mechanical.

In a network two mathematical structures are present:

(a) An underlying topological structure called a graph, e.g. the dead wires of an electrical network, or the stress free state of the structure.

(b) A superimposed algebraic structure corresponding to the live quantities of a network, e.g. electromagnetic in an electrical network, or the loaded state of a structure.

The corresponding mathematical structures will posses distinct properties common to the modelling of a variety of physical phenomena. The reason for the abstract description of these properties is to expose the essential elements of all types of network problem without being restricted to some particular physical phenomenon.

4.3.1 TOPOLOGICAL PROPERTIES OF NETWORKS

A network can be modelled as a graph. The connectivity relations of a network are the topological properties of its graph model. These properties were studied in Chapter 1 and are here re-introduced using slightly different notations.

Let S be the mathematical model of a network which is considered to be a directed graph. Consider $\overline{\mathbf{B}}$ as the node-member incidence matrix of S. Since each column of $\overline{\mathbf{B}}$ contains +1 and – 1, its rows are linearly dependent, i.e.

$$\sum_{i=1}^{N} \bar{b}_{ij} = 0 \quad \text{for } j = 1,2, ..., M \qquad (4\text{-}1)$$

Selecting any particular node of S as a datum (reference) node and assigning the ordinal number N to this node, one can write

$$\bar{b}_{Nj} = -\sum_{i=1}^{N-1} \bar{b}_{ij} \quad \text{for } j = 1,2, ...,M \qquad (4\text{-}2)$$

Thus the Nth row of $\overline{\mathbf{B}}$ can be deleted to obtain the incidence matrix $\mathbf{B}$ without loss of information, since

$$\bar{b}_{Nj} = -\sum_{i=1}^{N-1} \bar{b}_{ij} \quad \text{for } j = 1,2, ..., M, \qquad (4\text{-}3)$$

in which b_{ij} are the entries of matrix $\mathbf{B}$.

EXAMPLE: Consider a directed graph as shown in Fig. 4.1 for which M=6 and N=5. Matrix $\overline{\mathbf{B}}$ is formed as

	m_1	m_2	m_3	m_4	m_5	m_6
n_1	0	1	0	1	0	0
n_2	0	0	1	–1	1	0

$$\overline{\mathbf{B}} = \begin{array}{c|cccccc} n_3 & 1 & -1 & 0 & 0 & -1 & 1 \\ n_4 & 0 & 0 & -1 & 0 & 0 & -1 \\ n_5 & -1 & 0 & 0 & 0 & 0 & 0 \end{array} \qquad (4\text{-}4)$$

Eqs (4-1), (4-2) and (4-3) can now be verified as an exercise, using the above matrix.

Fig. 4.1 A directed graph S and numbers assigned to its nodes.

Consider an arbitrary tree T of S. If the branches of T are first ordered, followed by numbering its chords, then **B** will be partitioned into two submatrices $\mathbf{B}_T$ and $\mathbf{B}_C$.

$$\mathbf{B} = \begin{array}{c|cccccc} & m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \\ \hline n_1 & 0 & 1 & 0 & 1 & 0 & 0 \\ n_2 & 0 & 0 & 1 & -1 & 1 & 0 \\ n_3 & 1 & -1 & 0 & 0 & -1 & 1 \\ n_4 & 0 & 0 & -1 & 0 & 0 & -1 \\ & & \mathbf{B}_T & & & \mathbf{B}_C & \end{array} \qquad (4\text{-}5)$$

Submatrix $\mathbf{B}_T$ is an $(N - 1)\times(N - 1)$ matrix which is nonsingular. Its inverse $\mathbf{P}_T$ can be obtained topologically; it is the set of node-to-datum paths in the tree.

$$\mathbf{P}_T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 \end{bmatrix}. \qquad (4\text{-}6)$$

Now let **C** be the cycle basis incidence matrix of S as defined in Section 1.5.2.

$$\mathbf{C} = \begin{array}{c|cccccc} & m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \\ C_1 & 0 & -1 & 0 & 1 & 1 & 0 \\ C_2 & 0 & 1 & -1 & -1 & 0 & 1 \end{array} \qquad (4\text{-}7)$$

$$\mathbf{C}_T \qquad \mathbf{C}_c$$

The most important topological property of a graph is the orthogonality property, which can be expressed as

$$\mathbf{BC}^t = \mathbf{0} \quad \text{or} \quad \mathbf{CB}^t = \mathbf{0}. \qquad (4\text{-}8)$$

For a fundamental cycle basis, if tree members are numbered first, followed by numbering the chords, then Eq.(4-8) in partitioned form can be written as

$$\begin{bmatrix} \mathbf{B}_T & \mathbf{B}_c \end{bmatrix} \begin{bmatrix} \mathbf{C}_T^t \\ \mathbf{C}_c^t \end{bmatrix} = 0. \qquad (4\text{-}9)$$

Since $\mathbf{C}_c^t = \mathbf{I}$, thus

$$\mathbf{B}_T\mathbf{C}_T^t + \mathbf{B}_c = \mathbf{0},$$

or

$$\mathbf{C}_T^t = -\,\mathbf{B}_T^{-1}\,\mathbf{B}_c = -\,\mathbf{P}_T\mathbf{B}_c. \qquad (4\text{-}10)$$

These equations provide enough tools for the formulation of the network problem. The connectivity properties of a graph model are all contained in these relations. As pointed out in Chapter 1 , one can also construct and use a cut set basis incidence matrix in place of a cocycle basis (node)-member incidence matrix, but this is not discussed here.

4.3.2 ALGEBRAIC PROPERTIES OF NETWORKS

In algebraic topology and graph theory, it is customary to assign mathematical objects such as numbers to nodes (0-simplexes), members (1-simplexes), cut sets and cycles of a graph (1-complex). This leads to an algebraic structure associated with a graph , which is studied in the following.

Consider a graph S and associate a number with each of its nodes. These numbers are represented by an N-vector $\bar{\mathbf{u}}$. When $\bar{\mathbf{u}}$ is premultiplied by the transpose of the node-member incidence matrix of S, $\overline{\mathbf{B}}^t$, the transformation induces the assignment of some numbers to the members of S, determined by

$$\mathbf{u} = \overline{\mathbf{B}}^t \overline{\mathbf{u}}. \tag{4-11}$$

In this equation **u** is an M-vector containing the numbers assigned to members. If member m has nodes s and t as its starting and end nodes, respectively, then Eq.(4-8) yields

$$\overline{\mathbf{u}}_m = \sum_{j=1}^{N} \overline{b}_{mj}\, \overline{u}_j = \overline{u}_s - \overline{u}_t. \tag{4-12}$$

As an example, if $\overline{\mathbf{u}} = \{2,4,6,3,5\}^t$ contains the numbers assigned to the nodes of S in Fig. 4.1(a), and **B** is taken from the previous example (Eq.(4-4)), then Eq.(4-12) leads to

$$\mathbf{u} = \overline{\mathbf{B}}^t\, \overline{\mathbf{u}} = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 6-5 \\ 2-6 \\ 4-3 \\ 2-4 \\ 4-6 \\ 6-3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 1 \\ -2 \\ -2 \\ 3 \end{bmatrix}.$$

This shows that the number assigned to a member m is simply the difference between the quantities assigned to its end nodes. Now if the Nth node of S is taken as a datum node, then Eqs (4-11) and (4-12) can be written as

$$\mathbf{u}_m = \sum_{j=1}^{N-1} \overline{b}_{mj}\overline{u}_j + \overline{b}_{mN}\overline{u}_N = \sum_{j=1}^{N-1} b_{mj}(\overline{u}_i - \overline{u}_N). \tag{4-13}$$

This relation indicates that the member quantity $\mathbf{u}_m$ depends on the difference between the non-datum quantities and the datum quantities. Taking these differences as an (N–1)-vector **u**´, Eq.(4-13) can be written as

$$\mathbf{u} = \mathbf{B}^t \mathbf{u}', \tag{4-14}$$

in which **B** is the member-node incidence matrix.

For the previous example, if node 5 is taken as the datum node, then **u**´will be given as $\mathbf{u}' = \{2–5, 4–5, 6–5, 3–5\}^t = \{–3,–1,1,–2\}^t$ and **B** is obtained by deleting row 5 of $\overline{\mathbf{B}}$. Then Eq.(4-14) results in

$$\mathbf{u} = \mathbf{B}^t\,\mathbf{u}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 1 \\ -2 \\ -2 \\ 3 \end{bmatrix},$$

where **u** contains the same numbers as those assigned previously to the members of S.

The vector **u** is now premultiplied by the cycle basis incidence matrix **C** of S. Naturally this operation will assign numbers to the elements of the considered cycle basis of S. However, because of orthogonality of the cocycle basis and the cycle basis of a graph

$$\mathbf{C}\mathbf{B}^t = \mathbf{0}.$$

Thus the cycle quantities obtained in this way are all zeros. This is obvious since

$$\mathbf{C}\mathbf{u} = \mathbf{C}\mathbf{B}^t\mathbf{u}' = \mathbf{0}\mathbf{u}' = \mathbf{0}. \qquad (4\text{-}15)$$

In Fig. 4.2(a), for the cycle basis of S consisting of its regional cycles, we have

$$\mathbf{C} = \begin{bmatrix} 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix},$$

and

$$\mathbf{C}\mathbf{u} = \begin{bmatrix} 4 - 2 - 2 \\ 1 + 2 - 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which verifies Eq.(4-15).

(a) (b)

Fig. 4.2 Numbers assigned to the cycles and members of S.

For a member, apart from the number obtained by Eq.(4-11) as u_i, one can also assign an arbitrary number U_i. Collecting these values in an M-vector and premultiplying by **C**, leads to nonzero quantities assigned to the cycles, i.e.

$$\mathbf{C}\mathbf{U} = \mathbf{U}'. \tag{4-16}$$

For the previous example, consider $\mathbf{U} = \{6,10,15,5,12,8\}^t$, as shown in Fig. 4.2(b). Then Eq.(4-16) leads to

$$\mathbf{U}' = \mathbf{C}\mathbf{U} = \begin{bmatrix} -10 + 5 + 12 \\ 15 - 12 - 8 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}.$$

Adding the derived quantity **u** of a member to the arbitrary quantity **U** assigned to it results in

$$\mathbf{V} = \mathbf{U} + \mathbf{u}\ , \tag{4-17}$$

and Eqs (4-15) and (4-16) lead to

$$\mathbf{C}\mathbf{V} = \mathbf{U}'. \tag{4-18}$$

The relationships obtained up to the present are summarized as shown in Fig. 4.3.

Fig. 4.3 Transformation diagram of Type I variables.

Now a dual set of relations can be obtained by assigning an arbitrary number to each cycle of the graph S. These numbers can be collected in a vector $\mathbf{p}'$of length (dimension) $b_1(S) = M - N+1$ for a connected S. The transformation $\mathbf{C}^t\mathbf{p}'$ induces the assignment of a related set of numbers to the members of S according to

$$\mathbf{p} = \mathbf{C}^t\mathbf{p}'. \tag{4-19}$$

These member quantities induce quantities in the nodes of S by the following transformation

$$\mathbf{Bp} = \mathbf{BC}^t\mathbf{p}' = \mathbf{0p}' = \mathbf{0}. \qquad (4\text{-}20)$$

Again due to the orthogonality property ($\mathbf{BC}^t = \mathbf{0}$), the induced quantities in the nodes are zero.

It is possible to assign a certain arbitrary number P_i to each member, in addition to the derived member quantity p_i. Then the transformation **BP** will not lead to zero quantities for the nodes, i.e.

$$\mathbf{BP} = \mathbf{P}', \qquad (4\text{-}21)$$

in which **P´** is an (N – 1)-vector. Summing the derived and arbitrary quantities for the members, we have

$$\mathbf{R} = \mathbf{P} + \mathbf{p}. \qquad (4\text{-}22)$$

The above relations and transformations can be summarized as shown in Fig. 4.4.

Fig. 4.4 Transformation diagram of Type II variables.

At this point the purely topological aspects of the algebraic structure that characterizes the networks have been described. Two dual sets of variables **u´**, **u**, **U** and **U´** on one hand and **p´**, **p**, **P** and **P´** on the other hand have been independently studied. However, a distinct characteristic of a network problem is the existence of a relationship between the two sets of variables, expressed as

$$\mathbf{V} = \mathbf{fR}, \qquad (4\text{-}23)$$

or

$$\mathbf{U} + \mathbf{u} = \mathbf{f}(\mathbf{P}+\mathbf{p}). \tag{4-24}$$

The inverse relation also exists and can be written as

$$\mathbf{R} = \mathbf{kV}, \tag{4-25}$$

or

$$\mathbf{P} + \mathbf{p} = \mathbf{k}(\mathbf{U} + \mathbf{u}). \tag{4-26}$$

In the above relations **f** and **k** are $M \times M$ matrices, whose entries are real and complex numbers. **f** is the reverse of **k**, i.e.

$$\mathbf{f} = \mathbf{k}^{-1}. \tag{4-27}$$

Addition of the above transformation naturally reduces the types of mathematical object that can enter into this algebraic structure.

Combining the diagrams of Figs 4.3 and 4.4, together with Eqs (4-23) and (4-25), lead to the diagram shown in Fig. 4.5.

Fig. 4.5 Transformation diagram of the network problem.

Now the arbitrary vectors $\mathbf{u}'$, **U**, $\mathbf{p}'$ and **P** can no longer be chosen independent of each other, since these vectors are related through Eqs (4-13) , (4-19) , (4-23) and Eq.(4-25), i.e.

$$\mathbf{U} + \mathbf{B}^t\mathbf{u}' = \mathbf{f}(\mathbf{P} + \mathbf{C}^t\mathbf{p}'), \tag{4-28}$$

and

$$\mathbf{P} + \mathbf{C}^t\mathbf{p}' = \mathbf{k}(\mathbf{U} + \mathbf{B}^t\mathbf{u}'). \qquad (4\text{-}29)$$

Premultiplication of Eq.(4-25) by **C** and Eq.(4-26) by **B** results in

$$\mathbf{CU} = \mathbf{CfP} + \mathbf{CfC}^t\mathbf{p}', \qquad (4\text{-}30)$$

and

$$\mathbf{BP} = \mathbf{BkU} + \mathbf{BkB}^t\mathbf{u}'. \qquad (4\text{-}31)$$

The above relations should be satisfied for every admissible choice of vectors **u´**, **U**, **p´**and **P**.

Up to this stage no physical significance has been assigned to the network variables. However, there are many physical phenomena that exhibit two sets of variables (dual) as discussed above, and these sets will be inter-related in the same way. These phenomena can be treated as network problems, and subsequent sections of this volume will concentrate upon elastic networks.

4.3.3 FORMULATION OF NETWORK ANALYSIS

In a network problem the following data are normally given:

1. A graph model for which matrices **B** and **C** can easily be formed.

2. The transformation matrix **f** and/or matrix **k**.

3. The arbitrary vectors **U** and **P**.

The unknowns consists of **u** and **p** vectors, which should satisfy the following:

1. Eq.(4-24) and / or Eq.(4-26).

2. Eq.(4-20) and / or Eq.(4-19).

3. Eq.(4-15) and / or Eq.(4-14).

Two different methods can be used for solving the above problem, namely the mesh method and the node method. The first approach introduces auxiliary cycle (mesh) variables **p´** and the latter uses auxiliary node variables **u´**, respectively. The transformation diagram of Fig. 4.5 can easily be extended as illustrated in Fig. 4.6.

Fig. 4.6 Extended transformation diagram.

MESH METHOD: This method follows directly from Eq.(4-30). This equation satisfies conditions 1 and 3, and can be written as

$$\mathbf{C}(\mathbf{U} - \mathbf{fP}) = \mathbf{CfC}^t\mathbf{p}'. \qquad (4\text{-}32)$$

In this equation the left hand side is known and it may be solved for **p´** by the inversion of ($\mathbf{CfC}^t$), i.e.

$$\mathbf{p}' = (\mathbf{CfC}^t)^{-1}\mathbf{C}(\mathbf{U} - \mathbf{fP}). \qquad (4\text{-}33)$$

Now employing $\mathbf{p} = \mathbf{C}^t\mathbf{p}'$ and $\mathbf{u} = \mathbf{f}(\mathbf{P}+\mathbf{p}) - \mathbf{U}$, the solution is completed. It should be mentioned that condition 2 is also satisfied by Eq.(4-20).

When the computation of **R** and **V** in an explicit form is needed, then one can use

$$\mathbf{R} = \mathbf{C}^t (\mathbf{CfC}^t)\mathbf{CU} + [\,\mathbf{I} - \mathbf{C}^t(\mathbf{CfC}^t)^{-1}\mathbf{Cf}]\mathbf{P}, \qquad (4\text{-}34)$$

in which **I** is a unit matrix and

$$\mathbf{V} = \mathbf{fC}^t (\mathbf{CfC}^t)^{-1}\mathbf{CU} + [\mathbf{f} - \mathbf{fC}^t(\mathbf{CfC}^t)^{-1}\mathbf{Cf}]\mathbf{P}. \qquad (4\text{-}35)$$

NODE METHOD: This method uses Eq.(4-31), which satisfies conditions 1 and 2 of the solution and can be written as

$$\mathbf{B}(\mathbf{P} - \mathbf{kU}) = \mathbf{BkB}^t\mathbf{u}'. \qquad (4\text{-}36)$$

Solving for **u´**, we obtain

$$\mathbf{u}' = (\mathbf{BkB}^t)^{-1}\mathbf{B}(\mathbf{P} - \mathbf{kU}), \qquad (4\text{-}37)$$

and the solution is completed by employing $\mathbf{u} = \mathbf{B}^t\mathbf{u}'$ and $\mathbf{p} = \mathbf{k}(\mathbf{U}+\mathbf{u}) - \mathbf{P}$.

Vectors **V** and **R** can now be computed explicitly as

$$\mathbf{R} = \mathbf{kB}^t(\mathbf{BkB}^t)^{-1}\mathbf{BP} + [\mathbf{k} - k\mathbf{B}^t(\mathbf{BkB}^t)^{-1}\mathbf{Bk}]\mathbf{U}, \qquad (4\text{-}38)$$

and

$$\mathbf{V} = \mathbf{B}^t(\mathbf{BkB}^t)^{-1}\mathbf{BP} + [\mathbf{I} - \mathbf{B}^t(\mathbf{BkB}^t)^{-1}\mathbf{Bk}]\mathbf{U}. \qquad (4\text{-}39)$$

4.4 FORMULATION OF STRUCTURAL ANALYSIS

The following properties are common to networks and structures.

1. The underlying mathematical models for both are graphs.

2. The sum of the forces at each joint is zero.

3. Distortions in any cycle sum to zero.

Network variables have the following physical interpretations.

p´= cycle (redundant) forces.

p = member forces induced by redundants.

P = applied member forces (fixed end actions).

R = total member forces.

P´= joint loads.

U´= cycle distortions (lack of fit at closure).

U = applied member distortions (free-end distortion).

u = member distortions induced by joint displacements.

V = total member distortions.

u´= joint displacements.

k = unassembled stiffness matrix of the structure containing member stiffness matrices in a block diagonal form.

f = unassembled flexibility matrix of the structure containing member flexibility matrices in a block diagonal form.

Structural analysis is different from network analysis in that the elements corresponding to nodes, members and cycles are not scalars but vectors. Naturally, for different elements different vector bases are needed. This problem can be overcome by interpreting the elements of topological matrices as identity matrices of appropriate orders. The geometric transformations between vector bases, however, can be handled separately from the topological ones. Various methods are available for separating geometry from topology. As an example, using a global coordinate system for the entire structure, all variables can be referred to a common point and therefore become directly additive. An alternative vector basis of the member variables and a tree formation matrix between member and nodal quantities can be redefined, Spillers [170]. Finally, a slight modification can be made to the definition of the incidence matrix **B** whereby geometric transformations are effected only from the positive to the negative end of a member, Fenves and Branin [37]. It is now easy to show that the network problem corresponds to a complete formulation of structural analysis.

FORCE - DISPLACEMENT RELATIONSHIP

The equations **V** = **fR** and **R** = **kV** are force-displacement relationships which can be regarded as a generalized Hooke´s law. Matrices **f** and **k** contain the flexibility and stiffness matrices of individual members, respectively. These matrices are known as the unassembled flexibility and stiffness matrices of the structure, which are discussed in Chapters 5 and 6.

EQUILIBRIUM OF FORCES

The relationships $\mathbf{p} = \mathbf{C}^t\mathbf{p}'$ and $\mathbf{Bp} = \mathbf{0}$ are equilibrium equations. The first equation states that the member forces can be expressed as linear combinations of independent cycle forces or redundants. The latter indicates that the sum of member forces at every joint is equal to zero.

COMPATIBILITY OF DISPLACEMENTS

Equations $\mathbf{u} = \mathbf{B}^t\mathbf{u}'$ and $\mathbf{Cu} = \mathbf{0}$ are compatibility equations. The first indicates that the member distortions are linear combinations of joint displacements. The latter expresses this fact that the sum of distortions in a cycle is equal to zero.

The two methods of structural analysis, namely the force method and the displacement approach, are identical to the mesh and the node methods of network analysis, respectively. For comparison it is assumed that only applied joint loads $\mathbf{P}'$ are present and $\mathbf{U}$, $\mathbf{P}$ and $\mathbf{U}'$ are taken as zero.

FORCE METHOD: In this method the unknowns are taken as cycle forces (redundants) $\mathbf{p}'$. By the equilibrium of forces

$$\mathbf{R} = \mathbf{P} + \mathbf{p} = \mathbf{P} + \mathbf{C}^t\mathbf{p}'. \qquad (4\text{-}40)$$

$\mathbf{P}$ can be obtained by selecting a suitable tree and forming its node-to-datum path matrix $\mathbf{P}_T$. Obviously member forces in tree members are $\mathbf{P}_T\mathbf{P}'$ and zero in its chords. Therefore

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_T\,\mathbf{P}' \\ 0 \end{bmatrix}. \qquad (4\text{-}41)$$

Using the flexibility matrix, member distortions can be written as

$$\mathbf{V} = \mathbf{fR} = \mathbf{fP} + \mathbf{fC}^t\mathbf{p}'. \qquad (4\text{-}42)$$

Compatibility conditions can now be imposed, leading to

$$\mathbf{U}' = \mathbf{CV} = \mathbf{CfP} + \mathbf{CfC}^t\mathbf{p}' = \mathbf{0}. \qquad (4\text{-}43)$$

Solving for redundant forces, we obtain

$$\mathbf{p}' = -(\mathbf{CfC}^t)^{-1}\mathbf{CfP}. \qquad (4\text{-}44)$$

The member forces and distortions can be obtained as

$$\mathbf{R} = \mathbf{P} - \mathbf{C}^t(\mathbf{CfC}^t)^{-1}\mathbf{CfP}, \qquad (4\text{-}45)$$

and

$$\mathbf{V} = \mathbf{f}[\mathbf{P} - \mathbf{C}^t(\mathbf{CfC}^t)^{-1}\mathbf{CfP}], \qquad (4\text{-}46)$$

which completes the solution by the force method.

DISPLACEMENT METHOD: In this method the unknowns are joint displace - ments **u´**. Imposing the compatibility (continuity) conditions,

$$\mathbf{V} = \mathbf{U} + \mathbf{u} = \mathbf{0} + \mathbf{B}^t\mathbf{u}' = \mathbf{B}^t\mathbf{u}'. \qquad (4\text{-}47)$$

Using the unassembled stiffness matrix of the structure, we can write

$$\mathbf{R} = \mathbf{kV} = \mathbf{kB}^t\mathbf{u}'. \qquad (4\text{-}48)$$

Now by equilibrium, the total member forces in each joint of the structure must be equal to zero, i.e.

$$\mathbf{P}' = \mathbf{BR} = \mathbf{BkB}^t\mathbf{u}'. \qquad (4\text{-}49)$$

The joint displacements can now be calculated as

$$\mathbf{u}' = (\mathbf{BkB}^t)^{-1}\mathbf{P}', \qquad (4\text{-}50)$$

and the final member distortions and member forces can be obtained as

$$\mathbf{V} = \mathbf{B}^t(\mathbf{BkB}^t)^{-1}\mathbf{P}', \qquad (4\text{-}51)$$

and

$$\mathbf{R} = \mathbf{kB}^t(\mathbf{BkB}^t)^{-1}\mathbf{P}'. \qquad (4\text{-}52)$$

This completes the analysis of a structure by the displacement method.

At this stage we have shown that the two methods of structural analysis correspond directly to the general network solution as illustrated in the diagram of Fig. 4.6. Although a general formulation of the force and displacement methods has been presented in this chapter, an efficient analysis of a structure requires modifications and special considerations. These together with examples in illustration of the methods will be given in subsequent chapters.

EXERCISES

4.1 Consider the oriented graph model of a network as shown in Fig.(a). Associate the nodal quantities as depicted in Fig.(b). Find the corresponding member quantities.

(a) (b)

4.2 In the previous exercise find the quantities which are associated with the regional cycles of the graph.

The following exercises are intended to be treated as interesting research problems, and should not be considered as simple exercises.

4.3 Formulate the analysis of rigid-jointed skeletal structures in terms of vector spaces (see Ref. [127]).

4.4 Formulate the analysis of frame structures using algebraic topology (see Refs [117,165]).

4.5 Formulate the analysis of skeletal structures in terms of the theory of matroids (see Chapter 9 and Ref. [131] for equivalent network formulation).

4.6 How do you modify network analysis if there are some releases in the structure (see Ref. [126]).