Frequency Analysis of Relay Feedback Systems Based on Phase Properties

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Abstract— **Analyzes of the frequency variations of switching for relay feedback systems is considered in this paper. That is, the relationship between the frequency variations and the phase variations of the linear system, when used in relay feedback form, is derived. It will be shown that for linear systems that their phase angle does not crosses -180^0, the switching frequency increases without bound and there is no equilibrium point. Furthermore, it will be shown that for systems with the relative degree greater than two, i.e. systems that their phase angle crosses -180^0, the phase angle equal to -180^0 is an equilibrium point for the oscillations frequency. Moreover, the equation that indicates the relation between the switching frequency variations and characteristics of systems such as the phase and amplitude of the system will be derived.**

Keywords- Relay feedback systems; Phase angle; Switching frequency.

I. INTRODUCTION

Research on relay feedback systems is fast developing and recently several papers have been published in this area. This is mainly due to the complex behavior and interesting characteristics of relay feedback systems.

The first relay application was commenced in electromechanical systems by using simple models for dry frictions [1]. Then, due to interesting properties, the relay feedback systems were developed for aerospace applications [2] and later on for self-oscillating adaptive controllers [3].

A number of analysis methods and results for the modeling of relay feedback systems, existence of limit cycles and methods for estimating switching frequencies, and the amplitude of oscillations are discussed in [4-6].

It is well known that when relay feedback systems are operating in the sliding mode, they may have finite or infinite switching frequencies, which are called the limit cycle and the chattering phenomenon, respectively [7, 8].

The stability of relay feedback systems, when operating in the limit cycle, is shown by some researches [9-10]. Also, when the relay feedback system is operating in the sliding mode, one can use the amplitude and frequency of oscillations to determine some parameters of the system such as the amplitude of the system on that frequency. This property has been used for tuning parameters of PID controllers [11] and for identification of linear systems [12, 13]. In addition, relay feedback systems are designed using new structures to provide new tools in control engineering concepts [14-16]. For example, in [15] two relays are employed in parallel where the gain of relays can be determined such that the stability region of the system is improved. Some researchers have decomposed the relay model into the slow mode and the fast mode, which yield more accuracy as well as simpler implementation in some applications [17]. In [17], authors have used this kind of modeling for the observer design where the relay model in the slow frequency mode is replaced with the equivalent gain and have shown that it simplifies the stability of the closed-loop system.

In spite of these researches, there exists lack of analysis on the frequency variations of relay feedback systems. One of the important factors in relay feedback systems is to analyze the rate of frequency variations and its convergence rate. The main goal of this paper is to discuss the frequency behavior of relay feedback systems. It will be shown analytically, systems that their phase angles crosses -180^0 have an equilibrium point at this frequency. It should be noted that the phase variations around this equilibrium point is caused by the switching frequency. On the other hand, systems, whose phase does not cross -180^0 do not have any equilibrium point on their switching frequency.

This paper is organized as follows. Section 2 gives the problem statement. Analytical results on the frequency variations are shown in Section 3. Section 4 provides simulation results, followed by conclusion in Section 5.

II. PROBLEM STATEMENT

Let the linear plant be given by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x}, \end{aligned} \tag{1}$$

where u is the output of the relay (Fig. 1) and can be presented as

$$u = -c\,\mathrm{sgn}(y), \tag{2}$$

in which c is the gain of the relay. The harmonic balance equation of the relay feedback system can be written as [18]

$$G(j\Omega)N(a) = -1, \tag{3}$$

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where $G(j\Omega)$ is the system transfer function, $N(a)$ is the equivalent gain of the relay, and a is the amplitude of oscillations

$$y(t) = a\sin(\Omega t). \tag{4}$$

The equivalent gain of the relay feedback system is determined by the following equation [15, 16]:

$$N(a) = -\frac{4c}{\pi a}. \tag{5}$$

Assumption 1: The relay feedback considered in this paper is an ideal relay represented by (5).

Assumption 2: It is assumed that conditions for the stability of oscillations are satisfied. I.e. oscillations are stable.

These assumptions are taken in order to simplify the frequency analysis that will be discussed in the next section.

III. FREQUENCY ALANYSIS

The transfer function of the system (1) can be decomposed into the real $R(\Omega)$ and imaginary $I(\Omega)$ parts as

$$G(j\Omega) = R(\Omega) + jI(\Omega) \tag{6}$$

and in the polar from as

$$G(j\Omega) = M(\Omega)e^{j\Phi(\Omega)} \tag{7}$$

where $M(\Omega)$ is the amplitude and $\Phi(\Omega)$ is the phase angle of the system, respectively.

Theorem 1: Consider the linear system with the relay feedback given in (1) and (2). The frequency variation of oscillations $\Delta\Omega$ of the relay feedback system satisfies the following equations

$$\Delta\Omega = \frac{-\frac{\pi c}{4a} - R(\Omega)}{dR(\Omega)/d\Omega} \tag{8}$$

$$\Delta\Omega = \frac{-I(\Omega)}{dI(\Omega)/d\Omega} \tag{9}$$

Proof: Let rewrite (3) replacing Ω with $\Omega + \Delta\Omega$ as

$$G\left(j(\Omega + \Delta\Omega)\right) = \frac{-1}{N(a)}. \tag{10}$$

Using the Taylor expansion method, it gives

$$G(j\Omega) + j\Delta\Omega\frac{dG(j\Omega)}{jd\Omega} + \chi = \frac{-1}{N(a)}, \tag{11}$$

where χ indicates high order terms

$$\chi = \sum_{n=2}^{\infty}\left[(j\Delta\Omega)^n \frac{d^nG(j\Omega)}{jd\Omega^n}\right]. \tag{12}$$

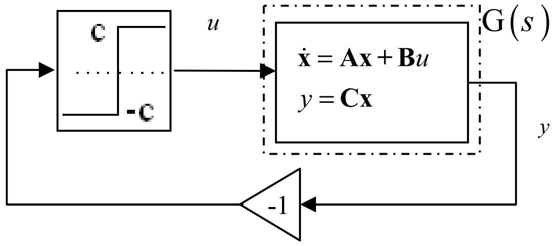

Fig. 1. Relay feedback system.

Since $\Delta\Omega$ is relatively small and in addition for linear systems $d^nG(j\Omega)/jd\Omega^n$ is negligible for $n > 2$, χ can be neglected without loss of generality. Using (5) and (6), (11) can be rewritten as

$$R(\Omega) + \Delta\Omega\frac{dR(\Omega)}{d\Omega} + j\left[I(\Omega) + \Delta\Omega\frac{dI(\Omega)}{d\Omega}\right] = -\frac{\pi a}{4c} \tag{13}$$

Separating the real and imaginary parts

$$R(\Omega) + \Delta\Omega\frac{dR(\Omega)}{d\Omega} = -\frac{\pi a}{4c} \tag{14}$$

$$I(\Omega) + \Delta\Omega\frac{dI(\Omega)}{d\Omega} = 0, \tag{15}$$

which are the same as (8) and (9). □

Remark 1: The frequency variation $\Delta\Omega$ must satisfy (8) and (9). Since (9) is independent of the amplitude of the limit cycle (*a*), it is easier to determine $\Delta\Omega$ using (9). Then, (8) can be used to validate the results from (9). Moreover, the amplitude of oscillations (*a*) obtained from (8) must be positive and finite. In other words, if $\Delta\Omega$ obtained form (9) and substituted into (8), gives negative value for the variable *a*, it is not acceptable; that is, such operating point does not exists for the system.

Remark 2: It can be concluded from (8) that when the amplitude of limit cycle *a* increases, the frequency variations $\Delta\Omega$ decreases.

Assumption 3: It is assumed that the sign of $dM(j\Omega)/d\Omega$ and $d\Phi(j\Omega)/d\Omega$ for the system (1) does not change when the switching occurs. This assumption is valid for almost all systems and does not restrict the validity of the analysis that follows.

Next, the equilibrium point of the frequency variations $\Delta\Omega$ is analyzed.

Proposition: if the following conditions are satisfied:

$$\operatorname{sgn}\left[\left(d\Phi(j\Omega)/d\Omega\right)\Delta\Omega\right] > 0 \quad \text{if} \quad \Phi \le \Phi_0 \tag{16}$$

$$\operatorname{sgn}\left[\left(d\Phi(j\Omega)/d\Omega\right)\Delta\Omega\right] < 0 \quad \text{if} \quad \Phi > \Phi_0, \tag{17}$$

then, the phase angle $\Phi = \Phi_0$, where $\Phi_0 = \Phi(\Omega_0)$, is an equilibrium point for the phase angle variations of the system caused by the switching frequency variations.

TABLE I. $\Delta\Omega$ OBTAINED USING (23) FOR SYSTEMS WITH RELATIVE DEGREE GREATER THAN 2

$\Phi(j\Omega)$	0^- $\Omega\to 0$	Φ_s	$-\pi/2^+$	$-\pi/2^-$	$-\pi^+$	$-\pi^-$	$-3\pi/2^+$	$-3\pi/2^-$
$-I(j\Omega)$	+	+	+	+	+	–	+	+
$\frac{dI(j\Omega)}{d\Omega}$	–	+	+	+	+	+	–	–
$\Delta\Omega$	–	+	+	+	+	–	–	–
a	–	+	+	+	+	+	+	+

⋆ The gray column indicates the case where $\Delta\Omega$ produces negative sign for a

Proof: By Assumption 3

$$\operatorname{sgn}\left[\left(d\Phi(j\Omega)/d\Omega\right)\Delta\Omega\right]=\operatorname{sgn}\left[\Delta\Omega/d\Omega\right]\operatorname{sgn}\left[d\Phi(j\Omega)\right] \quad (18)$$

which can be written as

$$\operatorname{sgn}\left[\left(d\Phi(j\Omega)/d\Omega\right)\Delta\Omega\right]=\operatorname{sgn}\left[d\Phi(j\Omega)\right].$$

Hence, for $\Phi\le\Phi_0$ if $\operatorname{sgn}\left[d\Phi(j\Omega)\right]>0$, then $\Phi(j\Omega)$ increases until it reaches the phase angle $\Phi=\Phi_0$; and for $\Phi>\Phi_0$ if $\operatorname{sgn}\left[d\Phi(j\Omega)\right]<0$, then $\Phi(j\Omega)$ decrease until it reaches the phase angle $\Phi=\Phi_0$. Therefore, under these conditions, $\Phi(j\Omega)$ converges to Φ_0. In other words, $\Phi_0=\Phi(\Omega_0)$ is an equilibrium point. □

Theorem 2: Consider the linear plant (1) with the relay feedback (2). The frequency variation $\Delta\Omega$ has an equilibrium point at the phase angle corresponding to -180^0 (i.e. a limit cycle exists). Moreover, if the system phase does not cross -180^0 at any frequency, then the frequency increases without bounds and the ideal sliding mode occurs.

Proof: Since

$$I(\Omega)=M(\Omega)\sin\left(\Phi(\Omega)\right) \quad (19)$$

$$R(\Omega)=M(\Omega)\cos\left(\Phi(\Omega)\right) \quad (20)$$

and

$$\frac{dI(\Omega)}{d\Omega}=\frac{dM(\Omega)}{d\Omega}\sin\left(\Phi(\Omega)\right)+M(\Omega)\frac{d\Phi(\Omega)}{d\Omega}\cos\left(\Phi(\Omega)\right) \quad (21)$$

$$\frac{dR(\Omega)}{d\Omega}=\frac{dM(\Omega)}{d\Omega}\cos\left(\Phi(\Omega)\right)-M(\Omega)\frac{d\Phi(\Omega)}{d\Omega}\sin\left(\Phi(\Omega)\right) \quad (22)$$

equation (9) becomes

$$\Delta\Omega=\frac{-M(\Omega)\sin\left(\Phi(\Omega)\right)}{\frac{dM(\Omega)}{d\Omega}\sin\left(\Phi(\Omega)\right)+M(\Omega)\frac{d\Phi(\Omega)}{d\Omega}\cos\left(\Phi(\Omega)\right)}. \quad (23)$$

In addition, substituting (21) into (9) yields

$$a=-\Delta\Omega\left(4c/\pi\right)\left[\frac{dM(\Omega)}{d\Omega}\cos\left(\Phi(\Omega)\right)-\right.$$
$$\left.-M(\Omega)\frac{d\Phi(\Omega)}{d\Omega}\sin\left(\Phi(\Omega)\right)\right]-\left(4c/\pi\right)M(s)\cos\left(\Phi(\Omega)\right) \quad (24)$$

According to Theorem 1 and Remark 1, the frequency variation $\Delta\Omega$, calculated using (23), is acceptable only when it is substituted in (24) it yields a positive and finite value for the amplitude of the limit cycle *a*. A negative value for *a* indicates that the limit cycle or the switching phenomenon does not occur for the relay feedback system.

In order to analyze the sign of $\Delta\Omega$ and finding the equilibrium point, the sign of the numerator and denominator of (23) must be determined. This will be determined for the following two cases.

Case 1*:* Assume that the relative degree of the system in (1) is greater than two. In other words, a system with generally reducing phase angle as the frequency grows and crosses the -180^o line. In this case, without loss of generality, it is assumed that during the switching, the slope of the phase and the amplitude is always negative; that is, $d\Phi(j\Omega)/d\Omega\le 0$ and $dM(j\Omega)/d\Omega\le 0$.

Table I shows the sign of $\Delta\Omega$ with respect to the frequency variations as the phase of the system decreases from 0^o to -270^o while the frequency Ω increases from 0 to $+\infty$.

If the limit cycle is stable at the starting moment of switching, there exists an increase in Ω. Hence, it is clear that the sign of $\Delta\Omega$ at this instance is positive. By observing Table 1, one can conclude that at the starting instance of the switching $\Phi\le\Phi_s$. As this table shows, the switching occurs in the range of $\Phi<\Phi_s$, where $0^o\le\Phi_s\le-90^o$. In this range of phase, $\Delta\Omega$ yields a positive value for a. Therefore, based on Remark 1, the sign of $\Delta\Omega$ can be determined for $\Phi<\Phi_s$.

One example from Table I can be shown as follows. Let $\Phi(j\Omega)\to-(\pi/2)^-$; then, from (23) and using $\sin\left(-\pi/2^-\right)=-1$ gives

$$\Delta\Omega\approx\frac{-M(\Omega)(-1)}{\frac{dM(\Omega)}{d\Omega}(-1)+M(\Omega)\frac{d\Phi(\Omega)}{d\Omega}\varepsilon}=\frac{\|M(\Omega)\|}{\left\|\frac{dM(\Omega)}{d\Omega}\right\|}>0 \quad (25)$$

Substituting (25) into (24) yields

$$a \approx -\Delta\Omega\left(\frac{4c}{\pi}\right)\left[-M(j\Omega)\frac{d\Phi(j\Omega)}{d\Omega}(-1)\right]$$
$$= \Delta\Omega\left(\frac{4c}{\pi}\right)\left\|M(j\Omega)\frac{d\Phi(j\Omega)}{d\Omega}\right\| > 0 \quad (26)$$

Based on Remark 1, it can be concluded that $\Delta\Omega > 0$ is true.

As Table I shows, the sign of $\Delta\Omega$ changes as the phase angle of the system crosses -180^{o}. Therefore, according to Proposition, -180^{o} is an equilibrium point for phase variations of the system caused by the switching frequency variations $\Delta\Omega$. Consequently, $\Delta\Omega \to 0$ as the phase angle of the system approaches -180^{o}. Hence, for systems with relative degree greater than two, a limit cycle with fixed frequency ($\Delta\Omega = 0$) exists.

Case 2: In this case, the frequency variation $\Delta\Omega$ is computed when the phase angle of the system converges to a certain value but does not cross -180^{o} as $\Omega \to \infty$. First, one can easily see from the theory of linear systems that

a) For systems with relative degree equal to zero, when $\Phi(j\Omega) \to 0^{+}$ as $\Omega \to \infty$, it yields

$$\frac{d\Phi(j\Omega)}{d\Omega} < 0 \text{ and } \frac{dM(j\Omega)}{d\Omega} > 0 .$$

Moreover, when $\Phi(j\Omega) \to 0^{-}$ as $\Omega \to \infty$, it gives

$$\frac{d\Phi(j\Omega)}{d\Omega} > 0 \text{ and } \frac{dM(j\Omega)}{d\Omega} < 0 .$$

b) For systems with relative degree equal to one and two (i.e. systems whose phase angle does not cross -180^{o}), when $\Phi(j\Omega) \to \Phi^{+}$ as $\Omega \to \infty$, where $\Phi = \{-90^{o}, -180^{o}\}$, it yields

$$\frac{d\Phi(j\Omega)}{d\Omega} < 0 \text{ and } \frac{dM(j\Omega)}{d\Omega} < 0 .$$

And when $\Phi(j\Omega) \to \Phi^{-}$ as $\Omega \to \infty$, where $\Phi = \{-90^{o}\}$, it gives

$$\frac{d\Phi(j\Omega)}{d\Omega} > 0 \text{ and } \frac{dM(j\Omega)}{d\Omega} < 0 .$$

Table II summarizes the results for different phase angles. For example, $\Delta\Omega$ can be computed for systems with relative degree equal to zero, when $\Phi(j\Omega) \to 0^{-}$ as $\Omega \to \infty$. From (23) and assuming that $\sin(0^{-}) = -\varepsilon < 0$

$$\Delta\Omega = \frac{-M(s)(-\varepsilon)}{\frac{dM(j\Omega)}{d\Omega}(-\varepsilon) + M(j\Omega)\frac{d\Phi(j\Omega)}{d\Omega}}$$
$$\approx \frac{M(s)\varepsilon}{M(j\Omega)\frac{d\Phi(j\Omega)}{d\Omega}} > 0 \quad (27)$$

TABLE II. $\Delta\Omega$ OBTAINED USING (23) FOR SYSTEMS WITH RELATIVE DEGREE SMALLER THAN 3.

$\Phi(j\Omega) \to_{\Omega\to\infty}$	0^{-}	$-\pi/2^{+}$	$-\pi/2^{-}$	$-\pi^{+}$
$-\mathrm{I}(j\Omega)$	+	+	+	+
$\frac{d\,\mathrm{I}(j\Omega)}{d\Omega}$	+	+	+	+
$\Delta\Omega$	+	+	+	+
a	+	+	+	+

Substituting (27) into (24) yields

$$a \approx +\Delta\Omega\left(\frac{4c}{\pi}\right)\left[\left\|\frac{dM(j\Omega)}{d\Omega}\cos(\Phi(j\Omega))\right\|\right] - \|M(s)\|\left(\frac{4c}{\pi}\right) \quad (28)$$

It is clear that if $\Delta\Omega > 0$, then a can become positive. Therefore, $\Delta\Omega > 0$ is acceptable. This means that the frequency of switching in systems with the relative degree equal to zero increases without bound.

Next, when $\Phi(j\Omega) \to -\pi^{+}$ as $\Omega \to \infty$ (i.e. systems with relative degree equal to two)

$$\Delta\Omega = \frac{-M(s)(-\varepsilon)}{\frac{dM(j\Omega)}{d\Omega}(-\varepsilon) + M(j\Omega)\frac{d\Phi(j\Omega)}{d\Omega}(-1)}$$
$$\approx \frac{M(s)\varepsilon}{\left\|\frac{dM(j\Omega)}{d\Omega}(\varepsilon)\right\| + \left\|M(j\Omega)\frac{d\Phi(j\Omega)}{d\Omega}\right\|} > 0 \quad (29)$$

Using (21), it gives

$$a \approx -\Delta\Omega\left(\frac{4c}{\pi}\right)\left[\frac{dM(j\Omega)}{d\Omega}(-1)\right] - \left(\frac{4c}{\pi}\right)M(s)(-1)$$
$$= -\Delta\Omega\left(\frac{4c}{\pi}\right)\left\|\frac{dM(j\Omega)}{d\Omega}\right\| + \left(\frac{4c}{\pi}\right)M(s) \quad (30)$$

It is clear that (30) can be positive for both $\Delta\Omega < 0$ and $\Delta\Omega > 0$. Positive $\Delta\Omega$ can produce positive a according to (30). Consequently, one can conclude that (29) is true. This completes the proof. □

IV. SIMULATING RESULTS

Example 1: Consider the following system with the relative degree equal to three:

$$\mathrm{G}_1(s) = \frac{50}{(s+0.2)(s+3)(s+4)} \quad (31)$$

The Bode diagram of this system is displayed in Fig. 2. Fig. 3 shows $\Delta\Omega$ using (23). As this figure shows, $\Delta\Omega$ exhibits a sign change at the frequency 3.66 rad/s , which corresponds to the phase angle equal to -180°. This, according to the Proposition, means that the system has a fixed oscillating frequency (i.e. a limit cycle) at 3.66 rad/s. Fig. 4 shows the time-domain response of the system using the closed-loop relay feedback configuration in Fig. 1. Fig. 4 confirms the limit cycle of 3.66 rad/s.

Example 2: Consider the following system with the relative degree equal to two:

$$\mathrm{G}_2(s)=\frac{5}{(s+0.2)(s+0.4)}. \tag{32}$$

Fig. 5 shows the Bode diagram of this system where the phase angle does not cross the -180° line. The frequency variation $\Delta\Omega$ is computed using (23) and is shown in Fig. 6. As this figure shows, $\Delta\Omega$ increases without bound; in other words, the frequency of switching approaches infinity. This fact is confirmed by inspecting the time-domain response of this system in the relay feedback form (Fig. 7).

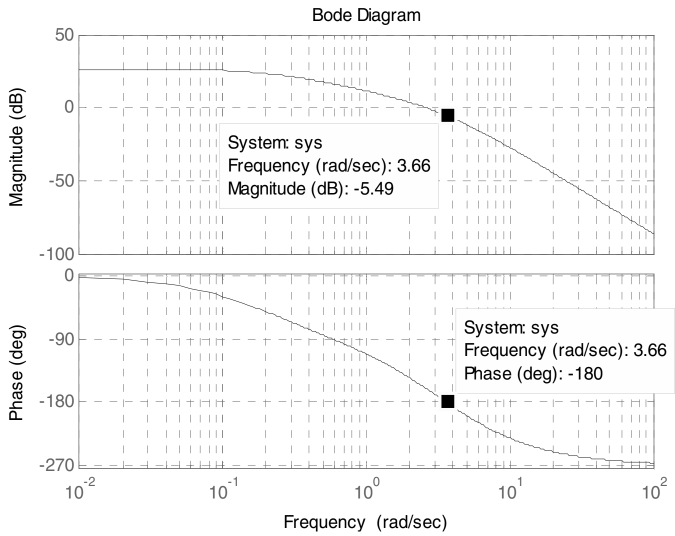

Fig. 2: Bode Diagram of $\mathrm{G}_1(s)$.

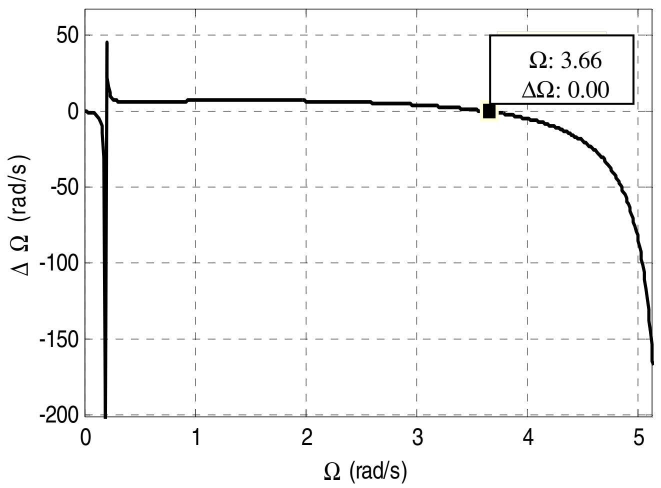

Fig. 3: $\Delta\Omega$ for $\mathrm{G}_1(s)$ using (18).

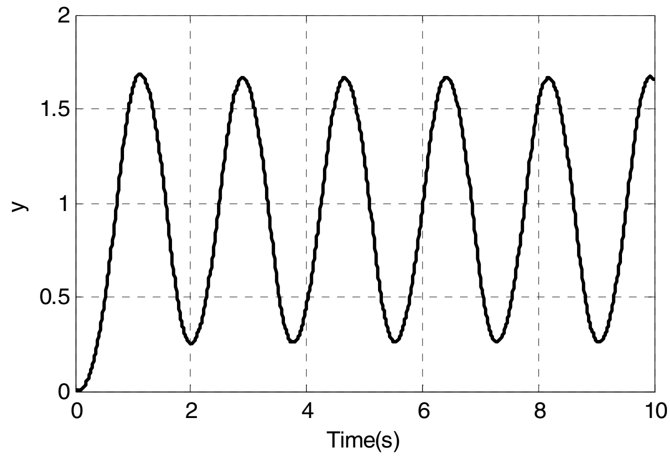

Fig. 4: Output of $\mathrm{G}_1(s)$ using relay feedback

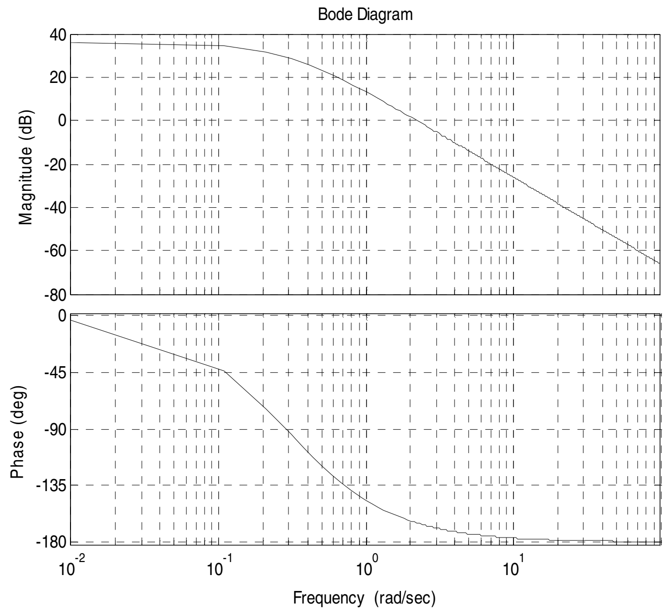

Fig. 5: Bode Diagram of $\mathrm{G}_2(s)$

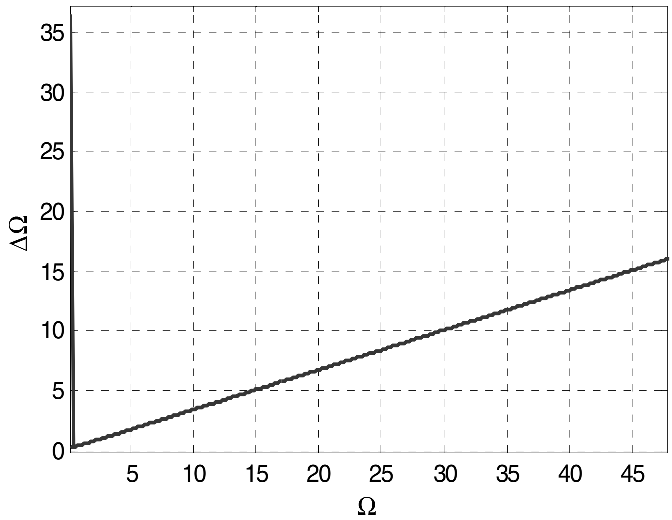

Fig. 6: $\Delta\Omega$ for $\mathrm{G}_2(s)$ using (18).

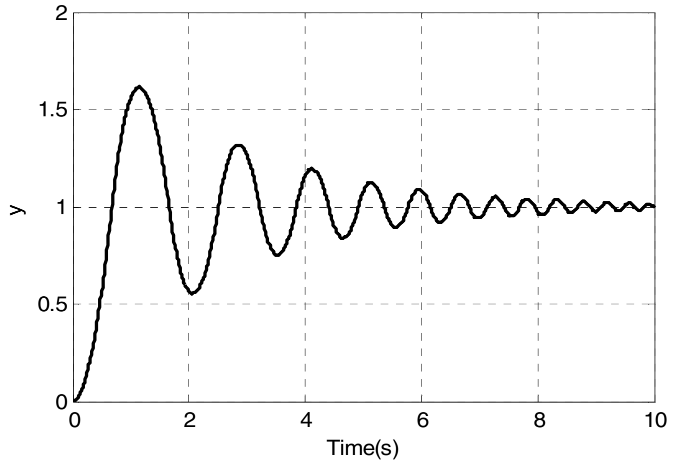

Fig. 7: Output of $G_2(s)$ using relay feedback.

V. CONCLUSION

In this paper, the switching frequency behavior of the relay feedback systems was considered. The relation of the frequency variations with respect to the amplitude and the phase angle of the system were derived. Moreover, it was shown that for systems with the relative degree greater than two, the frequency corresponding to the phase angle of -180^{o} is an equilibrium point. In the other words, for systems with the relative degree greater than two, the switching frequency increases until the phase angle of the system reaches -180^{o}.

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