Robust Adaptive Control of Non-linear Non-minimum Phase Systems with Uncertainties

S. M. Hoseini [a], M. Farrokhi [*a], A. J. Koshkouei [b]

[a] Department of Electrical Engineering,
Iran University of Science and Technology, Tehran 16846-13114, IRAN
[b] Control Theory and Applications Centre, Coventry University, Coventry, UK
E-mails: {sm_hoseini, farrokhi}@iust.ac.ir, a.koshkouei@coventry.ac.uk

Abstract

This paper, presents a robust adaptive control method for a class of non-linear non-minimum phase systems with uncertainties. The development of the control method comprises of two steps. First, stabilization of the system is considered based on the availability of the output and internal dynamics of the system. The reference signal is designed to stabilize the internal dynamics with respect to the output tracking error. Moreover, a combined neuro- adaptive controller is proposed to guarantee asymptotic stability of the tracking error. Then, the overall stability is achieved using the small gain theorem. Next, the availability of internal dynamics is relaxed by using a linear error observer. The unmatched uncertainty is compensated using a suitable reference signal. The ultimate boundedness of the reconstruction error signals is analytically shown using the extension of the Lyapunov theory. The theoretical results are applied to a translational oscillator/rotational actuator model to illustrate the effectiveness of the proposed scheme.

Key words: Adaptive Control, Non-minimum phase systems, Non-linear systems, Neural networks, Robust control.

1. Introduction

Control of nonlinear non-minimum phase systems is a challenging problem in control theory and has been an active research area for the last few decades. Several fundamental methods have been proposed in this area based on the state-feedback control, including the output redefinition and the zero assignment (Kazantzis, 2007; Talebi *et al.*, 2005), the stable inversion, and the iterative learning control for systems with predefined reference signals (Norrlof and Gunnarsson, 2001; Sogo *et al.* 2000). Moreover, the sliding mode control method (Yan *et al.*, 2006), neural networks and the fuzzy logic (Lee, 2004; Chen and Chen, 2003) have been successfully applied to control uncertain non-minimum phase systems.

In the case of output feedback control, the problem is more complicated. Contrary to linear systems, state observation of nonlinear systems is often not an easy task, even for many simple nonlinear systems. The main issue in output feedback control of non-minimum phase systems stems from the fact that information about state variables, associated with the zero dynamics, is vital in control design.

Recently, many methods have been proposed for output-feedback stabilization of uncertain non-minimum phase systems. Isidori (2000) has proposed a solution for semi-global output-feedback stabilization of non-minimum phase systems based on auxiliary constructions using a high-gain observer. Global output-feedback stabilization using the backstepping and the small-gain techniques have been employed by Karagiannis *et al.* (2005) and Wang *et al.* (2008). Ding (2005) has proposed a design method for the semi-global stabilization of a class of non-minimum phase non-linear systems that can be transformed to the global normal form as well as to the form of linear observer error dynamics. Sliding mode observers and output feedback sliding mode controllers for some classes of non-minimum phase non-linear systems have also been studied by many researchers including Yan *et al.* (2004). These methods have considered the stabilization problem for nonlinear systems in which their nonlinearities and the high frequency gain depend only on the system output.

Various results on the local and non-local stabilization of non-minimum phase non-linear systems have been presented that deal with the more general class of nonlinear systems using the universal approximation property of neural networks and fuzzy systems (Lee, 2004; Chen and Chen, 2003). However, in these works, it has been assumed that the system states are available. Hovakimyan et al. (2006) has been proposed a Gaussian Radial Basis Neural network (NN) using a tapped delay line of available measurement signals to compensate for modelling uncertainties, as proposed in Lavretsky et al. (2003). Their method is applicable to a class of non-minimum phase nonlinear systems with known relative degree and if the non-minimum phase zeros are modelled to a sufficient accuracy. In their work, the control is comprised of a linear controller and a neural network, and the adaptive laws have been given in terms of the output of a linear observer for the nominal system's error dynamics as in Hovakimyan et al. (2002). In addition, in their work, it was assumed that the augmentation of an arbitrary fixed gain linear controller must satisfy performance requirements in the absence of modelling errors. Their method is based on Lyapunov's direct method which guarantees local ultimate boundedness of error signals.

This paper presents an adaptive output-feedback control method for a class of observable and stabilizable nonlinear non-minimum phase systems. In the proposed method, only an approximate linear model of the nonlinear system is required with a few mild conditions. This linear model presents the non-minimum phase zeros of the nonlinear system with sufficient accuracy. In fact, there is a conic sector bound on the modeling error of the non-minimum phase zeros that is referred to as the unmatched uncertainty. Hence, the proposed approach can be applied to uncertain nonlinear systems, which have partially known Lipschitz continuous functions in their arguments. The system dynamics is described as two subsystems consisting of the internal and external dynamics. In Section 2, this class of nonlinear systems is introduced and a pseudo control is proposed to estimate the unknown external dynamics modeling or the matched uncertainty. The development of the control method is performed in two steps. First, the input-to-state stability of internal dynamic is studied based on availability assumption of the output. Then, the asymptotic stability of the output tracking error is proved using a combined output feedback controller. The stability of the closed-loop tracking error system is shown using the nonlinear small gain theorem (Jiang *et al.,* 1996; Karagiannis *et al.* 2005) which is presented in Section 3.. In contrast to the method presented by Lee (2004) and Chen and Chen (2003), the modeling error of internal dynamics is compensated to achieve a semi-global stability.
Therefore, only the information about the output and the internal dynamics are required to design an appropriate control which guarantees the asymptotic stability of the closed–loop tracking error system.

Next, in Section 4, the availability assumption on the internal dynamics is removed by designing an observer for the error tracking nonlinear system. In this section, under milder assumptions, the unmatched uncertainties are compensated using a reference signal and an adaptive robustifying term is designed to eliminate the approximation error of the NN. The robustifying term also guarantees the robustness against the parameter variations and small changes in the unmodeled dynamics. Moreover, a nonlinear parameterized NN is used to gain sufficient accuracy. In this case, it is proved that the states of the reconstruction error systems, created from the output tracking error system and observer, are ultimately bounded of the state is guaranteed. Therefore, a tracking output error depends on the observer and the approximation property of the NN. Hence, there is a trade-off between the relaxation of assumptions and the tracking output error.

Then in Section 5, simulations are carried out on the translational oscillator/rotational actuator (TORA) system to show the good performance of the proposed methods and to compare with newly proposed methods in the established literatures.

Finally conclusions are presented in Section 6.

2. Problem formulation

In this section a class of nonlinear systems which is considered in this paper is introduced. The dynamics of these systems are described by two subsystems, the so-called internal and external subsystems. A pseudo nonlinear control is proposed to estimate the unknown external dynamics modeling or the matched uncertainty. Consider the nonlinear system

$$\begin{cases} \dot{z}_i = z_{i+1} & 1 \le i \le r-1 \\ \dot{z}_r = f(\mathbf{z}, \boldsymbol{\eta}, u) & \\ \dot{\eta}_j = \eta_{j+1} & 1 \le j \le n-r-1 \\ \dot{\eta}_{n-r} = v(\mathbf{z}, \boldsymbol{\eta}) & \\ y = z_1, & \end{cases} \tag{1}$$

with the coordinates $[\mathbf{z}^T, \boldsymbol{\eta}^T] = [z_1, ..., z_r, \eta_1, ..., \eta_{n-r}]^T$, where r $(1 \le r < n)$ is the relative degree, $\boldsymbol{\eta} \in \Omega_\eta \subset R^{n-r}$ is the state vector associated with the internal dynamics, $\mathbf{z} \in \Omega_z \subset R^r$ where Ω_η and Ω_z are the compact sets associated with their corresponding operating regions, and u and y are the input and the output of the system, respectively. The mappings $f: R^{n+1} \to R$ and $v: R^n \to R$ are partially known and continuous Lipschitz functions with initial conditions $f(\mathbf{0}, \mathbf{0}, 0) = 0$ and $v(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. Note that the system (1) belongs to a class of nonlinear systems, the so-called normal (tracking) form (Isidori, 1995), and can be non-minimum phase. Hence, the stability assumption on the zero dynamics of the system is not required.

Assumption 1. Assume that for all $u \in R$, $f_u = \partial f(\mathbf{z}, \boldsymbol{\eta}, u)/\partial u \ne 0$. This condition implies that the smooth function f_u is strictly either positive or negative on the compact set $U = \{(\mathbf{z}, \boldsymbol{\eta}, u) | \mathbf{z} \in \Omega_z, \ \boldsymbol{\eta} \in \Omega_\eta, \ u \in R\}$.

Since the mappings $f: R^{n+1} \to R$ and $v: R^n \to R$ are partially known and continuous Lipschitz functions, the system (1) can be represented as the following expanded model:

$$\begin{cases} \dot{z}_i = z_{i+1} & 1 \le i \le r-1 \\ \dot{z}_r = \mathbf{m}^T \mathbf{z} + \mathbf{n}^T \boldsymbol{\eta} + b\psi(\mathbf{z}, \boldsymbol{\eta}, u) & \\ \dot{\eta}_j = \eta_{j+1} & 1 \le j \le n-r-1 \\ \dot{\eta}_{n-r} = \mathbf{f}^T \boldsymbol{\eta} + \mathbf{g}^T \mathbf{z} + \Delta_{\boldsymbol{\eta}}(\mathbf{z}, \boldsymbol{\eta}) & \end{cases} \tag{2}$$

where $\psi(\mathbf{z}, \boldsymbol{\eta}, u)$ and $\Delta_{\boldsymbol{\eta}}(\mathbf{z}, \boldsymbol{\eta})$ are the unknown external and internal dynamics modeling errors or the matched and unmatched uncertainty, respectively, and b is a scalar, $\mathbf{m} \in R^r$ and $\mathbf{n}, \mathbf{f} \in R^{n-r}$. Also $\mathbf{g} = [g_1 \cdots g_r]^T \in R^r$ and it is assumed that $D(s) = g_r s^{r-1} + g_{r-1} s^{r-2} + \cdots + g_1$ is a Hurwitz polynomial. Define the pseudo control

$$u_{ps} = \hat{\psi}(y, u), \tag{3}$$

where the invertible function $\hat{\psi}$ is the best available approximation of ψ. Therefore, from (3) $u = \hat{\psi}^{-1}(y, u_{ps})$. Define the modeling error of external dynamics as

$$\Delta(\mathbf{z}, \boldsymbol{\eta}, u) = \psi(\mathbf{z}, \boldsymbol{\eta}, u) - \hat{\psi}(y, u). \tag{4}$$

Substituting the $\psi(\mathbf{z}, \boldsymbol{\eta}, u)$ obtained from (4) into the nonlinear model (2) yields

$$\begin{cases} \dot{z}_i = z_{i+1} & 1 \le i \le r-1 \\ \dot{z}_r = \mathbf{m}^T \mathbf{z} + \mathbf{n}^T \boldsymbol{\eta} + b(\Delta(\mathbf{z}, \boldsymbol{\eta}, u) + u_{ps}) & \\ \dot{\eta}_j = \eta_{j+1} & 1 \le j \le n-r-1 \\ \dot{\eta}_{n-r} = \mathbf{f}^T \boldsymbol{\eta} + \mathbf{g}^T \mathbf{z} + \Delta_{\boldsymbol{\eta}}(\mathbf{z}, \boldsymbol{\eta}) & \end{cases} \tag{5}$$

3. Stabilization using output feedback and internal dynamics

Define the error signal as $e_1 = e := y_d - y$, $e_{i+1} = \dot{e}_i = y_d^{(i)} - z_{i+1}$ for $1 \le i \le r-1$ and the pseudo control signal as

$$u_{ps} = u_L - u_{ad} - u_R + b^{-1} y_d^{(r)} - b^{-1}\mathbf{m}^T \mathbf{y}_d \tag{6}$$

where $\mathbf{y}_d = [y_d \ \cdots \ y_d^{(r-1)}]^T$. Then, the system (5) can be described as the following two subsystems:

$$\Sigma_e : \begin{cases} \dot{e}_i = e_{i+1} & 1 \le i \le r-1 \\ \dot{e}_r = \mathbf{m}^T\mathbf{e} - b u_L - b[-u_{ad} + \underbrace{\Delta + \mu_\eta(\boldsymbol{\eta})}_{\Delta'(\mathbf{z},\boldsymbol{\eta},u)} - u_R] \end{cases} \tag{7}$$

$$\Sigma_\eta : \dot{\boldsymbol{\eta}} = \mathbf{F}\boldsymbol{\eta} + \mathbf{g}_1 y_d - \mathbf{g}_1 e + \mathbf{g}_1 \underbrace{\left(\Delta_\eta(\mathbf{z},\boldsymbol{\eta})/g_1 + \mu_z(\mathbf{z})\right)}_{\Delta'_\eta(\mathbf{z},\boldsymbol{\eta})} \tag{8}$$

where $\mu_\eta(\boldsymbol{\eta}) = \dfrac{\mathbf{n}^T}{b}\boldsymbol{\eta}$, $\mu_z(\mathbf{z}) = \dfrac{\mathbf{g}^T\mathbf{z}}{g_1} - z_1$, $\mathbf{F} = \begin{bmatrix} \mathbf{I}_{(n-r-1)\times(n-r)} \\ \mathbf{f}^T \end{bmatrix}$ and $\mathbf{g}_1 = [\mathbf{0}_{(n-r-1)\times 1} \ g_1]^T$.

The controller is designed in two phases. First, $y_d = y_d(\boldsymbol{\eta})$ is designed such that $\boldsymbol{\eta}$-subsystem (Σ_η) becomes input-to-state stable (ISS) with respect to the input *e*. Then, a combined adaptive output feedback control law that utilizes the available measurement $y(t)$, is used to obtain the system output tracking for the trajectory y_d, which is assumed to be *r*-times differentiable.

Assumption 2. The pair $(\mathbf{F}, \mathbf{g}_1)$ is stabilizable and the modeling error of the internal dynamics Δ'_η is bounded with a conic sector bound as

$$\left|\Delta'_\eta(\mathbf{z},\boldsymbol{\eta})\right| \le c_0 + c_1|z_1| + c_2\|\boldsymbol{\eta}\|, \tag{9}$$

where c_0 and c_2 are unknown constants and $0 \le c_1 < 1$ is a known positive constant.

3.1. Input-to-state stability of the $\boldsymbol{\eta}$-subsystem

Considering the internal subsystem in (8), $y_d(\boldsymbol{\eta})$ is introduced as

$$y_d(\boldsymbol{\eta}) := \mathbf{k}\boldsymbol{\eta} + \mathrm{v}(\boldsymbol{\eta}), \tag{10}$$

where $\mathrm{v}(\boldsymbol{\eta})$ is an auxiliary control and will be introduced later. Then, the closed-loop form of Σ_η can be written as

$$\dot{\boldsymbol{\eta}} = (\mathbf{F} + \mathbf{g}_1\mathbf{k})\boldsymbol{\eta} - \mathbf{g}_1 e + \mathbf{g}_1 \mathrm{v}(\boldsymbol{\eta}) + \mathbf{g}_1\Delta'_\eta(\mathbf{z},\boldsymbol{\eta}) \tag{11}$$

Assumption 2 ensures the existence of the gain vector $\mathbf{k}$ such that $\mathbf{F} + \mathbf{g}_1\mathbf{k}$ is Hurwitz, and guarantees the existence of a symmetric positive definite matrix $\mathbf{P}_1$, which satisfies

$$(\mathbf{F} + \mathbf{g}_1\mathbf{k})^T \mathbf{P}_1 + \mathbf{P}_1(\mathbf{F} + \mathbf{g}_1\mathbf{k}) = -\mathbf{Q}_1, \tag{12}$$

where $\mathbf{Q}_1$ is an arbitrary symmetric positive definite matrix. Using (10), the upper bound of the modeling error, introduced in (9), can be represented as

$$\left\|\Delta'_\eta(\mathbf{z},\boldsymbol{\eta})\right\| \le c_0 + \beta_1\|\boldsymbol{\eta}\| + c_1|\mathrm{v}| + c_1|e|, \tag{13}$$

where $\beta_1 = c_2 + c_1\|\mathbf{k}\|$.

Theorem 1. *Consider the control law* $\mathrm{v}(\boldsymbol{\eta})$ *as*

$$\mathrm{v}(\boldsymbol{\eta}) = \frac{-k_c}{1-c_1}\mathbf{g}_1^T\mathbf{w} \tag{14}$$

where $\mathbf{w}^T = \boldsymbol{\eta}^T\mathbf{P}_1$ *and* $k_c > 0$. *Then, the* η*-subsystem is ISS with respect to inputs e and* c_0.

Proof: Define the Lyapunov function

$$L_1 = \frac{1}{2}\boldsymbol{\eta}^T\mathbf{P}_1\boldsymbol{\eta}, \tag{15}$$

where matrix $\mathbf{P}_1$ is the unique positive-definite symmetric solution of (12). Using (11), (13) and (14), and adding and subtracting $3\lambda^2\|\mathbf{g}_1^T\mathbf{w}\|^2$ and completing of square terms, the time-derivative of L_1 becomes

$$\begin{aligned}\dot{L}_1 \le & -\frac{1}{2}q_{1m}\|\boldsymbol{\eta}\|^2 - (k_c - 3\lambda^2)\|\mathbf{g}_1^T\mathbf{w}\|^2 + (\beta_1/\lambda)^2\|\boldsymbol{\eta}\|^2 \\ & - \left(\lambda\|\mathbf{g}_1^T\mathbf{w}\| - \beta_1\lambda^{-1}\|\boldsymbol{\eta}\|\right)^2 - \left(\lambda\|\mathbf{g}_1^T\mathbf{w}\| - (c_1+1)\lambda^{-1}|e|\right)^2 \\ & + (c_1+1)^2\lambda^{-2}|e|^2 - \left(\lambda\|\mathbf{g}_1^T\mathbf{w}\| - \lambda^{-1}c_0\right)^2 + \left(\lambda^{-1}c_0\right)^2 \end{aligned} \tag{16}$$

where q_{1m} is the smallest eigenvalue of $\mathbf{Q}_1$ and λ is a sufficiently large positive constant. Now select the controller gain k_c such that $k_c > 3\lambda^2$. Then, by removing the negative terms from (16) it gives

$$\dot{L}_1 \le -\left(q_{1m}/2 - (\beta_1/\lambda)^2\right)\|\boldsymbol{\eta}\|^2 + (c_1+1)^2\lambda^{-2}|e|^2 + \lambda^{-2}c_0^2 \tag{17}$$

in which λ is selected sufficiently large such that it satisfies $q_{1m} > 2(\beta_1/\lambda)^2$. Hence, it can be concluded from (17) that Σ_η is ISS with respect to *e* and c_0 (Jiang *et al.*, 1996). Moreover, Σ_η may also be considered as input-to-state practical stable (ISPS) with respect to input *e*. □

3.2. Asymptotic stability of output tracking error

Using (7) and defining $\mathbf{E} := [e \ \dot{e} \ \cdots \ e^{(r)}]$, the error dynamic can be expressed as

$$\begin{cases} \dot{\mathbf{E}} = \mathbf{A}\mathbf{E} + \mathbf{b}u_L + \mathbf{b}\left((\Delta' - u_{ad}) - u_R\right) \\ e = \mathbf{c}\mathbf{E} \end{cases}, \tag{18}$$

where $\mathbf{A} = \begin{bmatrix} 0 & \mathbf{I}_{(r-1)\times(r-1)} \\ 0 & \mathbf{m}^T \end{bmatrix}$, $\mathbf{b} = [0 \cdots 0 \ -b]^T$ and $\mathbf{c} = [1 \ 0 \cdots 0]$; u_L is the output of a stabilizing linear dynamic compensator, which is designed to assure the boundedness of the closed-loop states in the presence of modeling errors.

3.2.1 Neural network-based adaptive control design

The control term u_{ad} in the control law (6), is included to approximate the modeling error $\Delta'(\mathbf{z}, \boldsymbol{\eta}, u)$. Hence, there exists a fixed-point problem as

$$u_{ad}(t) = \Delta'\left(\mathbf{z}, \boldsymbol{\eta}, \hat{\psi}^{-1}(y_r - u_{ad}(t) + u_L - u_R + b^{-1}y_d^{(r)} - b^{-1}\mathbf{m}^T\mathbf{y}_d)\right)$$

According to the contractive mapping theorem (Hunter and Nachtergaele, 2001), if the map $u_{ad} \to \Delta'$ is contractive over the entire input domain, then the above fixed point problem has a unique solution for u_{ad}. This map is contractive if it satisfies the following condition:

$$\left|\partial\Delta'/\partial u_{ad}\right| < 1. \tag{19}$$

Substituting (3), (4) and (6) into (19) yields

$$\begin{aligned}\left|\frac{\partial\Delta'}{\partial u_{ad}}\right| &= \left|\partial\left(\psi - \hat{\psi} + \mu_\eta(\boldsymbol{\eta})\right)/\partial u \times \partial u/\partial u_{ps} \times \partial u_{ps}/\partial u_{ad}\right| \\ &= \left|-\partial(\psi - \hat{\psi})/\partial u \times \partial u/\partial\hat{\psi}\right| < 1.\end{aligned}$$

This condition holds if and only if $(\partial\psi/\partial u)/(\partial\hat{\psi}/\partial u) < 2$ and $(\partial\psi/\partial u)/(\partial\hat{\psi}/\partial u) > 0$ which are equivalent to the following conditions:

$$|\partial\hat{\psi}/\partial u| > 0.5|\partial\psi/\partial u|,\ \operatorname{sgn}(\partial\psi/\partial u) = \operatorname{sgn}(\partial\hat{\psi}/\partial u) \quad (20)$$

If the conditions (20) are satisfied then based on the input-output data, the modeling error $\Delta'(\mathbf{z},\boldsymbol{\eta},u)$ can be approximated *with a bounded error* ε, *by a single hidden layer MultiLayer Perceptron (MLP) as*

$$\Delta' = \mathbf{w}^{*T}\boldsymbol{\sigma}(\mathbf{V}^{*T}\boldsymbol{\zeta}) + \varepsilon, \qquad \text{with } |\varepsilon| \le \varepsilon_M, \quad (21)$$

where ε_M is an appropriate bound on ε which is determined based on the network architecture, $\mathbf{w}^* \in R^m$ *is the vector containing synaptic weights of the output layer,* $\mathbf{V}^* \in R^{N\times m}$ *is the matrix containing the weights of the hidden layer,* $\boldsymbol{\sigma} = [\sigma_1 \cdots \sigma_m]^T$ *is the vector function containing the nonlinear function* $\tanh(\alpha x)$ *with* $\alpha>0$ *as the activation function of the hidden layer, and* $\boldsymbol{\zeta} = [1 \quad \bar{\mathbf{y}} \quad \bar{\mathbf{u}}_\alpha \quad \bar{\mathbf{u}}_{ad}]^T \in R^N$ *is the input vector where*

$$\begin{aligned}\bar{\mathbf{y}} &= [y(t) \ \cdots \ y(t - T_d(n_1 - 1))]\\ \bar{\mathbf{u}}_\alpha &= [u_\alpha(t) \ \cdots \ u_\alpha(t - T_d(n_1 - r - 1))]\\ \bar{\mathbf{u}}_{ad} &= [u_{ad}(t - T_d) \ \cdots \ u_{ad}(t - T_d(n_1 - r - 1))]\end{aligned}$$

and $u_\alpha = u_{ps} + u_{ad} = u_L - u_R + b^{-1}y_d^{(r)} - b^{-1}\mathbf{m}^T\mathbf{y}_d$ (Hoseini *et al*., 2009). Moreover, it is proved that if a non-linear system satisfies the conditions (20), then it is unnecessary to use $u_{ad}(t)$ as an input signal to the NN. Hence, the fixed point problem in the algebraic loop, which is created from feeding the output of NN back to its input, is eliminated.

Since Δ' can be modeled using a MLP, the adaptive control term is proposed as

$$u_{ad} := \mathbf{w}^T\boldsymbol{\sigma}(\mathbf{V}^T\boldsymbol{\zeta}), \quad (22)$$

where $\mathbf{w}$ and $\mathbf{V}$ are the actual weights of their corresponding ideal weights $\mathbf{w}^*$ and $\mathbf{V}^*$ which are defined as

$$(\mathbf{w}^*,\mathbf{V}^*) := \arg \min_{(\mathbf{w},\mathbf{V})\in\Omega_\mathbf{w}} \left\{ \sup_{\zeta\in\Omega_\zeta} \left|\mathbf{w}^T\boldsymbol{\sigma}(\mathbf{V}^T\boldsymbol{\zeta}) - \Delta'(.)\right| \right\}, \quad (23)$$

where $\Omega_\mathbf{w} = \{(\mathbf{w},\mathbf{V}) \mid \|\mathbf{w}\| \le M_\mathbf{w}, \|\mathbf{V}\|_F \le M_\mathbf{V}\}$, in which $M_\mathbf{w}$ and $M_\mathbf{V}$ are positive numbers and $\|\cdot\|_F$ denotes the Frobenius norm.

In practice, the weights of the NN may be different from the ideal ones. The approximation error, which arises from the difference between (21) and (22), satisfies the following equality:

$$\Delta'(\mathbf{z},\boldsymbol{\eta},u) - u_{ad} = \tilde{\mathbf{w}}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_\mathbf{V}\mathbf{V}^T\boldsymbol{\zeta}) + \mathbf{w}^T\boldsymbol{\sigma}_\mathbf{V}\tilde{\mathbf{V}}^T\boldsymbol{\zeta} + \delta(t), \quad (24)$$

where

$$\left.\begin{aligned}&|\delta(t)| \le (\varepsilon_M + 2\sqrt{m}M_\mathbf{w}) + \alpha M_\mathbf{w}\|\tilde{\mathbf{V}}\|_F\|\boldsymbol{\zeta}\| + \alpha M_\mathbf{V}\|\tilde{\mathbf{w}}\|\|\boldsymbol{\zeta}\|\\ &\tilde{\mathbf{w}} = \mathbf{w}^* - \mathbf{w}, \quad \tilde{\mathbf{V}} = \mathbf{V}^* - \mathbf{V}\end{aligned}\right\} \quad (25)$$

and $\boldsymbol{\sigma}_\mathbf{V} = \operatorname{diag}[\partial\sigma_1(v_1)/\partial v_1 \ \cdots \ \partial\sigma_m(v_m)/\partial v_m]$ is the derivative of $\boldsymbol{\sigma}$ with respect to the input signals v_i $(i = 1,\ldots,m)$, in which $[v_1 \cdots v_m]^T = \mathbf{V}^T\boldsymbol{\zeta}$; and m denotes the number of neurons in the hidden layer (Hoseini and Farrokhi, 2009).

3.2.2. Construction of SPR error dynamics

In this section, the strictly positive realness (SPR) property of the closed-loop error dynamic is studied (Calise *et al*., 2001; Astrom and Wittenmark, 1994). Assume that u_L is constructed using the following dynamic controller:

$$\begin{cases}\dot{\mathbf{x}}_\mathbf{c} = \mathbf{A}_c\mathbf{x}_\mathbf{c} + \mathbf{b}_c e\\ u_L = \mathbf{c}_\mathbf{c}\mathbf{x}_\mathbf{c} + d_c e\end{cases}$$

Applying this linear controller to the system (18) implies the following closed-loop system:

$$\begin{cases}\begin{pmatrix}\dot{\mathbf{E}}\\ \dot{\mathbf{x}}_\mathbf{c}\end{pmatrix} = \underbrace{\begin{pmatrix}\mathbf{A} - \mathbf{b}d_c\mathbf{c} & -\mathbf{b}\mathbf{c}_\mathbf{c}\\ \mathbf{b}_c\mathbf{c} & \mathbf{A}_c\end{pmatrix}}_{\mathbf{A}_0}\begin{pmatrix}\mathbf{E}\\ \mathbf{x}_\mathbf{c}\end{pmatrix} + \underbrace{\begin{pmatrix}\mathbf{b}\\ \mathbf{0}\end{pmatrix}}_{\mathbf{b}_0}((\Delta' - u_{ad}) - u_R)\\ e = \underbrace{(\mathbf{c} \quad \mathbf{0})}_{\mathbf{c}_0}\mathbf{E}\end{cases} \quad (26)$$

The controller u_L is designed such that the following closed-loop transfer function is stable and minimum phase:

$$G(s) = N_0/D_0 = \mathbf{c}_0(s\mathbf{I} - \mathbf{A}_0)^{-1}\mathbf{b}_0. \quad (27)$$

If the controller is proper, then the relative degree of $G(s)$ is r. Now define the filtered error signal e_f as

$$e_f = G_{ad}(s)e = (N_{ad}/D_{ad})e, \quad (28)$$

where $G_{ad}(s)$ is selected such that $G_{ad}(0) \ne 0$ and $\deg(D_{ad}) = \deg(N_{ad})$. The error signal (28) is used to adapt the NN weights. Using (26), (27) and (28), the closed-loop transfer function of the system can be written as

$$e_f(s) = \frac{N_0N_{ad}}{D_0D_{ad}}((\Delta'(\mathbf{x},u) - u_{ad}) - u_R)(s) \quad (29)$$

As it is shown in the next section, for realization of the adaptation rule of the NN weights (i.e. using only the available data), the transfer function N_0N_{ad}/D_0D_{ad} must be strictly positive real (SPR). When the relative degree of N_0N_{ad}/D_0D_{ad} is equal to one (i.e. r =1), this transfer function can be made SPR by a proper selection of $N_{ad}(s)$. However, when $r > 1$, it cannot be made SPR (Narendra and Annaswamy, 1989). To achieve a SPR transfer function for $r > 1$, a stable low pass filter $T(s)$ is introduced such that $r - 1 \le \deg(T(s)) \le r$. Thus, the new filtered error dynamics is

$$e_f(s) = G_T(s)T^{-1}(s)((\Delta'(\mathbf{x},u) - u_{ad}) - u_R)(s), \quad (30)$$

where $G_T(s) = N_0N_{ad}T(s)/(D_0D_{ad})$.

Since $G_T(s)$ is a stable transfer function, its zeros (roots of N_{ad} and $T(s)$) can be easily placed to make it SPR. Moreover, it is important to note that $T(s)$ is designed such that the step response of $T^{-1}(s)$ has no overshoot and $|T^{-1}(s)| \le 1$. Hence, the state space model of the closed-loop error dynamics given in (30) can be represented as

$$\begin{cases}\dot{\boldsymbol{\xi}} = \mathbf{A}_{cl}\boldsymbol{\xi} + \mathbf{b}_{cl}\left[T^{-1}(s)((\Delta'(\mathbf{z},\boldsymbol{\eta},u) - u_{ad}) - u_R)\right]\\ e_f = \mathbf{c}_{cl}^T\boldsymbol{\xi}\end{cases} \quad (31)$$

According to the Kalman-Yakubovich lemma, the strictly positive realness of $G_T(s)$ assures the existence of a symmetric positive definite matrix $\mathbf{P}_2$ which satisfies

$$\left.\begin{array}{l}\mathbf{A}_{cl}^T\mathbf{P}_2+\mathbf{P}_2\mathbf{A}_{cl}=-\mathbf{Q}_2\\ \mathbf{P}_2\mathbf{b}_{cl}=\mathbf{c}_{cl}\end{array}\right\} \quad (32)$$

where $\mathbf{Q}_2=\mathbf{Q}_2^T>0$.

3.3. Stability analysis

In this section, first, the asymptotic stability of the tracking error is proved and then the stability of the overall system using the small gain theorem is presented.

Substituting (24) into (31), the closed-loop error dynamic can be represented as

$$\dot{\boldsymbol{\xi}}=\mathbf{A}_{cl}\boldsymbol{\xi}+\mathbf{b}_{cl}\left(T^{-1}\tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{\mathrm{V}}\mathbf{V}^T\boldsymbol{\zeta}\right)+T^{-1}\mathbf{w}^T\boldsymbol{\sigma}_{\mathrm{V}}\tilde{\mathbf{V}}^T\boldsymbol{\zeta}+\delta_f(t)-u_{Rf}\right)$$

where $\delta_f(t)=T^{-1}(s)\delta(t)$ and $u_{Rf}(t)=T^{-1}(s)u_R(t)$.

Now define $\boldsymbol{\psi}:=\boldsymbol{\sigma}-\boldsymbol{\sigma}_{\mathrm{V}}\mathbf{V}^T\boldsymbol{\zeta}$ and $\boldsymbol{\Psi}:=\boldsymbol{\zeta}\,\mathbf{w}^T\boldsymbol{\sigma}_{\mathrm{V}}$, and consider the discontinues control signal

$$u_R=\chi\,\varphi\,\operatorname{sgn}\left(e_f\right), \quad (33)$$

where φ is an adaptive gain and χ is a function of the NN weights and input vector $\boldsymbol{\zeta}$. Using the fact $\mathbf{w}^T\boldsymbol{\sigma}_{\mathrm{V}}\tilde{\mathbf{V}}^T\boldsymbol{\zeta}=\operatorname{tr}(\tilde{\mathbf{V}}^T\boldsymbol{\zeta}\mathbf{w}^T\boldsymbol{\sigma}_{\mathrm{V}})$, the closed-loop error dynamics can be written as

$$\begin{aligned}\dot{\boldsymbol{\xi}}=\mathbf{A}_{cl}\boldsymbol{\xi}+\mathbf{b}_{cl}\Big[&T^{-1}(s)\tilde{\mathbf{w}}^T\boldsymbol{\psi}+\operatorname{tr}\left(T^{-1}(s)\tilde{\mathbf{V}}^T\boldsymbol{\Psi}\right)+\delta_f(t)\\ &-T^{-1}(s)\chi\varphi\operatorname{sgn}(e_f)\Big]\end{aligned} \quad (34)$$

The NN weights $\tilde{\mathbf{V}}$ and $\tilde{\mathbf{w}}$, and $\varphi\operatorname{sgn}(e_f)$ are time-varying signals. Hence, the transfer function operator in (34) is not commutable. Now consider the following error terms:

$$\begin{aligned}\delta_w&:=T^{-1}(s)\tilde{\mathbf{w}}^T\boldsymbol{\psi}-\tilde{\mathbf{w}}^TT^{-1}(s)\boldsymbol{\psi}\\ \delta_V&:=\operatorname{tr}\left(T^{-1}(s)\tilde{\mathbf{V}}^T\boldsymbol{\Psi}\right)-\operatorname{tr}\left(\tilde{\mathbf{V}}^TT^{-1}(s)\boldsymbol{\Psi}\right)\\ \delta_\varphi&:=T^{-1}(s)\chi\varphi\operatorname{sgn}(e_f)-\varphi\operatorname{sgn}(e_f)T^{-1}(s)\chi\end{aligned} \quad (35)$$

for which the following bounds can be assumed

$$\left|\delta_{\mathrm{w}}\right|\le c_3,\quad \left|\delta_{\mathrm{V}}\right|\le c_4,\quad \left|\delta_\varphi\right|\le c_5 \quad (36)$$

where c_3, c_4 and c_5 are positive numbers. Substituting (35) into (34) yields

$$\begin{aligned}\dot{\boldsymbol{\xi}}=\mathbf{A}_{cl}\boldsymbol{\xi}+\mathbf{b}_{cl}\Big[&\tilde{\mathbf{w}}^T\boldsymbol{\psi}_f+\operatorname{tr}(\tilde{\mathbf{V}}^T\boldsymbol{\Psi}_f)-\varphi\operatorname{sgn}(e_f)T^{-1}\chi\\ &+\delta_{\mathrm{w}}+\delta_{\mathrm{V}}+\delta_f-\delta_\varphi\Big],\end{aligned} \quad (37)$$

where $\boldsymbol{\psi}_f=T^{-1}(s)\,\boldsymbol{\psi}$ and $\boldsymbol{\Psi}_f=T^{-1}(s)\,\boldsymbol{\Psi}$

In order to show that the error dynamics (37) is asymptotically stable, the following lemma is needed.

Lemma 1. *The following inequality holds:*

$$\left|\delta_f+\delta_{\mathrm{w}}+\delta_{\mathrm{V}}-\delta_\varphi\right|\le\left(\varphi/\mu\right)\left|T^{-1}(s)\right|\chi, \quad (38)$$

where

$$\varphi_1=\max\left\{\left(\varepsilon_M+2\sqrt{m}M_{\mathrm{w}}+c\right),2\alpha M_{\mathrm{V}}M_{\mathrm{w}},\alpha M_{\mathrm{V}},\alpha M_w\right\},$$

with $c=\sum_{i=1}^3c_i$, $0<\mu<1$ and $\chi=4\left(1+\|\boldsymbol{\zeta}\|\left(1+\|\mathbf{w}\|+\|\mathbf{V}\|_{\mathrm{F}}\right)\right)$.

Proof: Using (25) and (36), and considering the condition $\left|T^{-1}(s)\right|\le1$, it is obtained

$$\begin{aligned}\left|\delta_f+\delta_{\mathrm{w}}+\delta_{\mathrm{V}}-\delta_\varphi\right|&\le\left(\varepsilon_M+2\sqrt{m}M_{\mathrm{w}}\right)+\alpha M_{\mathrm{V}}\left\|\mathbf{w}^*-\mathbf{w}\right\|\|\boldsymbol{\zeta}\|\\ &\quad+\alpha M_{\mathrm{w}}\left\|\mathbf{V}^*-\mathbf{V}\right\|_{\mathrm{F}}\|\boldsymbol{\zeta}\|+c_3+c_4+c_5\\ &\le\varphi_1\left(1+\|\boldsymbol{\zeta}\|+\|\mathbf{w}\|\|\boldsymbol{\zeta}\|+\|\mathbf{V}\|_{\mathrm{F}}\|\boldsymbol{\zeta}\|\right)\\ &=\varphi_1\chi.\end{aligned}$$

Since the step response of $T^{-1}(s)$ has no overshoot and $\left|T^{-1}(s)\right|\le1$ and χ is a positive signal, then by selecting suitable initials for filter states, there exists a $0<\mu<1$ such that $\mu\chi\le T^{-1}(s)\chi$. Consequently,

$$\left|\delta_f+\delta_{\mathrm{w}}+\delta_{\mathrm{V}}-\delta_\varphi\right|\le(\varphi/\mu)T^{-1}(s)\chi.$$

Remark 1. Note that a suitable linear controller u_L could stabilize the system (Hoseini *et al.*, 2009). Therefore, the closed-loop system is stable even via only an appropriate linear controller. Hence, before the adaptive control parts (NN and u_R) are included to the control law, it may be assumed that the state variables are bounded. These control parts are included to obtain a lower error bound and to ensure the closed-loop system is robust against changes in the system parameters. Hence, even before considering the adaptive control parts, it may be assumed that $\Delta'(\cdot)$ is bounded. Therefore, the ideal weights; (V^*,w^*,φ^*) are bounded, and initializing (V,w,φ) to small values, implies the boundedness of $(\tilde{V},\tilde{w},\tilde{\varphi})$. Therefore, the bounds defined in (36) are always valid. □

Theorem 2. *Considering the discontinuous control (33) and selecting the adaptation laws for the NN weights and the gain of the robustifying term φ as*

$$\dot{\mathbf{w}}=\gamma_{\mathrm{w}}e_f\,\boldsymbol{\psi}_f,\quad \dot{\mathbf{V}}=\gamma_{\mathrm{V}}e_f\,\boldsymbol{\Psi}_f,\quad \dot{\varphi}=\gamma_\varphi\left|e_f\right|\left(T^{-1}\chi\right), \quad (39)$$

Then the closed-loop tracking error (37) is asymptotically stable and the weights of the NN remain bounded.

Proof: Consider the Lyapunov function

$$L_2:=0.5\boldsymbol{\xi}^T\mathbf{P}_2\boldsymbol{\xi}+0.5\gamma_w^{-1}\|\tilde{\mathbf{w}}\|^2+0.5\gamma_V^{-1}\|\tilde{\mathbf{V}}\|_{\mathrm{F}}^2+0.5\gamma_\varphi^{-1}|\tilde{\varphi}|^2 \quad (40)$$

where $\mathbf{P}_2$ is the unique symmetric positive-definite solution of (32) and $\tilde{\varphi}=\varphi^*-\varphi$ with φ^* as an estimate of φ. Moreover, assume that $\mathbf{w}^*$ and $\mathbf{V}^*$ are the ideal constant weights defined in (23); then, from (25) $\dot{\mathbf{V}}=-\dot{\tilde{\mathbf{V}}}$ and $\dot{\mathbf{w}}=-\dot{\tilde{\mathbf{w}}}$. Using (37), the time-derivative of L becomes

$$\begin{aligned}\dot{L}_2\le-0.5q_{2m}\|\boldsymbol{\xi}\|^2+\boldsymbol{\xi}^T\mathbf{P}_2\mathbf{b}_{cl}\Big[&\tilde{\mathbf{w}}^T\boldsymbol{\psi}_f-\varphi\operatorname{sgn}(e_f)T^{-1}\chi+\delta_{\mathrm{w}}+\delta_{\mathrm{V}}\\ &+\delta_f+\operatorname{tr}\left(\tilde{\mathbf{V}}^T\boldsymbol{\Psi}_f\right)-\delta_\varphi\Big]-\gamma_w^{-1}\tilde{\mathbf{w}}^T\dot{\mathbf{w}}-\gamma_V^{-1}\operatorname{tr}\left(\tilde{\mathbf{V}}^T\dot{\mathbf{V}}\right)-\gamma_\varphi^{-1}\tilde{\varphi}\dot{\varphi},\end{aligned}$$

where q_{2m} is the smallest eigenvalue of $\mathbf{Q}_2$. Substituting $e_f=\boldsymbol{\xi}^T\mathbf{P}\mathbf{b}_{cl}$, obtained from (31) and (32), and using Lemma 1, yield

$$\begin{aligned}\dot{L}_2\le&\,\tilde{\mathbf{w}}^T\left(e_f\boldsymbol{\psi}_f-\gamma_w^{-1}\dot{\mathbf{w}}\right)+\operatorname{tr}\left(\tilde{\mathbf{V}}^T\left(e_f\boldsymbol{\Psi}_f-\gamma_V^{-1}\dot{\mathbf{V}}\right)\right)-0.5q_{2m}\|\boldsymbol{\xi}\|^2\\ &-e_f\varphi\operatorname{sgn}(e_f)T^{-1}(s)\chi+\left|e_f\right|\varphi^*T^{-1}(s)\chi-\gamma_\varphi^{-1}\tilde{\varphi}\dot{\varphi}.\end{aligned}$$

Using the adaptation laws (38), it is followed that

$$\begin{aligned}\dot{L}_2&\le-0.5q_{2m}\|\boldsymbol{\xi}\|^2+\left|e_f\right|\varphi^*T^{-1}\chi-\left|e_f\right|\varphi\,T^{-1}\chi-\gamma_\varphi^{-1}\tilde{\varphi}\dot{\varphi}\\ &=-0.5q_{2m}\|\boldsymbol{\xi}\|^2+\left(\left|e_f\right|T^{-1}\chi-\gamma_\varphi^{-1}\dot{\varphi}\right)\tilde{\varphi}\\ &=-0.5q_{2m}\|\boldsymbol{\xi}\|^2\le-0.5\|\mathbf{c}_{cl}\|^{-2}q_{2m}\left|e_f\right|^2\end{aligned} \quad (41)$$

which shows that the closed-loop tracking error is asymptotically stable. Moreover, since L_2 is a positive function and $\dot{L}_2 \le 0$, one can conclude that $\|\xi\|$, $\|\tilde{\mathbf{V}}\|$, $\|\tilde{\mathbf{w}}\|$ and $|\tilde{\varphi}|$ are bounded. In addition, (23) shows that $\mathbf{V}^*$ and $\mathbf{w}^*$ are also bounded. Therefore, according to (25), $\mathbf{V}$ and $\mathbf{w}$ remain bounded. Moreover, integrating from (41) gives

$$\int_0^\infty \|\xi(t)\|^2 dt \le 2q_{2m}^{-1}\left(L_2(t)|_{t=0} - L_2(t)|_{t=\infty}\right). \qquad (42)$$

The right-hand side of (42) is bounded, therefore, according to Barbalet's lemma $\lim_{t\to\infty}\|\xi\|^2 = 0$. Since $e_f = \mathbf{c}_{cl}^T\xi$, then $\lim_{t\to\infty} e_f(t) = 0$. Now applying the final value theorem and using (28) yield

$$\lim_{s\to 0} s\, e_f(s) = \lim_{s\to 0} s\, G_{ad}(s) e(s) = 0. \qquad (43)$$

Since $G_{ad}(0) \ne 0$, one can conclude that $\lim_{s\to 0} s\, e(s) = 0$ and hence, $\lim_{t\to\infty} e(t) = 0$. □

Now based on the results in Karagiannis *et al.* (2005), it can be concluded that the interconnected systems (7) and (8) are ISS with respect to c_0. Therefore, the error trajectories are ultimately bounded. Moreover, since in the system (1), $v(\mathbf{0},\mathbf{0}) = 0$, the bound defined on $\Delta'_\eta(\mathbf{z},\boldsymbol{\eta})$ may be satisfied even with $c_0 = 0$. In this case, the asymptotic stability of the overall system can be achieved.

4. Observer-based output feedback stabilization

In this section, the assumption of the availability of the internal dynamics is relaxed by designing an error observer. Moreover, a suitable reference signal is designed to compensate the modeling error of internal dynamics or the unmatched uncertainty.

Consider the system dynamics given in (5) and define the pseudo control as in (6). Then, the error dynamics (7)-(8) can now be rewritten as

$$\begin{cases} \dot{e}_i = e_{i+1} & 1 \le i \le r-1 \\ \dot{e}_r = \mathbf{m}^T\mathbf{e} - \mathbf{n}^T\boldsymbol{\eta} - b u_L + b\left(u_{ad} - \Delta + u_R\right) \\ \dot{\eta}_j = \dot{\eta}_{j+1} & 1 \le j \le n-r-1 \\ \dot{\eta}_{n-r} = \mathbf{f}^T\boldsymbol{\eta} - \mathbf{g}^T\mathbf{e} + y^* + \Delta_\eta(\mathbf{z},\boldsymbol{\eta}) \end{cases}, \qquad (44)$$

where $y^* := \mathbf{g}^T\,\mathbf{y}_d = g_1 y_d + g_2 \dot{y}_d + \cdots + g_r y_d^{(r-1)}$.

Assumption 3. The signal y_d and its derivatives are bounded. Moreover, the unmatched uncertainty $\Delta_\eta(\mathbf{z},\boldsymbol{\eta})$ is bounded with a constant and conic sector bound. That is

$$\begin{aligned} &\left|\Delta_\eta(\mathbf{z},\boldsymbol{\eta})\right| \le c_0^* + c_1\|\mathbf{z}\| + c_2^*\|\boldsymbol{\eta}\| \\ &\sum_{i=0}^{r}\left|y_d^{(i)}\right| \le c_3^*, \end{aligned} \qquad (45)$$

where c_0^*, c_2^* and c_3^* are unknown constants and c_1 is a known positive constant such that $c_1 < 1$.

Note that here the bound on $\Delta_\eta(\mathbf{z},\boldsymbol{\eta})$ as defined in (45) is less restrictive than the one given in (9) in Assumption 2.

Let $\xi_e := [\mathbf{e}^T, \boldsymbol{\eta}^T]^T$. Then the error dynamics of the nonlinear system (44) can be represented as

$$\begin{cases} \dot{\xi}_e = \mathbf{A}\,\xi_e + \mathbf{b}\; u_L + \mathbf{b}\left(\Delta - u_{ad} - u_R\right) + \mathbf{q}\left(y^* + \Delta_\eta\right) \\ e = \mathbf{c}\,\xi_e \end{cases} \qquad (46)$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{M} & -\mathbf{N} \\ -\mathbf{G} & \mathbf{F}_a \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} \mathbf{0}_{1\times(r-1)} & -b & \mathbf{0}_{1\times(n-r)} \end{bmatrix}^T,\ \mathbf{q} = \begin{bmatrix} \mathbf{0}_{n-1} & 1 \end{bmatrix}^T$$

$$\mathbf{c} = \begin{bmatrix} 1 & \mathbf{0}_{n-1} \end{bmatrix},\ \mathbf{M} = \begin{bmatrix} \mathbf{0}_{(r-1)\times 1} & \mathbf{I}_{(r-1)\times(r-1)} \\ \hline \multicolumn{2}{c}{\mathbf{m}^T} \end{bmatrix},\ \mathbf{N} = \begin{bmatrix} \mathbf{0}_{(r-1)\times(n-r)} \\ \mathbf{n}^T \end{bmatrix},$$

$$\mathbf{F}_a = \begin{bmatrix} \mathbf{0}_{(n-r-1)\times 1} & \mathbf{I}_{(n-r-1)\times(n-r-1)} \\ \hline \multicolumn{2}{c}{\mathbf{f}^T} \end{bmatrix},\ \mathbf{G} = \begin{bmatrix} \mathbf{0}_{(n-r-1)\times r} \\ \mathbf{g}^T \end{bmatrix}$$

Since the system is non-minimum phase, the matrix $\mathbf{A}$ has at least one eigenvalue with positive real part. Therefore, the linear control u_L is first designed to stabilize the linearized system.

4.1. Linear control design

Since $(\mathbf{A},\mathbf{b})$ is a controllable pair, the following algebraic Riccati equation:

$$\mathbf{P}_3\mathbf{A} + \mathbf{A}^T\mathbf{P}_3 + \mathbf{Q}_3 - 2\mathbf{P}_3\mathbf{b}\mathbf{b}^T\mathbf{P}_3 = 0, \qquad (47)$$

where $\mathbf{Q}_3$ is a symmetric positive-definite matrix, has a unique symmetric positive-definite solution $\mathbf{P}_3$. The optimal linear control is

$$u_L = -\rho_1 = -\mathbf{k}_c\,\hat{\xi}_e, \qquad (48)$$

where $\hat{\xi}$ denotes estimation of ξ_e and the vector gain $\mathbf{k}_c$ is

$$\mathbf{k}_c^T = \mathbf{P}_3\mathbf{b}. \qquad (49)$$

Substituting (49) into (47) gives

$$\left(\mathbf{A} - \mathbf{b}\mathbf{k}_c\right)^T\mathbf{P}_3 + \mathbf{P}_3\left(\mathbf{A} - \mathbf{b}\mathbf{k}_c\right) + \mathbf{Q}_3 = 0. \qquad (50)$$

Hence, $\mathbf{A} - \mathbf{b}\mathbf{k}_c$ is a stable matrix and u_L stabilizes the system in the presence of uncertainties (Hoseini *et al.*, 2009).

4.2. Adaptive control term

The neuro-adaptive law is used to cancel out the matched uncertainty Δ. The adaptation rules for the weights of NN in this case are defined as follows:

$$\left.\begin{aligned} \dot{\mathbf{w}} &= \gamma_{\mathrm{w}}\left(\rho_1\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathrm{V}}\mathbf{V}^T\boldsymbol{\zeta}\right) - k_w\mathbf{w}\right) \\ \dot{\mathbf{V}} &= \gamma_{\mathrm{V}}\left(\rho_1\boldsymbol{\zeta}\mathbf{w}^T\boldsymbol{\sigma}_{\mathrm{V}} - k_V\mathbf{V}\right) \end{aligned}\right\}, \qquad (51)$$

where ρ_1 is the same as in (48), γ_{w} and γ_{V} are learning coefficients, and k_{w} and k_{V} are σ-modification gains.

Remark 2. As it is shown in Section 4.5, the stability analysis relies on an extension of the Lyapunov theory. The derivate of this Lyapunov function is negative outside a compact set. In this case, to avoid any persistent excitation condition on the NN inputs and to guarantee the boundedness of $\tilde{\mathbf{w}}$ and $\tilde{\mathbf{V}}$, the σ-modification terms are considered in the adaptation rules (Ioannou and Kokotovic, 1983; Lewis *et al.*, 1996; Yesildirek and Lewis, 1995).

Using Lemmas 1 and the discussion in Section 3.2.1, the approximation error of the NN can be bounded as $|\delta| \le \varphi^*\chi$, where χ is the same as defined in (38) and

$$\varphi^* = \max\left\{\varepsilon_{1M} + 2\sqrt{m}M_{\mathrm{w}},\, 2\alpha M_{\mathrm{V}} M_{\mathrm{w}},\, \alpha M_{\mathrm{V}},\, \alpha M_w\right\}.$$

To compensate the NN approximation error, the following adaptive robustifying control term is added to the control law

$$u_R = \chi\, \varphi\, sign(\rho_1), \tag{52}$$

with the following adaptation rule

$$\dot{\varphi} = \gamma_\varphi\, \chi\, |\rho_1|, \tag{53}$$

where γ_φ is the learning coefficient. Using (21), (22), (23) and $\|\boldsymbol{\sigma}\| \le \sqrt{m}$, the following conservative upper bound of the approximation error is obtained

$$\begin{aligned} |\Delta(\mathbf{z},\boldsymbol{\eta},u) - u_{ad}| &= \left|\mathbf{w}^{*T}\boldsymbol{\sigma}(\mathbf{V}^{*T}\zeta) + \varepsilon_1 - \mathbf{w}^T\boldsymbol{\sigma}(\mathbf{V}^T\zeta)\right| \\ &\le \left|\mathbf{w}^{*T}\boldsymbol{\sigma}(\mathbf{V}^{*T}\zeta)\right| + \left|\mathbf{w}^T\boldsymbol{\sigma}(\mathbf{V}^T\zeta)\right| + |\varepsilon_1| \\ &\le \|\mathbf{w}^*\|\|\boldsymbol{\sigma}\| + \|\mathbf{w}\|\|\boldsymbol{\sigma}\| + |\varepsilon_1| \le 2\sqrt{m}\, M_W + \varepsilon_M \end{aligned} \tag{54}$$

4.3. Observer design

For realization of weight adaptation laws, given in (51) and (53), (i.e. dependency only on the measurable system output), the following linear state estimator is proposed

$$\dot{\hat{\boldsymbol{\xi}}}_e = \mathbf{A}\hat{\boldsymbol{\xi}}_e + \mathbf{b}\, u_L + \mathbf{k}_o\left(e - \mathbf{c}\hat{\boldsymbol{\xi}}_e\right), \tag{55}$$

where $\mathbf{b}$ and $\mathbf{c}$ are the same as in (46) and the observer gain $\mathbf{k}_o = [k_1 \ \cdots \ k_n]^T$ is selected such that $\mathbf{A}-\mathbf{k}_o\mathbf{c}$ is stable. Moreover, the stability of $\mathbf{A}-\mathbf{k}_o\mathbf{c}$ assures the existence of the symmetric positive definite solution $\mathbf{P}_4$ of the following algebraic Riccati equation:

$$\mathbf{P}_4(\mathbf{A}-\mathbf{k}_o\mathbf{c}) + (\mathbf{A}-\mathbf{k}_o\mathbf{c})^T\mathbf{P}_4 = -\mathbf{Q}_4 - \mathbf{c}^T\mathbf{k}_o^T\mathbf{P}_3\mathbf{Q}_3^{-1}\mathbf{P}_3\mathbf{k}_o\mathbf{c} \tag{56}$$

where $\mathbf{Q}_4$ is a symmetric positive definite matrix. This observer is incorporated to the nonlinear system (5).

Define the state estimation error as $\tilde{\boldsymbol{\xi}}_e := \hat{\boldsymbol{\xi}}_e - \boldsymbol{\xi}_e$ and

$$\mathbf{E}_\xi := \begin{bmatrix} \boldsymbol{\xi}_e^T & \tilde{\boldsymbol{\xi}}_e^T \end{bmatrix}^T \tag{57}$$

Then, the augmented system dynamics can be described as

$$\begin{aligned} \dot{\mathbf{E}}_{\xi_e} = \underbrace{\begin{bmatrix} \mathbf{A}-\mathbf{b}\mathbf{k}_c & -\mathbf{b}\mathbf{k}_c \\ 0 & \mathbf{A}-\mathbf{k}_o\mathbf{c} \end{bmatrix}}_{:=\mathbf{A}_0}\mathbf{E}_{\xi_e} + \underbrace{\begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}}_{:=\mathbf{b}_0}\left(u_L + \mathbf{k}_c\hat{\boldsymbol{\xi}}_e + \beta\right) \\ + \underbrace{\begin{bmatrix} \mathbf{q} \\ 0 \end{bmatrix}}_{:=\mathbf{q}_0}\gamma - \underbrace{\begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}}_{:=\mathbf{b}_1}\beta - \underbrace{\begin{bmatrix} 0 \\ \mathbf{q} \end{bmatrix}}_{:=\mathbf{q}_1}\gamma \end{aligned} \tag{58}$$

where $\beta := \Delta - u_{ad} - u_R$ and $\gamma := y^* + \Delta_{\boldsymbol{\eta}}$. Therefore, the augmented system dynamics can be described as

$$\dot{\mathbf{E}}_{\xi_e} = \mathbf{A}_0\mathbf{E}_{\xi_e} + \mathbf{b}_0\left(u_L + \mathbf{k}_c\hat{\boldsymbol{\xi}}_e + \beta\right) + \mathbf{q}_0\gamma - \mathbf{b}_1\beta - \mathbf{q}_1\gamma. \tag{59}$$

Also the available output signals are introduced as

$$\begin{aligned} \rho_1 &= \mathbf{k}_c\hat{\boldsymbol{\xi}}_e = [\mathbf{k}_c \quad \mathbf{k}_c]\mathbf{E}_\xi \\ \rho_2 &= \mathbf{q}^T\mathbf{P}_3\hat{\boldsymbol{\xi}}_e = \left[\mathbf{q}^T\mathbf{P}_3 \quad \mathbf{q}^T\mathbf{P}_3\right]\mathbf{E}_\xi. \end{aligned} \tag{60}$$

4.4. Reference signal construction

The reference signal y_d is designed to cancel out the unmatched uncertainty $\Delta_{\boldsymbol{\eta}}$. Using the error $e := y_d - y$, the upper bound of the modeling error, defined in (45), can be represented as

$$\left|\Delta_{\boldsymbol{\eta}}(\mathbf{z},\boldsymbol{\eta})\right| \le c_0^* + c_1\left(\|\mathbf{e}\| + \|\mathbf{y}_d\| + \left|y_d^{(r)}\right|\right) + c_2^*\|\boldsymbol{\eta}\| \tag{61}$$

where c_0^* and c_2^* are estimates of c_0 and c_2, respectively. On the other hand, from (44) and (45) the following bounds can be derived:

$$\|\mathbf{y}_d\| + \left|y_d^{(r)}\right| = \sqrt{\sum_{i=0}^{r-1}\left(y_d^{(i)}\right)^2} + \left|y_d^{(r)}\right| \le \sum_{i=0}^{r}\left|y_d^{(i)}\right|$$

$$\left|y^*\right| = \left|\sum_{i=0}^{r-1} g_{i+1}y_d^{(i)}\right| \le \sum_{i=0}^{r-1}|g_{i+1}|\left|y_d^{(i)}\right|$$

Then,

$$\|\mathbf{y}_d\| + \left|y_d^{(r)}\right| \le \left|y^*\right| - \left|\sum_{i=0}^{r-1} g_{i+1}y_d^{(i)}\right| + \sum_{i=0}^{r}\left|y_d^{(i)}\right| \le \left|y^*\right| + c_3^* p, \tag{62}$$

where $p \le 1$ is a nonnegative real number and c_3^* is defined in (45). Substituting (62) into (61) yields

$$\left|\Delta_{\boldsymbol{\eta}}(\mathbf{z},\boldsymbol{\eta})\right| \le c_4^* + c_1\left|y^*\right| + c_5^*\|\boldsymbol{\xi}_e\|,$$

where $c_5^* = c_1 + c_2^*$ and $c_4^* = c_0^* + c_1 c_3^* p$. Now, define $\boldsymbol{\lambda}^* := [c_4^* \quad c_5^*]^T$; then,

$$\left|\Delta_{\boldsymbol{\eta}}(\mathbf{z},\boldsymbol{\eta})\right| \le c_1\left|y^*\right| + \boldsymbol{\lambda}^{*T}\left[1 \quad \|\boldsymbol{\xi}_e\|\right]^T \tag{63}$$

Let $\boldsymbol{\lambda}$ be an estimate of the unknown parameter $\boldsymbol{\lambda}^*$. An adaptive reference signal is proposed as

$$y_d = \frac{1}{D(s)}y^* = \frac{1}{D(s)}\left(-\frac{\boldsymbol{\lambda}^T\left[1 \quad \|\hat{\boldsymbol{\xi}}_e\|\right]^T}{1-c_1}\tanh\left(\rho_2/\mu_y\right)\right) \tag{64}$$

with the following adaptation rule:

$$\dot{\boldsymbol{\lambda}} = [\dot{c}_4 \quad \dot{c}_5]^T = \Gamma_\lambda\left[1 \quad \|\hat{\boldsymbol{\xi}}_e\|\right]^T|\rho_2|, \tag{65}$$

where μ_y is a positive constant, Γ_λ is the learning coefficient matrix and

$$D(s) = g_r s^{r-1} + g_{r-1}s^{r-2} + \cdots + g_1$$

is a Hurwitz polynomial in which $g_i\ (i = 1, \ldots, r)$ were defined in (2).

Remark 3. In practice, small positive numbers can be selected as initial values for $[c_4 \quad c_5]$. Then, according to (65) these gains increase and approach to $[c_4^* \quad c_5^*]$. Hence, always $c_4 \le c_4^*$. Moreover, using the approximation error $\gamma = y^* + \Delta_{\boldsymbol{\eta}}$, and equations (63) and (64), the following bound can be derived:

$$|\gamma| \le c_4^*(1+d) + c_5^* d\|\tilde{\boldsymbol{\xi}}_e\| + c_5^*(1+d)\|\boldsymbol{\xi}_e\|$$

where $d = (1+c_1)/(1-c_1)$. Substituting (57) into the above equation yields

$$|\gamma| \le \alpha_0 + \alpha_1\|\mathbf{E}_{\xi_e}\|$$

(66)

where $\alpha_0 = c_4^*(1+d)$ and $\alpha_1 = c_5^*(1+2d)$.

4.5. Stability analysis

In this section, the ultimately boundedness of the error trajectories $\mathbf{E}_\xi$, $\tilde{\mathbf{w}}$ and $\tilde{\mathbf{V}}$ are shown using the Lyapunov stability approach.

Definition 1. Let Ω_Δ be the compact set in which the NN approximates Δ, and Ω_{r_Δ} be the largest hypersphere within the error space $\mathbf{E}_a = [\mathbf{E}_\xi, \|\tilde{\mathbf{w}}\|, \|\tilde{\mathbf{V}}\|_F]$ defined as

$$\Omega_{r_\Delta} := \left\{\mathbf{E}_a \,\middle|\, \|\mathbf{E}_a\| \le r_\Delta\right\}, \tag{67}$$

where r_Δ is a positive number, such that for every $\mathbf{E}_a \in \Omega r_\Delta$ there exists $(\mathbf{z}, \boldsymbol{\eta}, u) \in \Omega_\Delta$.

Assumption 4. There exists a positive number $r_{\max}$ which satisfies the following inequality

$$r_{\max} < \sqrt{S_m / S_M}\, r_\Delta , \tag{68}$$

where S_m and S_M are the minimum and the maximum eigenvalues of the following matrix, respectively:

$$\mathbf{S} = \frac{1}{2}\begin{bmatrix} \mathbf{P} & \mathbf{0} & 0 \\ \mathbf{0} & \gamma_{\mathbf{w}}^{-1} & 0 \\ \mathbf{0} & 0 & \gamma_{\mathbf{V}}^{-1} \end{bmatrix} \text{ with } \mathbf{P} = \begin{bmatrix} \mathbf{P}_3 & \mathbf{P}_3 \\ \mathbf{P}_3 & \mathbf{P}_3 + \mathbf{P}_4 \end{bmatrix}.$$

Theorem 3. *Consider the linear controller (48), the neuro-adaptive controller in (22) with adaptation rules (51), the robustifying controller (52) with adaptation rule (53) and the reference signal y_d as defined in (64). Then, if Assumptions 3 and 4 hold and $\mathbf{E_a}(0)$ belongs to the compact set $\mathrm{B}_{r\Delta} \subset \Omega_{r\Delta}$, the errors $\mathbf{E}_{\xi_e}$, $\tilde{\mathbf{w}}$ and $\tilde{\mathbf{V}}$ in the closed-loop system are uniformly ultimately bounded.*

Proof. Consider the Lyapunov function

$$L = \frac{1}{2}\hat{\boldsymbol{\xi}}_e^T \mathbf{P}_3 \hat{\boldsymbol{\xi}}_e + \frac{1}{2}\hat{\boldsymbol{\xi}}_e^T \mathbf{P}_4 \hat{\boldsymbol{\xi}}_e + \frac{1}{2\gamma_{\mathbf{w}}}\|\tilde{\mathbf{w}}\|^2 + \frac{1}{2\gamma_V}\|\tilde{\mathbf{V}}\|_F^2 + \frac{1}{2\gamma_\varphi}|\tilde{\varphi}|^2 + \frac{1}{2}\tilde{\boldsymbol{\lambda}}^T \Gamma_\lambda^{-1}\tilde{\boldsymbol{\lambda}}$$

where $\tilde{\varphi} := \varphi^* - \varphi$ and $\tilde{\boldsymbol{\lambda}} := \boldsymbol{\lambda}^* - \boldsymbol{\lambda}$, in which φ^* and $\boldsymbol{\lambda}^*$ are the ideal gains of their corresponding estimated values φ and $\boldsymbol{\lambda}$, respectively. Using (57), this Lyapunov function can be represented as

$$L = \frac{1}{2}\mathbf{E}_{\xi_e}^T \mathbf{P}\mathbf{E}_{\xi_e} + \frac{1}{2\gamma_{\mathbf{w}}}\|\tilde{\mathbf{w}}\|^2 + \frac{1}{2\gamma_V}\|\tilde{\mathbf{V}}\|_F^2 + \frac{1}{2\gamma_\varphi}|\tilde{\varphi}|^2 + \frac{1}{2}\tilde{\boldsymbol{\lambda}}^T \Gamma_\lambda^{-1}\tilde{\boldsymbol{\lambda}} \tag{69}$$

Recall that $\dot{\mathbf{w}} = -\dot{\tilde{\mathbf{w}}}$ and $\dot{\mathbf{V}} = -\dot{\tilde{\mathbf{V}}}$. Using (59), the time-derivative of the Lyapunov function (69) becomes

$$\begin{aligned}\dot{L} = \frac{1}{2}\mathbf{E}_{\xi_e}^T &\left(\begin{bmatrix} \mathbf{P}_3 & \mathbf{P}_3 \\ \mathbf{P}_3 & \mathbf{P}_3 + \mathbf{P}_4 \end{bmatrix} \begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k}_c & -\mathbf{b}\mathbf{k}_c \\ 0 & \mathbf{A} - \mathbf{k}_o\mathbf{c} \end{bmatrix} \right. \\ &\left. + \begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k}_c & -\mathbf{b}\mathbf{k}_c \\ 0 & \mathbf{A} - \mathbf{k}_o\mathbf{c} \end{bmatrix}^T \begin{bmatrix} \mathbf{P}_3 & \mathbf{P}_3 \\ \mathbf{P}_3 & \mathbf{P}_3 + \mathbf{P}_4 \end{bmatrix} \right) \mathbf{E}_{\xi_e} \\ &+ \mathbf{E}_{\xi_e}^T \begin{bmatrix} \mathbf{P}_3 & \mathbf{P}_3 \\ \mathbf{P}_3 & \mathbf{P}_3 + \mathbf{P}_4 \end{bmatrix} \mathbf{b}_0 \left(\beta + u_L + \mathbf{k}_c \hat{\boldsymbol{\xi}}_e\right) \\ &+ \mathbf{E}_{\xi_e}^T \begin{bmatrix} \mathbf{P}_3 & \mathbf{P}_3 \\ \mathbf{P}_3 & \mathbf{P}_3 + \mathbf{P}_4 \end{bmatrix} \mathbf{q}_0 \gamma - \mathbf{E}_{\xi_e}^T \mathbf{P}\mathbf{b}_1 \beta - \mathbf{E}_{\xi_e}^T \mathbf{P}\mathbf{q}_1 \gamma \\ &- \gamma_w^{-1}\tilde{\mathbf{w}}^T \dot{\mathbf{w}} - \gamma_V^{-1}\operatorname{tr}\left(\tilde{\mathbf{V}}^T \dot{\mathbf{V}}\right) - \gamma_\varphi^{-1}\tilde{\varphi}\dot{\varphi} - \tilde{\boldsymbol{\lambda}}^T \Gamma_\lambda^{-1}\dot{\boldsymbol{\lambda}}\end{aligned}$$

Using(49), (58) and (60), it is obtained

$$\begin{aligned}\mathbf{E}_{\xi_e}^T \mathbf{P}\mathbf{b}_0 &= \mathbf{E}_{\xi_e}^T \begin{bmatrix} \mathbf{P}_3 & \mathbf{P}_3 \\ \mathbf{P}_3 & \mathbf{P}_3 + \mathbf{P}_4 \end{bmatrix} \mathbf{b}_0 = \mathbf{E}_{\xi_e}^T \begin{bmatrix} \mathbf{P}_3\mathbf{b} \\ \mathbf{P}_3\mathbf{b} \end{bmatrix} = \mathbf{E}_{\xi_e}^T \begin{bmatrix} \mathbf{k}_c^T \\ \mathbf{k}_c^T \end{bmatrix} = \rho_1 \\ \mathbf{E}_{\xi_e}^T \mathbf{P}\mathbf{q}_0 &= \mathbf{E}_{\xi_e}^T \begin{bmatrix} \mathbf{P}_3 & \mathbf{P}_3 \\ \mathbf{P}_3 & \mathbf{P}_3 + \mathbf{P}_4 \end{bmatrix} \mathbf{q}_0 = \mathbf{E}_{\xi_e}^T \begin{bmatrix} \mathbf{P}_3\mathbf{q} \\ \mathbf{P}_3\mathbf{q} \end{bmatrix} = \rho_2\end{aligned} \tag{70}$$

Then using (48), (50), (56) and (70), and substituting $\beta = \tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma} - \dot{\boldsymbol{\sigma}}\mathbf{V}^T\zeta\right) + \operatorname{tr}\left(\tilde{\mathbf{V}}^T\zeta\mathbf{w}^T\dot{\boldsymbol{\sigma}}\right) + \delta - u_R$, after some mathematical manipulations $\dot{L}$ becomes

$$\begin{aligned}\dot{L} = &-\frac{1}{2}\mathbf{E}_{\xi_e}^T \mathbf{Q}\mathbf{E}_{\xi_e} + \rho_1\left(\tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma} - \dot{\boldsymbol{\sigma}}\mathbf{V}^T\zeta\right) + \operatorname{tr}\left(\tilde{\mathbf{V}}^T\zeta\mathbf{w}^T\dot{\boldsymbol{\sigma}}\right)\right) \\ &- \gamma_w^{-1}\tilde{\mathbf{w}}^T\dot{\mathbf{w}} - \gamma_V^{-1}\operatorname{tr}\left(\tilde{\mathbf{V}}^T\dot{\mathbf{V}}\right) + \rho_1\left(\delta - u_R\right) - \gamma_\varphi^{-1}\tilde{\varphi}\dot{\varphi} \\ &+ \rho_2\left(y^* + \Delta_{\boldsymbol{\eta}}\right) - \tilde{\boldsymbol{\lambda}}^T\Gamma_\lambda^{-1}\dot{\boldsymbol{\lambda}} - \mathbf{E}_{\xi_e}^T\mathbf{P}\mathbf{b}_1\beta - \mathbf{E}_{\xi_e}^T\mathbf{P}\mathbf{q}_1\gamma.\end{aligned}$$

where

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_3 & \mathbf{Q}_3 + \mathbf{P}_3\mathbf{k}_o\mathbf{c} \\ \mathbf{Q}_3 + \mathbf{c}^T\mathbf{k}_o^T\mathbf{P}_3 & \mathbf{Q}_4 + \left(\mathbf{Q}_2 + \mathbf{c}^T\mathbf{k}_o^T\mathbf{P}_3\right)\mathbf{Q}_3^{-1}\left(\mathbf{Q}_3 + \mathbf{P}_3\mathbf{k}_o\mathbf{c}\right) \end{bmatrix}.$$

Since $\mathbf{Q}_3$ and $\mathbf{Q}_4$ are symmetric positive-definite matrices, $\mathbf{Q}$ is also a symmetric positive-definite matrix.

Now, from the bounds (63), (66), $|\delta| \le \varphi^*\chi$, the robustifying control term (52) and the reference signal (64), and considering the fact that $-x\tanh(x/\mu_x) \le -|x| + k\mu_x$ with $k = 0.2785$, the time derivative of L satisfies the following inequality:

$$\begin{aligned}\dot{L} \le &-\frac{1}{2}q_{\min}\left\|\mathbf{E}_{\xi_e}^T\right\|^2 + \tilde{\mathbf{w}}^T\left(\rho_1\boldsymbol{\psi} - k_w\mathbf{w} - \gamma_w^{-1}\dot{\mathbf{w}}\right) + k_w\tilde{\mathbf{w}}^T\mathbf{w} \\ &+ \operatorname{tr}\left(\tilde{\mathbf{V}}^T\left(\rho_1\boldsymbol{\Psi} - k_V\mathbf{V} - \gamma_V^{-1}\dot{\mathbf{V}}\right)\right) + k_V\operatorname{tr}\left(\tilde{\mathbf{V}}^T\mathbf{V}\right) - \tilde{\boldsymbol{\lambda}}^T\Gamma_\lambda^{-1}\dot{\boldsymbol{\lambda}} \\ &+ |\rho_1|\left(\varphi^* - \varphi\right)\chi - \gamma_\varphi^{-1}\tilde{\varphi}\dot{\varphi} + \left\|\mathbf{E}_{\xi_e}\right\|\left\|\mathbf{P}\mathbf{q}_1\right\|\left(\alpha_0 + \alpha_1\left\|\mathbf{E}_{\xi_e}\right\|\right) \\ &+ |\rho_2|\boldsymbol{\lambda}^T\begin{bmatrix}1 & \left\|\hat{\boldsymbol{\xi}}_e\right\|\end{bmatrix}^T\begin{bmatrix}\frac{c_1}{1-c_1} - \frac{1}{1-c_1}\end{bmatrix} + |\rho_2|\boldsymbol{\lambda}^{*T}\begin{bmatrix}1 & \left\|\hat{\boldsymbol{\xi}}_e\right\|\end{bmatrix}^T \\ &+ \frac{\boldsymbol{\lambda}^{*T}k\mu_y}{1-c_1}\begin{bmatrix}1 & \left\|\hat{\boldsymbol{\xi}}_e\right\|\end{bmatrix}^T + c_5^*|\rho_2|\left\|\tilde{\boldsymbol{\xi}}_e\right\| + \|\mathbf{E}\|\left\|\mathbf{P}\mathbf{b}_1\right\|\beta_M\end{aligned}$$

where $q_{\min}$ denotes the minimum eigenvalue of $\mathbf{Q}$ and $\beta_M := 2\sqrt{m}M_w + \varepsilon_M + U_M$ is the upper bound of $|\beta|$ in which U_M is a positive constant such $|u_R| \le U_M$. Note that because of the universal approximation property of NNs, the approximation error is bounded. Hence, it is always possible to find such a positive constant.

Next, let $\varepsilon := k\mu_y/(1-c_1)$. Using the inequalities $|\rho_2| \le \|\mathbf{P}\mathbf{q}_1\|\left\|\hat{\boldsymbol{\xi}}_e\right\|$, $\left\|\hat{\boldsymbol{\xi}}_e\right\| \le \sqrt{2}\left\|\mathbf{E}_\varsigma\right\|$ and $\left\|\tilde{\boldsymbol{\xi}}_e\right\| \le \left\|\mathbf{E}_{\varsigma_e}\right\|$, and applying the adaptation rules (51)

$$\begin{aligned}\dot{L} \le &-\frac{1}{2}q_{\min}\left\|\mathbf{E}_{\xi_e}^T\right\|^2 - k_w\|\tilde{\mathbf{w}}\|^2 + k_wM_w\|\tilde{\mathbf{w}}\| - k_V\left\|\tilde{\mathbf{V}}\right\|_F^2 \\ &+ k_VM_V\left\|\tilde{\mathbf{V}}\right\|_F + \tilde{\varphi}\left(|\rho_1|\chi - \gamma_\varphi^{-1}\dot{\varphi}\right) + \varepsilon c_4^* + \sqrt{2}\varepsilon c_5^*\left\|\mathbf{E}_{\xi_e}\right\| \\ &+ \tilde{\boldsymbol{\lambda}}^T\left(|\rho_2|\begin{bmatrix}1 & \left\|\hat{\boldsymbol{\xi}}_e\right\|\end{bmatrix}^T - \Gamma_\lambda^{-1}\dot{\boldsymbol{\lambda}}\right) + \sqrt{2}c_5^*\left\|\mathbf{P}\mathbf{q}_1\right\|\left\|\mathbf{E}_{\varsigma_e}\right\|^2 \\ &+ \left\|\mathbf{E}_{\xi_e}\right\|\left\|\mathbf{P}\mathbf{b}_1\right\|\beta_M + \left\|\mathbf{E}_{\xi_e}\right\|\left\|\mathbf{P}\mathbf{q}_1\right\|\left(\alpha_0 + \alpha_1\left\|\mathbf{E}_{\xi_e}\right\|\right).\end{aligned}$$

Now the adaptation rules (53) and (65), and completing the square terms yield

$$\dot{L} \le -A_E\left\|\mathbf{E}_{\varsigma_e}\right\|^2 - (k_w - 1)\|\tilde{\mathbf{w}}\|^2 - (k_V - 1)\left\|\tilde{\mathbf{V}}\right\|^2 + R, \tag{71}$$

where

$$A_E := \left(\frac{1}{2}q_{\min} - \left(\alpha_1 + \sqrt{2}c_5^*\right)\left\|\mathbf{P}\mathbf{q}_1\right\| - 1\right)$$

$$R := \frac{(k_wM_w)^2}{4} + \frac{(k_VM_V)^2}{4} + \frac{\left(\alpha_0\left\|\mathbf{P}\mathbf{q}_1\right\| + \beta_M\left\|\mathbf{P}\mathbf{b}_1\right\| + \varepsilon c_5^*\right)^2}{4} + \varepsilon c_4^* \tag{72}$$

Select $k_w > 1$ and $k_V > 1$ and let the constants α_1 and c_5^* be such that the following condition is satisfied

$$q_{\min} > 2\left(\alpha_1 + \sqrt{2}c_5^*\right)\left\|\mathbf{P}\mathbf{q}_1\right\| + 2. \tag{73}$$

Define a compact set around the origin as $\Omega := \left\{\mathbf{E}_a \,\middle|\, \left\|\mathbf{E}_a\right\| \le r_{\max}\right\}$, where

$$r_{\max} = \max\left\{\sqrt{R/A_E},\ \sqrt{R/(k_w - 1)},\ \sqrt{R/(k_V - 1)}\right\}.$$

As Fig. 1 shows, $\dot{L} \le 0$ if the errors are the outside of the compact set Ω. Next, consider the Lyapunov function (69), which can alternatively be written as $L = \mathbf{E}_a{}^T \mathbf{S} \mathbf{E}_a + L_{\lambda\varphi}$, with $L_{\lambda\varphi} = 0.5\gamma_\varphi^{-1} |\tilde{\varphi}|^2 + 0.5\tilde{\boldsymbol{\lambda}}^T \Gamma_\lambda^{-1} \tilde{\boldsymbol{\lambda}}$ and

$$S_m \left\| \mathbf{E}_{\xi_e} \right\|^2 + L_{\lambda\varphi} \le L\left(\mathbf{E}_{\xi_e}\right) \le S_M \left\| \mathbf{E}_{\xi_e} \right\|^2 + L_{\lambda\varphi},$$

where S_m and S_M are the smallest and the largest eigenvalues of $\mathbf{S}$, respectively. Let L_M be the maximum value of the Lyapunov function L on the boundary of Ω, i.e. $L_M = S_M r_{\max}^2$ and L_m be its minimum value on the boundary of Ω_{r_Δ}, i.e. $L_m = S_m r_\Delta^2$. Therefore, if $L_M > L_m$ then the error trajectory initialized in the shadow area may leave Ω_{r_Δ}. See Fig. 2(a). On the other hand, if $L_M < L_m$ then Assumption 4 holds, and therefore, $\Omega \subset \Omega_{r_\Delta}$. See Fig. 2(b). Moreover, consider the compact set $\mathrm{B}_{r\Delta} := \left\{ \mathbf{E}_a \in \Omega_{r\Delta} \,\middle|\, L(\mathbf{E}_a) \le L_m \right\}$ as depicted in Fig. 2(c). One can conclude that if an error trajectory starts form a point inside $\mathrm{B}_{r\Delta}$ (i.e. $\mathbf{E}_a(0) \in \mathrm{B}_{r_\Delta}$), then according to the standard Lyapunov theorem extension, the error trajectory $\mathbf{E}_a(t)$ is ultimately bounded (Lewis *et al.* 1996; Yesildirek and Lewis, 1995; Ge and Zhang, 2003). □

The block diagram of the closed-loop system is depicted in Fig. 3.

5. Example

A TORA model is considered to illustrate the performance of the proposed controllers (Karagiannis *et al.*, 2005; Lee,

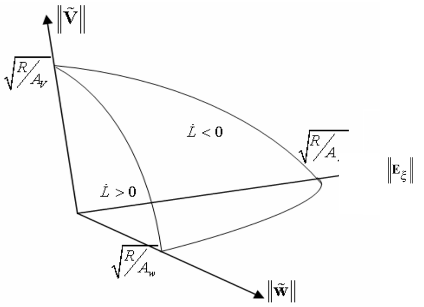

Fig. 1. The surface $A_E \left\| \mathbf{E}_\varsigma \right\|^2 + A_w \left\| \tilde{\mathbf{w}} \right\|^2 + A_V \left\| \tilde{\mathbf{V}} \right\|_F^2 = R$

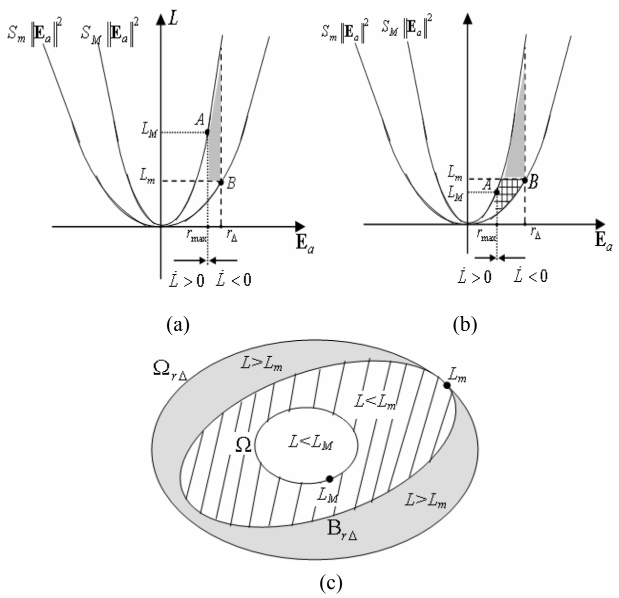

Fig. 2. Semi-global ultimately error boundedness of the proposed method

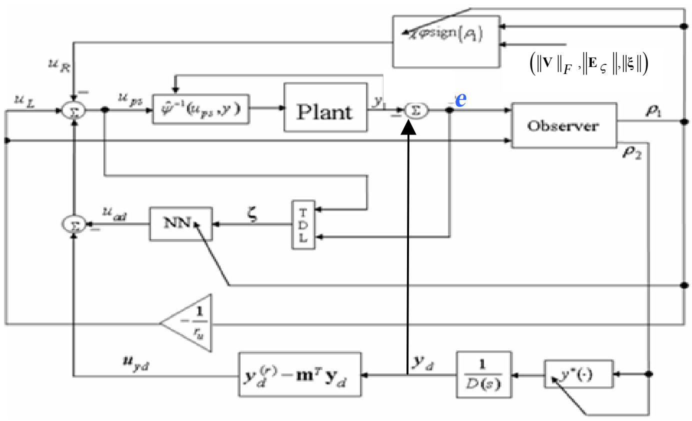

Fig. 3. Block diagram of the proposed controller

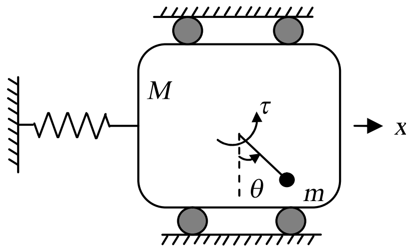

Fig. 4. A translational oscillator with a rotational actuator (TORA)

2004), see Fig. 4. The system dynamics is governed by the following differential equations:

$$(M + m)\ddot{x} + m\,l(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta) = -k\,x$$

$$(J + m\,l^2)\ddot{\theta} + m\,l\cos\theta\,\ddot{x} = \tau$$

where θ is the angle of rotation, x is the translational displacement, and τ is the control torque. The positive constants k, l, J, M and m denote the spring stiffness, the radius of rotation, the moment of inertia, the mass of the cart, and the eccentric mass, respectively. Define the states and the input variables as

$$\eta_1 = x + m\,l\sin\theta/(M+m), \eta_2 = \dot{x} + m\,l\dot{\theta}\cos\theta/(M+m)$$

$$z_1 = \theta, \quad z_2 = \dot{\theta}, \quad u = \tau.$$

In these coordinates, the system can be described in the following normal form:

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = (\phi(z_1))^{-1}\left(ka_1 \cos z_1 \eta_1 - a_1^2 a_2 \sin z_1 \cos z_1 \right. \\ \qquad \left. - m^2 l^2 z_2^2 \sin z_1 \cos z_1 + (M+m)\,u\right) \\ \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = -a_2\eta_1 + a_3 \sin z_1 \end{cases}$$

where $\phi(z_1) = (M+m)(J+m\,l^2) - m^2 l^2 \cos^2\theta$, $a_1 = ml$, $a_2 = k/(M+m)$ and $a_3 = k\,ml/(M+m)^2$.

The output of the system is $y = z_1$. Therefore, the zero dynamics of this system is

$$\begin{cases} \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = -a_2\eta_1. \end{cases}$$

Since $a_2 > 0$, the zero dynamics is unstable and the system is non-minimum phase. The following linear model of the TORA system is available:

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -\hat{a}_1^2 \hat{a}_2 \left(\hat{\phi}(0)\right)^{-1} z_1 + \hat{k}\hat{a}_1 \left(\hat{\phi}(0)\right)^{-1} \eta_1 + (M+\hat{m})\left(\hat{\phi}(0)\right)^{-1} u \\ \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = -\hat{a}_2 \eta_1 + \hat{a}_3 z_1, \end{cases}$$

where $\hat{m}$, $\hat{k}$, $\hat{J}$, $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$ and $\hat{\phi}$ are the estimates of the parameters m, k, J, a_1, a_2, a_3 and ϕ respectively.

Note that Assumption 1 is satisfied; that is

$$\partial f(z,\eta,u)/\partial u = (M+m)\left(\phi(z_1)\right)^{-1} > 0.$$

Also consider the best available approximation of ψ as $\hat{\psi} = u_{ps} = cu$, where c should be selected such that conditions (20) hold; i.e.

$$c \geq 0.5 \frac{(M+m)\hat{\phi}(0)}{(M+\hat{m})\phi(z_1)} > 0 \qquad \forall z_1 \in \Omega_z .$$

To ensure that this condition holds for $\hat{m} < 2m$ and $\hat{J} < 2J$, it is assumed that $c = 1$. For comparison, simulations have been carried out using the same parameters and initial conditions as in Karagiannis *et al.* (2005):

$J = 0.0002175$ kg/m^2, $M = 1.3608$ kg, $m = 0.096$ kg, $l = 0.0592$ m, and $k = 186.3$ N/m, $z_2(0) = 0$ rad/sec, $\eta_1(0) = 0.025$ m, $\eta_2(0) = 0$ m/sec., $z_1(0) = 0$ rad.

The procedure of the control design is as follows: First the system is stabilized assuming that the internal dynamics are available according to the method proposed in Section 3. The reference signal is designed using the following parameters $\mathbf{k} = [-234 \quad 0.67]$, $k_c = 0.12$. The NN is an MLP and comprises of 10 neurons in one hidden layer with tangent hyperbolic as the activation functions, and the weights are initialized randomly using small numbers. The input vector to the NN is

$$\xi = [1, y(t), y(t-T_d), y(t-2T_d), y(t-3T_d), u(t-T_d), u(t-2T_d)]^T$$

and the learning coefficients are $\gamma_{\mathbf{w}} = \gamma_{\mathbf{V}} = 0.03$.

Simulations are first performed using $y_d = 0$. As Fig. 5 shows, the system states oscillate and converge very slowly; however, when the desired reference signal is applied to the system, the states converge faster.

Simulations are then carried out using the error observer proposed in Section 4. The controller and observer gains are $\mathbf{k}_c = [-4.6, -1, -298.6, 6.9]$, $\mathbf{k}_o = [32, 594.2, -2.14, 38.4]$.

The learning coefficients are selected as $\gamma_{\mathbf{w}} = \gamma_{\mathbf{V}} = 3$, $\gamma_\varphi = 2$, $\Gamma_\lambda = \mathrm{diag}[0.05 \quad 1]$, and $k_{\mathbf{w}} = k_{\mathbf{V}} = 1.2$. The results are depicted in Figs. 6–8. First, only the proposed combined control law has been used without the unmatched uncertainty approximation (i.e. $y_d = 0$).

Then, the proposed y_d is employed (See Fig. 6). Note that when the unmatched uncertainty is compensated by y_d the responses converge faster. Fig. 7 shows the comparison of the simulation results between the proposed approach and the backstepping-based controller proposed by Karagiannis *et al.* (2005). Note that the convergence rate of the proposed approach is faster. Fig. 8 presents the approximation of the matched uncertainty Δ using $u_{ad} + u_R$, the normalized norm of adaptive weights and the state estimation errors.

6. Conclusions

In this paper, an adaptive control method for a class of non-minimum phase nonlinear systems has been developed. First, stabilization problem of the system was considered assuming that the internal dynamics are available. Theses dynamics were applied to construct the reference signal, which guarantees the input to state stability of the internal dynamics. Then the assumption availability of the internal dynamics was removed by designing a suitable linear error observer. Simulation results show good performance of the proposed methods in comparison with other traditional methods such as the backstepping method.

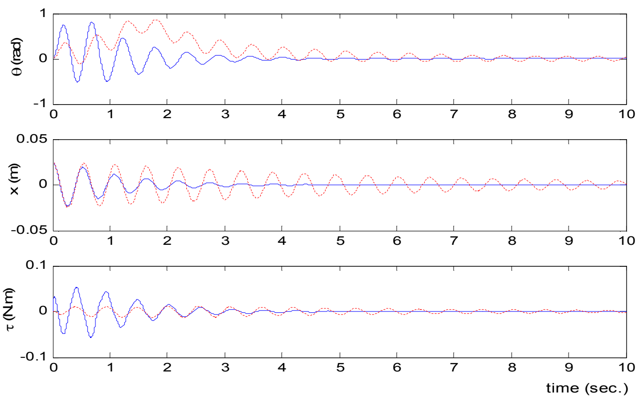

Fig. 5. Response of the TORA system, dashed line: without the reference signal ($y_d = 0$); solid line: with the reference signal.

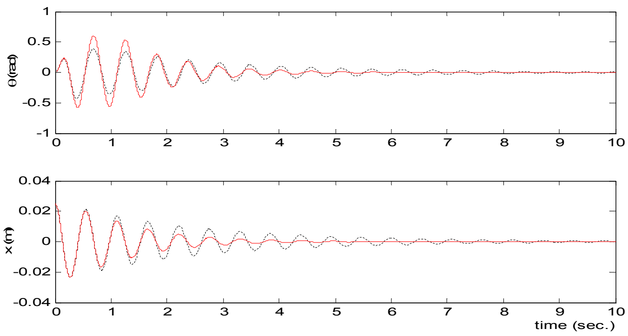

Fig. 6. Response of the TORA system, dashed line: without unmatched uncertainty compensation ($y_d = 0$); solid line: with unmatched uncertainty cancellation using the proposed y_d.

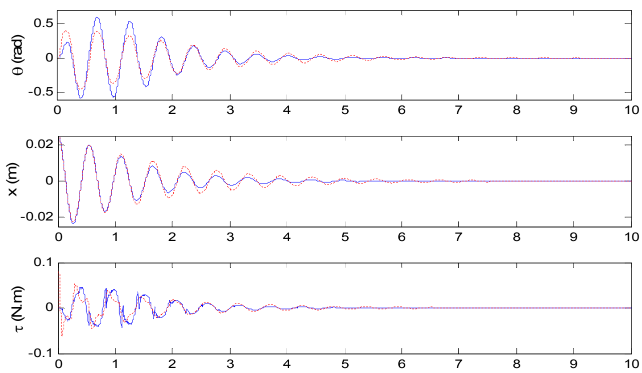

Fig. 7. Response of the TORA system without parameters uncertainties; Solid line: the proposed method; dashed line: the backstepping controller.

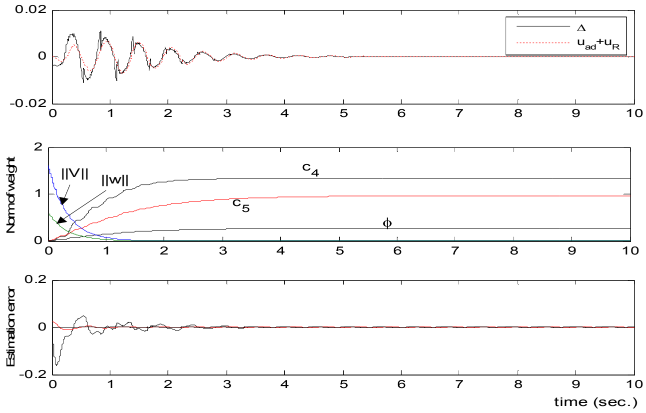

Fig.8. The closed-loop signals of the TORA system: (a) Matched uncertainty cancellation; (b) Normalized norm of weights; (c) States estimation error

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