International Journal of Civil Engineering

Structural Engineering

Analysis of near-regular structures with node irregularity using SVD of equilibrium matrix

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Received: June 2013, Accepted:October 2013

Abstract

In the process of structural analysis we often come to structures that can be analyzed with simpler methods than the standard approaches. For these structures, known as regular structures, the matrices involved are in canonical forms and their eigen-solution can be performed in a simple manner. However, by adding or removing some elements or nodes, such methods cannot be utilized. Here, an efficient method is developed for the analysis of irregular structures in the form a regular structure with additional or missing nodes or with additional or missing supports. The power of the method becomes apparent when the analysis should be repeated many times as it is the case in optimal design.

Keywords: *Regular graph, Irregular graph, Equilibrium matrix, Eigen-Solution, Analysis of structure, Singular value decomposition.*

1. Introduction

Pellegrino and Calladine [1] and Pellegrino [2] in order to avoid the inversion of the stiffness matrix, used singular value decomposition of the equilibrium matrix. Guest [3] found that exact tangent stiffness matrix of structure being described by the equilibrium and stress matrix. He presented a simple derivation of the tangent stiffness matrix for a pre-stressed pin-jointed structure and compared to some formulations that could be found in the literature for finding the structural response of pre-stressed structures.

Pellegrino and Calladine classified structural assemblies. Based on singular value decomposition, the criterion for geometrical stability of mechanism was introduced by Pellegrino [4].

Singhal and Singhal [5] and Katz and Singhal [6] used compatibility matrix (transpose of equilibrium matrix) in design process of substructures for providing compatibility between parts. Lu et al. [7] presented a matrix-based method for the determination of the mobility and stable equilibrium mechanisms according to the effects of the external loads. In this process, the first and second variations of the potential energy function of mechanisms under conservative force field were analyzed. Based on this method, singular value decomposition of equilibrium matrix was presented as a new criterion for the mobility and stable equilibrium mechanisms.

Recently the authors presented a method to solve the problem of member irregularity [8]. This method was based on singular value decomposition of the equilibrium matrix. In the follow up, in this article the method is extended to provide efficient tool to solve the problem of node irregularity and consequently the problem of multi-irregularities is solved. Therefore, we consider the basic relations of the previous article and make only the necessary additions to cover the new irregular forms. Further details of the new method can be found in the previous article of the authors.

Here a method is developed for the analysis of regular structures with additional nodes, missing nodes or supports. Then an algorithm is present for solving different types of irregularity. In this method we use singular value decomposition of the equilibrium matrix for solution of the structures. Thus, the method can solve geometrically unstable structures that become stable by applying special external forces. Hence, this method has special applications, because of having these two useful abilities, namely simultaneously solving different type of irregular unstable structures. This method is applicable to various types of structures such as frame, truss and finite element models.

One of the main applications of this paper is its use in optimal design of structures. In fact in each step of the optimization instead of analysis of the entire structure only a small part can be analyzed and combined with results of the previous step. This results in saving a considerable amount of computational time.

After the introduction in Section 2 of this article, the

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basic relationships of the equilibrium matrix are presented. In Section 3, the analysis method of structures with additional or lack of nodes is provided. In this section, essential tips of the equilibrium matrix relations are expressed as a step by step process. In Section 4, the method is extended for structures with irregular supports. In Section 5 some practical examples are presented to show the efficiency of the method in improving the rate of analysis. Section 6 concludes the article.

In Appendix A the notations are presented and in Appendix B the important equations used in the paper are described.

2. Basic Definitions and Concepts

An irregular structure is decomposed into a regular structure, and an irregular part containing nodes and elements changing a regular structure into an irregular one. In this method, we utilize the equilibrium matrix to analyze two separated parts of structure independently. At the end, the final results can gain by assembling the results of the two parts in an appropriate form.

On this basis, the structure components are named as follows:

"Irregular structure" is a structure that has lost its simple analysis form by adding or removing some parts (elements, nodes or supports). Elements that by removing them, irregular structure will have simple analysis form are named "irregular Element". "Regular structure" is a part of an irregular structure that is obtained by eliminating the irregular elements and it independently has simple analysis form. "Irregular nodes" are some irregular structural nodes that do not belong to the regular structural nodes. It is obvious that by adding regular structure and irregular elements and nodes, we will obtain irregular structure.

This paper uses the method of Ref. [2] and Ref. [8] for structural analysis by singular value decomposition (SVD) of the equilibrium matrix. However, the fundamental changes have been made in using the equilibrium matrix. In [2] this method is employed for analyzing the structure directly; while here, we utilize it to fulfill the equilibrium and compatibility conditions between two regular structure and irregular elements of the structure.

In the past, extensive research has been carried out to analyze the structures having regular forms. Thus, finding the inverse of the stiffness matrix of a regular structure is easily possible. In this regard, we can mention canonical forms that use "Kronecker product" to provide block-diagonal form of stiffness matrix [15, 16, 19-22, 27, 28]. In the other hand, by group theory, we can also simplify calulating the eigen-values and then find the inverse of the stiffness matrix of regular structures in some cases [17, 18, 23].

2.1. Structural analysis by equilibrium matrix

In general, one can decompose a structure into its constituting components. In the following, nodal force equilibrium equations are created between the internal forces of structure's components. Thus, the equilibrium matrix, A, is formed as

$$A.Q = P \tag{1}$$

P is the vector of external loads and Q is the vector of internal forces of the components. The relation between the displacement vectors in local and global coordinate systems is presented in the following form:

$$A^t.\Delta = \delta \tag{2}$$

Δ and $\boldsymbol{\delta}$ are the nodal displacement vectors in the global and local coordinates system, respectively.

As is described in Ref. [8], analysis results can be drived using SVD of equilibrium matrix easily. Hence, the internal forces of structural components can be presented by the following relation:

$$Q = pinv\,(A).P - V_z.(V_z'.F.V_z)^{-1}.V_z'.F.pinv\,(A).p \tag{3}$$

V_z contains the right singular vectors of the equilibrium matrix corresponding to zero singular values. $Pinv$ is the notation used for the pseudo-inverse. F is the block-diagonal flexibility matrix of the structural components.

Following, the governing equilibrium equation in local coordinate system is as follows:

$$\delta = F.Q = F(pinv\,(A).P - V_s.(V_s'.F.V_s)^{-1}.V_s'.F.pinv\,(A).p) \tag{4}$$

Here, nodal displacement vector $\boldsymbol{\Delta}$ is obtained as:

$$\Delta = pinv\,(A^t).\delta - U_z.(G^t.U_z)^{-1}.G^t.pinv\,(A^t).\delta \tag{5}$$

U_z contains the left singular vectors of the equilibrium matrix corresponding to zero singular values. The columns of the matrix G are equivalent to the vector of geometric loads of the structure Ref. [4].

Considering the concept of the matrix $\mathbf{U}_z$ the following equation satisfies the equilibrium of the forces in the structure.

$$U_z^t.P = Z \tag{6}$$

The matrix U_z contains some modal forces which do not satisfy the equilibrium. This means no multiple of these modes can be sustained by the structure. Thus for the above relationship to hold, the structure should either be stable or it can be unstable but with the help of some external forces it should have been made stable.

In unstable structures, if Eq. (6) is satisfied, then G can be provided by the following orthogonal relationship that is taken from Chapter 4 of Ref. [4].

In this relation, the concept of vector of geometric load is utilized. This means a set of additional equations obtained from the equilibrating forces is used for calculating the vector of nodal displacements.

$$G^t.\Delta = Z \tag{7}$$

This is the same as the stiffness method; i.e. we assumed the compatibility to hold and we satisfy the equilibrium.

If the structure is stable, then $U_z = []$. . Therefore, the vector of nodal displacements of the structure is found as:

$$\Delta = pinv\,(A^t).\delta = pinv(A').F.(pinv(A).P - V_s.(V_s'.F.V_s)^{-1}.V_s'.F.pinv\,(A).P) \tag{8}$$

In the next section, the analysis of the structures with node irregularity is presented.

3. Analysis of Regular Structures with Nodal Irregularity

In the present section, analysis of regular structures with additional or lack of nodes is provided. On this basis, the previous analysis method will utilize for separate analysis of regular and irregular parts of structure.

The difference between the current paper and the previous one [8] by these authors is in the generalization of the structural irregularity from element irregularity to nodes and supports. However, this article expands element irregularity and analyzes structures that include elements and the nodes irregularities simultaneously; the basic concepts of both papers are similar.

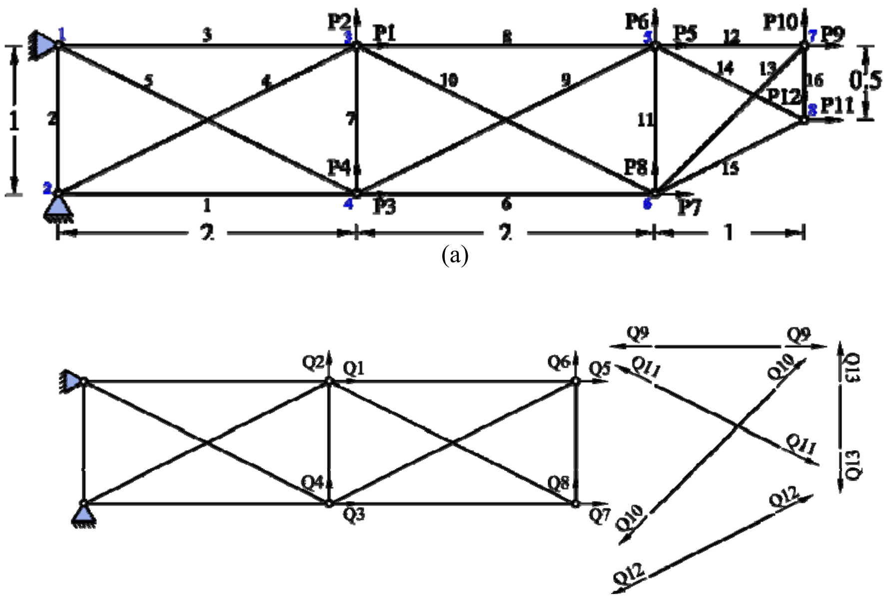
3.1. Analysis of regular structure with additional nodes

In this section, we describe an analysis method where nodes and elements are added to a regular structure.

Example: Consider a 16-bar truss structure as shown in Fig. 1(a). If nodes 7 and 8, along with elements 12 to 16 are separated from the structure, then the structure will be transformed into a regular structure form. Here, we consider the structure of Fig. 1(a) as an "irregular structure", nodes 7 and 8 as "irregular nodes" and elements 12 to 16 as "irregular elements". Therefore, the structural form resulted by the separation of the irregular nodes and elements from the irregular structure is called "regular structure" (Fig. 1(b)).

The external load of structure is assumed as following:

$$P = [0 \quad -10 \quad 0 \quad 0 \quad 0 \quad -20 \quad 0 \quad 0 \quad 40 \quad -30 \quad 0 \quad 0]^t$$



(a)

(b)

Fig. 1 (a) A truss structure with 16 bars and its DOFs in the global coordinates system. (b) Display of the irregular elements on the right hand side and regular structure on the left hand side of the figure.

Here, P_i represents the DOFs of the irregular structure and Q_i represents the DOFs of the regular structure in global coordinate system and irregular elements in local coordinate system.

3.1.1. Formation of the equilibrium matrix

The equilibrium matrix is created for providing equilibrium and compatibility conditions between two parts of the irregular elements and regular structure. To achieve this, the force equilibrium equations between irregular elements and the regular structure are formed.

The equilibrium matrix and its SVD matrices follow certain block form. It will be shown that the decomposition of whole parts of the equilibrium matrix is not required. In fact, the final results can be achieved by the analysis of a small part of this matrix. More, general form of the equilibrium matrix and simple method for its formation is provided.

3.1.2. General form of the equilibrium matrix

General form of the equilibrium matrix analyzed in this article is illustrated in Fig. 2.

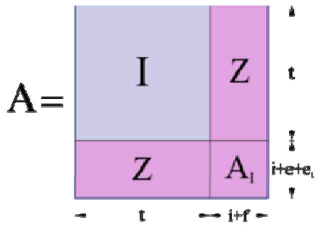


Fig. 2 The general equilibrium matrix form used in the analytical method of this article

Thus, the dimension of the equilibrium matrix can be introduced by the following equations:

$$m = t + i + e + e_1 \,,\; n = t + i + f \tag{9}$$

e is the number of DOFs of the additional nodes of the irregular structure. e_1 is number of lack of DOFs that are required to form a regular structure. t and i are the numbers of non-related and related DOFs of regular structure to irregular elements, respectively. f is the number of internal forces of the irregular elements. m is the number of DOFs of the irregular structure in the global coordinate system. n is total of number of DOFs of the regular structure in global coordinate and the number of DOFs of the irregular elements in local coordinate system.

A_1 is part of the equilibrium matrix which corresponds to the DOFs of the irregular elements and the related DOFs of the regular structure.

According to Fig. 2, instead of SVD of whole parts of the equilibrium matrix, we can decompose only the matrix A_1 Indeed, by analyzing a small part of the equilibrium matrix, the analysis of irregular structure can be performed. General form of matrix A_1 is as the form shown in Fig. 3.

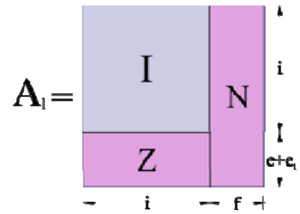


Fig. 3 General form of the matrix $\mathbf{A}_1$

It should be noted that the matrix A_1 compared to the previous article of these authors [8] has changed, due to the added or lost degrees from the regular structure. In this method by extending the concept of matrix N, one can generate the process of irregular structure analysis to a situation in which the structural DOFs can be altered.

3.1.3. The rapid formation of matrix A_1

Here, simple formation of the matrix A_1 is provided. A_1 can be formed easily by assembling the rotation matrix of the irregular elements. According to the form of matrix A_1 presented in Fig. 3, for the formation of this matrix only the assembly of matrix N is required. Therefore, this matrix can be assembled as follows:

If h is the DOFs of the two end nodes of the j th irregular element in local coordinates and g is the DOFs of the two ends of jth irregular element in the global coordinates (when it is connected to the irregular structure), then the columns of matrix N correspond to the degrees h as follows:

$$N(g,h) = T_j^t \tag{10}$$

Other rows of these columns are zero. We should repeat this process for all the irregular elements. T_j is the modified rotation matrix of the jth irregular element. For truss elements, this matrix contains the cosine of the element conductors. This matrix is displayed in the following form:

For space truss elements

$$T_j = [T_1 | -T_1];\, T_1 = [COS\theta \quad COS\beta \quad COS\gamma]$$

For two-dimensional frames

$$\mathbf{T}_j = \left[\mathbf{T}_1 \mid \mathbf{T}_3\right] \quad ; \quad \mathbf{T}_3 = \mathbf{s}_1^{-1}.\mathbf{s}_2.\mathbf{T}_2 \quad ; \quad \mathbf{s}_j = \left[\begin{array}{c|c} \mathbf{s}_1 & \mathbf{s}_2 \\ \hline \mathbf{s}_2 & \mathbf{s}_1 \end{array}\right]$$

$$\mathbf{T}_1 = \begin{bmatrix} Cos\theta & Sin\theta & 0 \\ -Sin\theta & Cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \; ; \; \mathbf{T}_2 = \begin{bmatrix} -Cos\theta & -Sin\theta & 0 \\ Sin\theta & -Cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

$\mathbf{s}_j$ is the stiffness matrix of j th element in the local coordinates system. θ, β and γ are the element angles to x, y and z coordinate axes, respectively. By SVD of the matrix $\mathbf{A}_1$, we can easily reach to decomposition of the equilibrium matrix and then analysis results of the irregular structure. For this purpose, we act as described in the following section.

3.1.4. The SVD of the equilibrium matrix by decomposition of A_1

In Fig. 2, it was observed that the equilibrium matrix has a block diagonal form and its main part is a unit matrix. On this basis, decomposition of the equilibrium matrix can be limited into the decomposition of A_1. For this purpose, we from the SVD of the A_1 using the following equation:

$$\mathbf{A}_1 = \mathbf{U}_1 . \mathbf{W}_1 . \mathbf{V}_1^t = \left[\mathbf{U}_{11} \mid \mathbf{U}_{12} \right] . \left[\begin{array}{c|c} \mathbf{D}_1 & \mathbf{Z} \\ \hline \mathbf{Z} & \mathbf{Z} \end{array} \right] . \left[\begin{array}{c} \mathbf{V}_{11}^t \\ \hline \mathbf{V}_{12}^t \end{array} \right] \tag{12}$$

By substituting these matrices in the following forms, the SVD of the equilibrium matrix can be obtained as illustrated in Fig. 4.

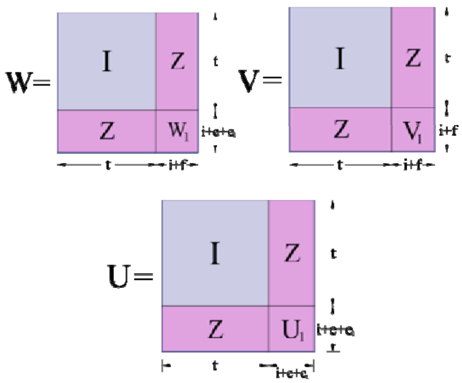


Fig. 4 Matrices containing the left and right singular vectors and the singular values matrix of the equilibrium matrix

Partitioning of the singular matrices V_1 and U_1 accordance with zero columns and rows of the matrix W_1, respectively, as presented in Fig. 5.

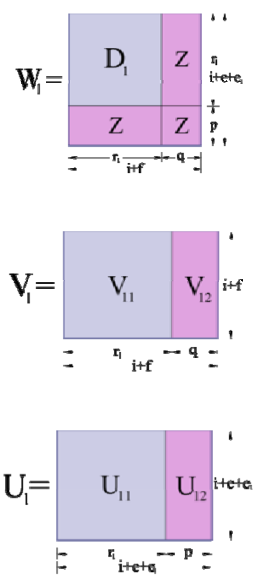


Fig. 5 Block presentation of the singular matrices for SVD of A_1

In the above figure, we have $r_1 = rank(A_1)$ and obviously, if the structure is stable, p=0. D_1 is a diagonal square matrix containing the nonzero singular values of the matrix W_1 Matrices V_{11} and V_{12} are the set of columns of the matrix V_1 corresponding to non-zero and zero singular values of the matrix A_1, respectively. The matrix, $pinv(A_1)$ can be obtained from the SVD of the matrix A_1 as:

$$pinv\ (A_1) = V_{11} . D_1 . U_{11}^t \tag{13}$$

Similarly, the general form of the pseudo-inverse of the equilibrium matrix can be derived by the form shown in Fig. 6.

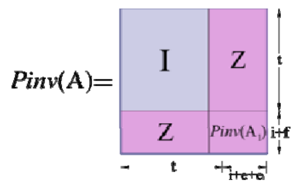


Fig. 6 Display of the pseudo-inverse of the equilibrium matrix

In the coming sections of the article, depending on the irregularity of the problem, merely by decomposition of the matrix A_1 and determining the required parameters (t, i, e, e_1, q and p), the above matrices can easily be formed for any types of irregularity.

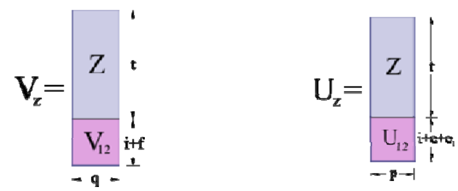


Fig. 7 Formation of $\mathbf{V}_z$ and $\mathbf{U}_z$ accordance with $\mathbf{V}_{12}$ and $\mathbf{U}_{12}$

For the example of this section, the required parameters are $t = 4$, $i = 4$, $e = 4$, $e_1 = 0$ and $f = 5$. Thus from Figs. 2 and 3 for this structure we have:

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{I}_4 & \mathbf{Z} \\ \hline \mathbf{Z} & \mathbf{A}_1 \end{array} \right] \quad ; \quad \mathbf{A}_1 = \left[\begin{array}{c|c} \mathbf{I}_4 & \mathbf{N} \\ \mathbf{Z}_4 & \end{array} \right]$$

$$\mathbf{N} = \begin{bmatrix} -1 & -0.8944 & 0 & 0 & 0 \\ 0 & 0.4472 & 0 & 0 & 0 \\ 0 & 0 & -0.7071 & -0.8944 & 0 \\ 0 & 0 & -0.7071 & -0.4472 & 0 \\ 1 & 0 & 0.7071 & 0 & 0 \\ 0 & 0 & 0.7071 & 0 & 1 \\ 0 & 0.9844 & 0 & 0.8944 & 0 \\ 0 & -0.4472 & 0 & 0.4472 & -1 \end{bmatrix}$$

We perform the SVD of the matrix A_1. Matrices obtained are as follows:

$$\mathbf{V}_1 = \left[\begin{array}{cccccccc|c}
0.2620 & -0.3629 & 0.1367 & -0.1189 & 0.0848 & -0.1597 & 0.6273 & 0.5835 & 0 \\
-0.0612 & 0.0880 & -0.0473 & -0.3281 & 0.4945 & -0.7532 & -0.0989 & -0.1453 & 0.1889 \\
0.2247 & 0.3136 & 0.0720 & 0.0715 & 0.4639 & 0.3080 & 0.5478 & -0.4802 & 0 \\
0.1665 & 0.2194 & -0.0500 & -0.1030 & -0.6844 & -0.4174 & 0.3466 & -0.3351 & -0.1889 \\
-0.4515 & 0.4049 & -0.0600 & 0.5609 & -0.0506 & -0.1796 & 0.2607 & 0.2610 & 0.3779 \\
-0.4430 & 0.4272 & -0.1516 & -0.5309 & 0.0521 & 0.1815 & 0.1342 & 0.2898 & -0.4225 \\
-0.4962 & -0.3841 & 0.3485 & 0.2840 & 0.1222 & -0.1742 & 0.1249 & -0.2396 & -0.5345 \\
-0.4205 & -0.4568 & -0.3907 & -0.2824 & -0.1211 & 0.1748 & 0.2729 & -0.2892 & 0.4225 \\
-0.1614 & 0.1058 & 0.8189 & -0.3245 & -0.1721 & 0.1085 & -0.0163 & -0.0473 & 0.3779
\end{array}\right]$$

$$\mathbf{V}_{12} = \begin{bmatrix} 0 & 0.1889 & 0 & -0.1889 & 0.3779 & -0.4225 & -0.5345 & 0.4225 & 0.3779 \end{bmatrix}^{t}$$

$$\mathbf{W}_1 = \left[\mathbf{D}_1 \mid \mathbf{Z}_{8\times 1} \right] \quad ; \quad \mathbf{D}_1 = diag\ \{2.0579 \quad 1.7801 \quad 1.5593 \quad 1.3128 \quad 1.0233 \quad 0.9445 \quad 0.6269 \quad 0.3293\}$$

It can be seen that the matrix W_1 has no zero rows. Indeed structure is stable and $p = 0$, $q = 1$. The value of q can be obtained as $q = 5 - 4 = 1$. Using the matrices obtained by the above decomposition and Eq. (13), the pseudo-inverse of the matrix A_1 can be obtained as follows:

$$Pinv\,(\mathbf{A}_1) = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & -1 & 0.5 & -1 \\
0 & 0.9642 & 0 & 0.0357 & -0.0714 & 0.25 & -0.2321 & 0.3214 \\
0 & 0 & 1 & 0 & 0 & 1 & 0.5 & 1 \\
0 & 0.0357 & 0 & 0.9642 & 0.0714 & 0.75 & 0.2321 & 0.6785 \\
0 & -0.0714 & 0 & 0.0714 & 0.8571 & -0.5 & 0.0357 & -0.3571 \\
0 & 0.0798 & 0 & -0.0798 & 0.1597 & -0.5590 & 0.5190 & -0.7187 \\
0 & 0.1010 & 0 & -0.1010 & 0.2020 & 0.7071 & -0.0505 & 0.5050 \\
0 & -0.0798 & 0 & 0.0798 & -0.1597 & 0.5590 & 0.5989 & 0.7187 \\
0 & -0.0714 & 0 & 0.0714 & -0.1428 & 0.5 & 0.0357 & -0.3571
\end{bmatrix}$$

By Figs. 4, 5 and 6, the singular value decomposition and pseudo-inverse of the equilibrium matrix (A) are obtained as follows:

$$\mathbf{U} = \left[\begin{array}{c|c} \mathbf{I}_4 & \mathbf{Z}_{4\times 8} \\ \hline \mathbf{Z}_{8\times 4} & (\mathbf{U}_1)_8 \end{array}\right] \ ; \ \mathbf{W} = \left[\begin{array}{c|c} \mathbf{I}_4 & \mathbf{Z}_{4\times 9} \\ \hline \mathbf{Z}_{8\times 4} & (\mathbf{W}_1)_{8\times 9} \end{array}\right] \ ; \ \mathbf{V} = \left[\begin{array}{c|c} \mathbf{I}_4 & \mathbf{Z}_{4\times 9} \\ \hline \mathbf{Z}_{9\times 4} & (\mathbf{V}_1)_9 \end{array}\right]$$

$$Pinv\,(\mathbf{A}) = \left[\begin{array}{c|c} \mathbf{I}_4 & \mathbf{Z}_{4\times 8} \\ \hline \mathbf{Z}_{9\times 4} & Pinv\,(\mathbf{A}_1)_{9\times 8} \end{array}\right] \quad ; \quad Pinv\,(\mathbf{A}^{t}) = \left[Pinv\,(\mathbf{A}) \right]^{t}$$

After the decomposition of the equilibrium matrix, the formation of the flexibility matrix of the structure is carried out.

3.1.5. Structural flexibility matrix

F is the block diagonal flexibility matrix of the structure. As it was stated before, the purpose of the analysis by the equilibrium matrix is the separate analysis of the regular and irregular parts of structure.

Indeed F contains the flexibility matrix of the regular structure and the irregular elements. Hence, we consider the inverse of structural stiffness matrix as regular structure felexibility matrix. Indeed, in this process, the regular structure is analyzed. Thus, flexibility matrix can be formed as:

$$\mathbf{F} = \left[\begin{array}{c|c} \mathbf{S}^{-1} & \mathbf{Z} \\ \hline \mathbf{Z} & \mathbf{F}_e \end{array}\right] \qquad (14)$$

Where S^{-1} is inverse of the stiffness matrix of the regular structure and F_e is the block-diagonal flexibility matrix of the irregular elements with dimension $f \times f$.

In this example, the inverse of the structural stiffness matrix, S^{-1}, has dimension 8×8 and the matrix F_e is as follows:

$$F_e = diag\{1 \quad 1.4142 \quad 1.1180 \quad 1.1180 \quad 0.5\}$$

According to the block diagonal form of F, we can conclude that the regular and irregular parts of structure are analyzed independently. To form the inverse of the stiffness matrix of the regular structure we could use the presented methods in references [19].

3.1.6. Concluding points about the method

According to the matrices obtained, we use equilibrium matrix SVD to calculate the nodal displacement of the irregular structure. Using the matrices V_z, F, $pinv(A)$ and P, Q is formed as follows:

$$\mathbf{Q} = \begin{array}{c} \begin{matrix} \ 1 & \ 2 & 3 & 4 & 5 & & 9 & 10 & 11 & 12 & 13 \end{matrix} \\ \begin{bmatrix} 0 & -10 & 0 & 0 & 70 & \ldots & 50.7356 & 21.5382 & -15.1825 & -21.5382 & -19.2644 \end{bmatrix}^{t} \end{array}$$

By Eq. (4), the vector **δ** is obtained similarly:

$$\boldsymbol{\delta}=\begin{array}{c}\begin{array}{ccccccccc}1&2&3&4&&10&11&12&13\end{array}\\ \left[\begin{array}{ccccccccc}444.61&-1181.65&-395.38&-1169.17&\dots&30.45&-16.97&-24.08&-9.63\end{array}\right]^{t}\end{array}$$

Finally, by vector **δ** and Eq. (8), the nodal displacement vector of structure can be derived as follows:

$$\Delta=\begin{array}{c}\begin{array}{ccccccccc}1&2&3&4&&9&10&11&12\end{array}\\ \left[\begin{array}{ccccccccc}444.61&-1181.65&-395.38&-1169.17&\dots&720.83&-4845.53&55.72&-4835.9\end{array}\right]^{t}\end{array}$$

3.1.7. An application

In optimization techniques based on iteration in each step the variables change according to some pattern and in order to control the effects analysis is repeated. Thus in optimization a great deal of time is allocated to the analysis. Using the present method, the analysis can be simplified, since in each step the results of the previous step can be incorporated in the analysis. In fact the analysis of each step can be considered as the results of the regular structure of the subsequent step and only the additional changes can be incorporated. In this way a limited amount of operations will be needed for performing the analysis.

3.2. Analysis of regular structures having node shortage

To analyze the structures that have lost their regularity due to the lack of nodes, we take the following procedure:

In place of the node shortage, we put two pairs of imaginary nodes. Instead of the required structural elements, two sets of appropriate elements with asymmetrical module of elasticity are inserted. Each set of elements with positive and negative modulus of elasticity will be connected to one of the assumed pairs of node sets. The appropriate degrees of freedom are considered for each of the assumed.

In the next step, the regular structure is formed. We take this action by separating the elements with negative modulus and its related nodes from the already formed structure. Elements with negative modulus are considered as "Irregular elements" and their related nodes are as "Irregular nodes". Equilibrium matrix of the structure is established based on the equilibrium equations between regular and irregular parts of structure. By the above process, $2e_1$ degrees of freedom should be added to the degrees of freedom of the irregular structure, where e_1 is the number of degrees of freedom that is needed to alter the structure into a regular form.

4. Analysis of regular structure with irregular supports

Sometimes, structural irregularity is not related to the structural geometry, but it is because of support conditions. Indeed, by modifying the support conditions, the structure will have simple analysis method. In this section, structures with this type of irregularity are analyzed by improving the support conditions.

The main process of this analysis is similar to that of the previous section. But at some points there are differences that are described through a simple example. In the following, first the analysis of structures with irregular or lack of supports is studied. Then, the analysis of structures with additional support is described.

4.1. Lack of supports or inappropriate supports

If there are disproportionate supports in a regular structure, we can assume these supports as "irregular nodes" and by separating them from the structure, and we can replace new support in their places. For this purpose, we act similar to the procedure of suggested for the analysis of the regular structure with nodes irregularity. Here, first the irregular supports with their connected elements are separated. We assume two pairs of appropriate imaginary support nodes in place of irregular support nodes. Then, we should put pairs of elements with asymmetric modulus of elasticity in place of the required elements of new supports. Sets of elements with positive and negative module of elasticity should be connected to the new support nodes, separately. Elements with negative modulus and their corresponding support nodes are "irregular elements and nodes". Here, the parameter e is the number of DOFs of the separated irregular supports and e_1 is the number of DOFs of the added supports.

Note: if there is a lack of structural support, we should act similar to the above process. However, there are no inappropriate supports to be separated from structure. This means that $e=0$. In the following, an example is presented for illustration of this problem.

Example: Consider truss structure with 11 bars as shown in Fig. 8(a). If the support node 5 becomes fixed, then the analysis will be simplified.

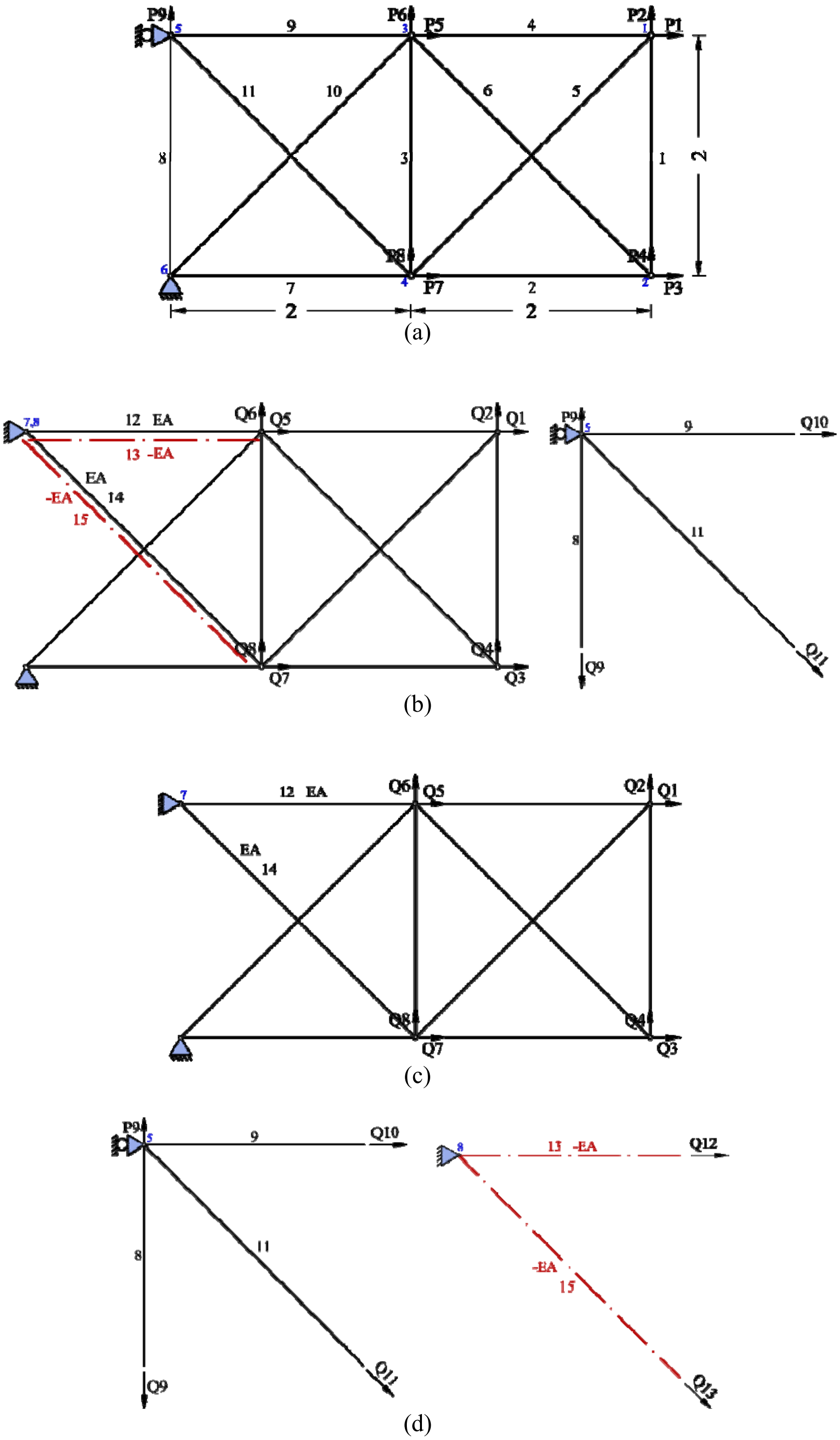


Fig. 8 (a) A truss structure with 11 bars (b) Improved structure by imaginary nodes and elements along with the irregular separated support and its connected elements (c) DOF of the regular structure (d) Separated parts of the structure with their DOFs in local and global coordinate systems.

For the analysis of this structure, first we separate the irregular support in node 5 and its connected elements. An appropriate pair of support nodes (nodes with numbers 7 and 8) and two pairs of elements with asymmetric elasticity modulus are replaced (Fig. 8 (b)).

Therefore, the irregular elements consist of the elements 8, 9, 11, 13 and 15, and the irregular nodes are as nodes 5 and 8. According to the described details, used parameters in the method will be as:

$e = 1$, $e_1 = 0$, $f = 5$ and $i = t = 4$

The matrix A_1 can simply be formed:

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{I}_{4\times4} & \mathbf{Z} \\ \hline \mathbf{Z} & \mathbf{A}_1 \end{array}\right] \quad ; \quad \mathbf{A}_1 = \left[\begin{array}{c|c} \mathbf{I}_{4\times4} & \mathbf{N} \\ \hline \mathbf{Z}_{1\times4} & \end{array}\right]_{5\times9}$$

$$\mathbf{N} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7071 & 0 & 0.7071 \\ 0 & 0 & -0.7071 & 0 & -0.7071 \\ 1 & 0 & 0.7071 & 0 & 0 \end{bmatrix}$$

We find that p=0 and q=4, thus the structure is stable.

In the following, we calculate the SVD of the matrix V_1. With this matrix and Fig. 6, $PinvA_1$ is provided as a matrix of dimension of 13×9 and the matrix V_z is obtained with dimension 13×4. Flexibility matrix (F) is obtained by inversion of the stiffness matrix of regular structure and the flexibility matrix of the truss element is a 1×1 matrix:

$$\mathbf{F} = \left[\begin{array}{c|c} \mathbf{S}^{-1} & \mathbf{Z} \\ \hline \mathbf{Z} & \mathbf{F}_e \end{array}\right] \quad ; \quad \mathbf{F}_e = diag\{2 \quad 2 \quad 2.8284 \quad -2 \quad -2.8284\}$$

Using matrices $\mathbf{V}_z$, $Pinv(A)$, F and P, we can obtain the vector Q and vector δ. The vector Δ is obtained by the above and Eq. (8) as follows:

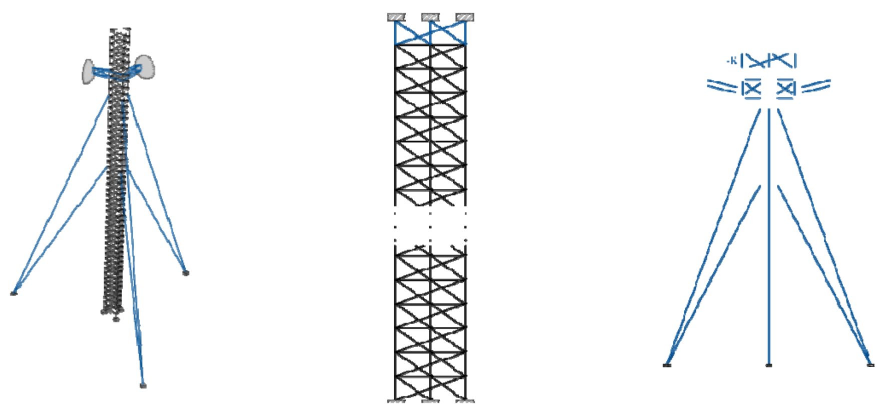
$$\Delta = [41.876 \quad -165.630 \quad -38.123 \quad -15.569 \quad 30.938 \quad -61.876 \quad -29.061 \quad -63.753 \quad -9.061]^t$$

4.2 Analysis of regular structures with additional supports

Analysis of a regular structure with additional supports is a particular case of analysis method of a regular structure with irregular supports. Thus, in this case we should separate additional supports and their connected elements from the irregular structure. These are introduced as "irregular nodes and elements". Since in this method we do not add any node to the structure, therefore we have always e_1=0. The process of the formation of the equilibrium matrix and the analysis of regular and irregular separated parts of structure are similar to the method of the previous section. In the following, the algorithm for the analysis of regular structure with variety of irregularities is provided.

5. Practical examples

Example 1: Consider a telecommunications antenna shown in Fig. 9(a). The central truss part of the structure has a regular form. However, the attached parts (for the dish's installation) and also the cables have made structure irregular. Here, the regular and irregular parts of structure are analyzed separately. Irregular structure has 154 free nodes and 470 truss elements.



(a) (b) (c)

Fig. 9 (a) A three dimensional view of the antenna truss structure with 154 free nodes. (b) Regular structure with its ends closed by 6 imaginary elements. (c) Sets of irregular elements separated from the irregular structure. In top of figure, elements with negative modules of elasticity are shown.

External load of the structure in dish connection nodes are equal to P=10 N. And for all elements we assume $E\times A=1$. We separate the dish holder elements and cables as irregular elements from structure. If the end nodes at the top of the regular structure are fixed, then it can easily be analyzed by the method of Ref. [19]. For creating this case, we assume 6 pairs of elements with asymmetric modulus of elasticity (similar to the elements of structure) at the end nodes of the structure. One end of these elements is connected to the structural nodes and the other end is connected to the fixed support. Thus, the end nodes of structure will be fixed.

The elements with negative modules of elasticity added to the irregular elements. Thus, the central part of the structure becomes equivalent to a regular structure of Ref. [19] (Fig. 9(b)). The irregular elements consist of 14 holder truss elements, 6 cable elements and 6 elements with negative modules of elasticity (Fig. 9(c)). The corresponding parameters are equal to i=45, t=405, e=12, e_1=0 and f=26.

Matrix A_1 with dimension 57×71 can be formed by rotation matrix of irregular elements and Eq. (10) and Fig. 3. Then its singular value decomposition is performed. Thus, we have q=14 and p=0. On this basis, the rank of the matrix A_1 is 57 (r_1=57) and the structure is stable. The matrix V_1 can be partitioned according to the zero columns of the matrix A_1. Resulted matrices V_{11} and V_{12}have dimensions 71×57 and 71×14, respectively.

Using the pervious matrices, we obtain the $Pinv(A_1)$ as a 71×57 matrix from Eq. (13). Hence, the matrices V_z and $PinvA$ can be formed with dimensions 476×14 and 476×462, respectively, by substituting matrices V_{12} and $Pinv(A_1)$ in the forms of Figs. 6 and 7.

The stiffness matrix of the regular structure will have the following form [19]:

$$S = F_{50}(2K, -K, 2K)$$

Inverse of the stiffness matrix of the regular structure with dimension 450×450 can be obtained by calculating the eigen-values of fifty 9×9 blocks matrices [19]. These eigen-values can be formed as follows:

$$\lambda = \bigcup_{i=1}^{50} \lambda_i.K; \lambda_i = 2\left(1 - Cos\left(\frac{i\mu}{51}\right)\right)$$

K is a 9×9 square matrix. Thus, the flexibility matrix F with dimension 476×476 can be obtained by inverse of the stiffness matrix of regular structure and the flexibility matrix of the irregular elements (F_e).

By Eq. (3), the vector Q and by Eq. (4), we will have the vector δ. Both of these are of dimension 476×1.

$$\mathbf{Q} = \begin{array}{c} \begin{array}{cccccccccc} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & & {\scriptstyle 473} & {\scriptstyle 474} & {\scriptstyle 475} & {\scriptstyle 476} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & -236.565 & -307.858 & 317.1463 & -6.2087 \end{bmatrix}^t \end{array}$$

$$\boldsymbol{\delta} = \begin{array}{c} \begin{array}{cccccccccc} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & & {\scriptstyle 473} & {\scriptstyle 474} & {\scriptstyle 475} & {\scriptstyle 476} \end{array} \\ \begin{bmatrix} -0.5713 & 22.6313 & -1.8584 & 0.0953 & \ldots & 709.6964 & 3214.135 & -3311.1 & 64.8211 \end{bmatrix}^t \end{array}$$

The nodal displacement is obtained by Eq. (8) as follows:

$$\Delta = \begin{array}{c} \begin{array}{cccccccccc} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & & {\scriptstyle 459} & {\scriptstyle 460} & {\scriptstyle 461} & {\scriptstyle 462} \end{array} \\ \begin{bmatrix} -0.5713 & 22.6313 & -1.8584 & 0.0953 & \ldots & -3523.77 & -61.5583 & -2045.8 & -3538.48 \end{bmatrix}^t \end{array}$$

For direct analysis, we should find the inverse of a matrix of dimension 462×462. While by the presented method, we need only the SVD of a 57×71 matrix, inversion of a 14×14 matrix, and finding the eigen-values of 50 matrices with dimensions 9×9. Indeed, in this example, instead of analyzing an irregular structure with large degrees of freedom, we convert the problem to the analysis of sets of small substructures and small matrices for assembling the results of the separated parts of the structure.

Example 2: Consider a bending structure as shown in Fig. 10. This structure is composed by 24 stories and each story has 91 columns and 162 beams. These are the same for all the stories. Parts of the structure in the last two floors are not extended, and therefore it becomes irregular. On the other hand, 48 truss bars, for increasing the lateral stiffness of structure are installed. Thus, this structure is affected by two types of irregularities in its form. First, that truss bars that do not follow a specific regularity form and the other, no extension of the top part of the building that is added to structural irregularities.

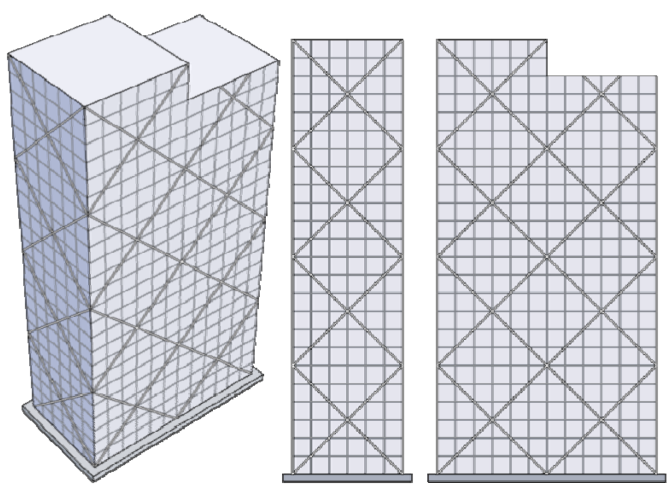
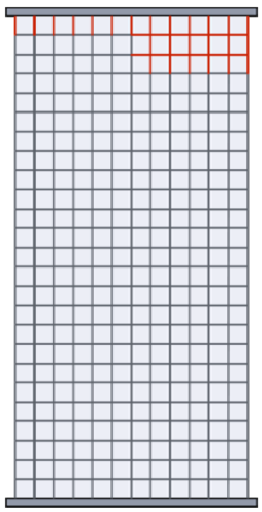


Fig. 10 A three-dimensional view and the side and front views of a 24-storey tower structure

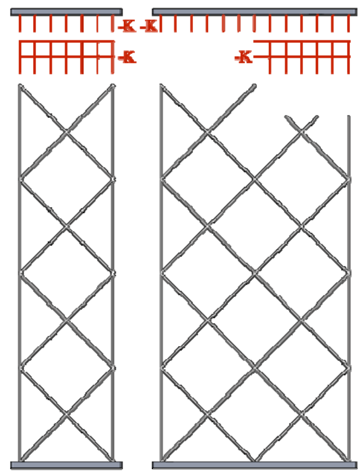
Notice that the irregularities in the upper part of the structure can be considered as lack of nodes. Thus, we can assume pairs of nodes and elements with asymmetric modulus of elasticity in the appropriate places and call them "first assumed part". We separate the elements with negative modulus in this part and truss elements from structure. Thus, the lack of elements in end part of the structure is compensated resolved.

If the end nodes of the bending structure in its top were fixed, inversion of the stiffness matrix could easily be by reference [19]. For this purpose, we attach to end of the structure, set of pairs of bending elements (columns) with asymmetric modulus of elasticity. These are connected to the rigid support from their next ends. Thus, we call them "second assumed part".

We separate elements with negative modulus of this part from the structure as irregular elements. Thus, a regular structure that has simple analysis method is formed (Fig. 11(a)). Irregular elements (those elements with negative modulus of the first and second assumed parts of structure and truss elements) are shown in Fig. 11(b).



(a)



(b)

Fig. 11 (a) A regular bending structure that becomes regular by adding imaginary elements. (b) Front and side views of the irregular elements of the structure (truss bars in bottom, the first assumed part in the middle, and the second assumed part at the top of the figure are shown).

The number of elements with negative modulus in the first assumed part is 240 and in the second assumed part it is 91. Thus the discussed parameters in this method are i=840, t=11760, $f=2034$, $e=0$, $p=0$ and $e_1=504$.

Matrix A_1 established by the rotation matrix of the irregular elements in Eq. (10) and Fig. 3 with dimension 1344×2874.

We calculate the SVD of this matrix. In this process, we have $q=1530$, $p=0$ and the non-zero singular value matrix (D_1) is of dimension 1344×1344. For the formation of vector Q, inversing a matrix 1530×1530 will be needed. According to Eq. (13), the matrix $Pinv(A_1)$ is formed with dimension 2874×1344. Thus the matrix V_z with dimension 15138×1530 and matrix $Pinv(A)$ with dimension 15138×13608 can be established by Figs. 6 and 7.

For the formation of the flexibility matrix (F) by Eq. (14) we form the inverse of the stiffness matrix of the regular structure. This matrix can be obtained by calculating the eigen-values of 24 matrices with dimension 546×546 utilizing the method of Ref. [19]. The value 546 in dimension of these matrices represents the number of DOFs of each floor of the regular structure (that is as repeated substructure of regular structure). Thus it can be observed that in this example, we transform the analysis of "irregular structure" to the analysis of "substructures of the regular structure".

In Eq. (14), the matrix of F_e is with dimension 2034×2034. Thus, the vector Q from Eq. (3) and similarly the vector δ from Eq. (4) are established. Both of these vectors are of dimension 15138×1. In Eq. (8), the nodal displacement vector is obtained as 13608×1 a vector.

Here, we only attend to form the rotation matrix and the stiffness matrix of bending elements in Eq. (11). Thus, instead of direct analysis that require the inversion of a 12600×12600 matrix, we analyze the irregular structure by a more simple method through calculating the SVD of a 1344×2874 matrix, inversing of a 1530×1530 matrix, and finding the eigen-values of 24 matrices of dimension 546×546.

Example 3: Consider cable bridge as shown in Fig. 12(a). The deck of the bridge is a truss structure that is formed by repetition of a substructure. We can analyze this part of the structure simply by [19], if it be independent of other parts. Thus the holder cables make the structure irregular. Structure is symmetric, and two side parts are identical. Here, we call the side parts as "first part" and the middle part as "second part" and these are analyzed simultaneously.

First part of structure has 110 free nodes of the regular structure and 4 irregular nodes and 10 cable irregular elements, while the second part has 470 free nodes of the regular structure, 22 irregular nodes and 66 cable irregular elements.

Based on the above figures, parameters of the first part of the structure are $t = 318$, $i = 12$, $f = 10$, $e = 12$ and for the second part we have $t = 1344$, $i = 66$, $f = 66$, $e = 66$.

The matrix A_1 for the first and second parts of structure with dimension 24×22 and 132×132, respectively, and we calculate their singular value decomposition.

However, since the external load of the structure is the gravity load, the structure will be stable (gravity load and loads caused by elongation of the cables have no components on vertical direct to the page). On this basis, we can analyze the structure by the present method.

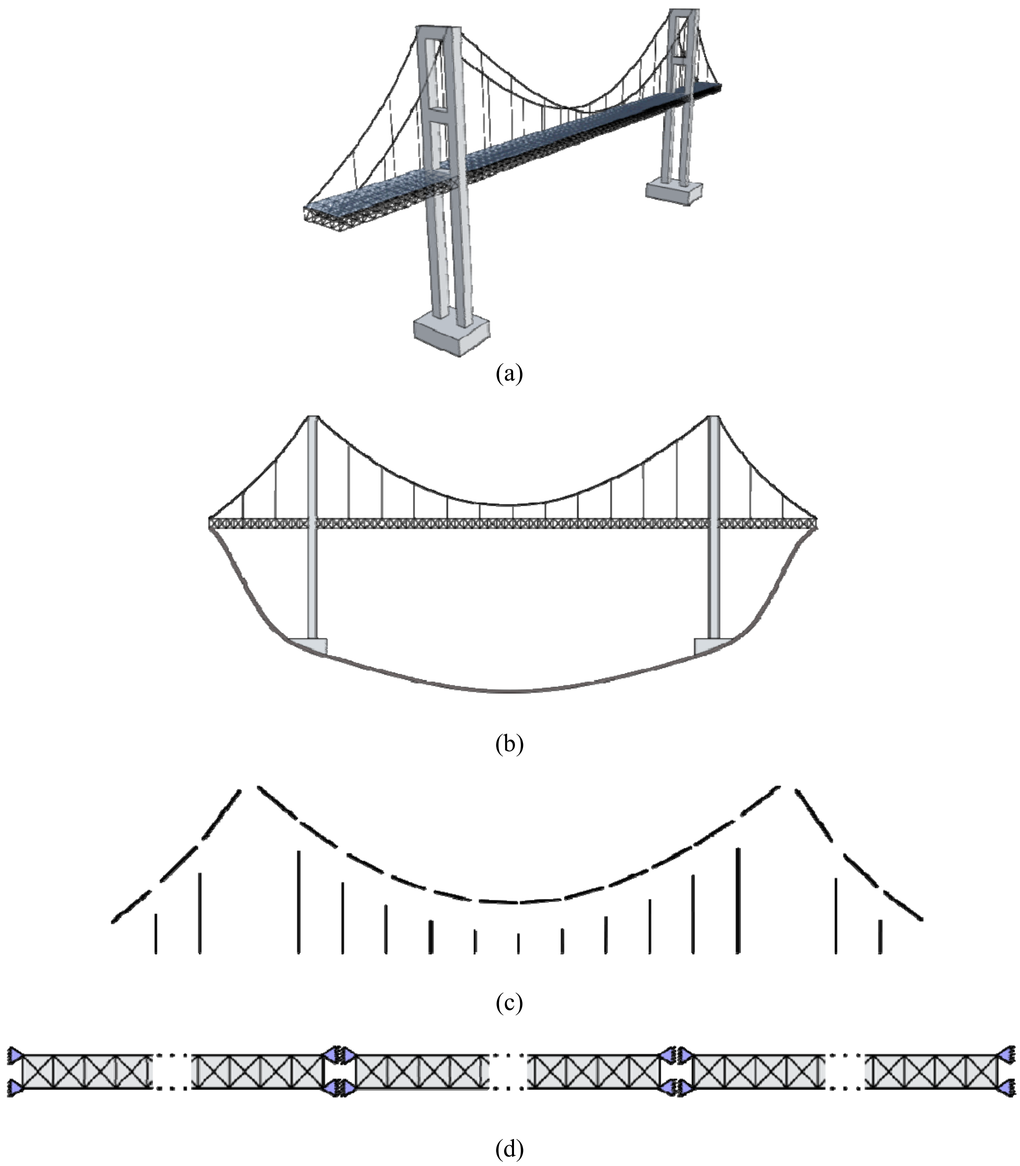


Fig. 12 (a) A display of the three-dimensional structure of the cable bridge (irregular structure) discussed in Example 3. (b) Front view of the irregular structure. (c) Display of the irregular elements (cables) that are separated from the structure. (d) View of the separated regular structure, including the side and middle parts.

By decomposition of A_1, for the first part of the structure we have $r_1 = 20$, $q = 2$, $p = 4$ and for the second part, we have $r_1 = 110$, $q = 22$, $p = 22$. Since $p \neq 0$, therefore both parts of structure are unstable. It is interesting to note that the value of p is equal to the number of DOFs orthogonal to the page in irregular nodes of the structure in Fig. 12(b).

Hence, according to Eq. (12) and Fig. 5, by partitioning the matrices according to the rank of matrices A_1, we have:

$$\mathbf{V}_1 = \left[(\mathbf{V}_{11})_{22\times 20} \mid (\mathbf{V}_{12})_{22\times 2} \right] \;\; ; \;\; \mathbf{U}_1 = \left[(\mathbf{U}_{11})_{24\times 20} \mid (\mathbf{U}_{12})_{24\times 4} \right] \qquad \text{For second part}$$

$$\mathbf{V}_1 = \left[(\mathbf{V}_{11})_{132\times 110} \mid (\mathbf{V}_{12})_{132\times 22} \right] \;\; ; \;\; \mathbf{U}_1 = \left[(\mathbf{U}_{11})_{132\times 110} \mid (\mathbf{U}_{12})_{132\times 22} \right] \qquad \text{For second part}$$

Pseudo-inverses of the matrices A_1 are obtained by Eq. (13) for all parts with dimensions 22×24 and 132×132, respectively. we can form V_z U_z and $Pinv(A)$ as follows:

$$\mathbf{V}_z = \left[\begin{array}{c} \mathbf{Z}_{318\times 2} \\ \hline (\mathbf{V}_{12})_{22\times 2} \end{array} \right] \;\; ; \;\; \mathbf{U}_z = \left[\begin{array}{c} \mathbf{Z}_{318\times 4} \\ \hline (\mathbf{U}_{12})_{24\times 4} \end{array} \right] \;\; ; \;\; Pinv\,(\mathbf{A}) = \left[\begin{array}{c|c} \mathbf{I}_{318} & \mathbf{Z} \\ \hline \mathbf{Z} & (Pinv\,(\mathbf{A}_1))_{22\times 24} \end{array} \right]$$

For the second part:

$$\mathbf{V}_z = \left[\begin{array}{c} \mathbf{Z}_{1344\times 22} \\ \hline (\mathbf{V}_{12})_{132\times 22} \end{array} \right] \;\; ; \;\; \mathbf{U}_z = \left[\begin{array}{c} \mathbf{Z}_{318\times 22} \\ \hline (\mathbf{U}_{12})_{132\times 22} \end{array} \right] \;\; ; \;\; Pinv\,(\mathbf{A}) = \left[\begin{array}{c|c} \mathbf{I}_{1344} & \mathbf{Z} \\ \hline \mathbf{Z} & (Pinv\,(\mathbf{A}_1))_{132\times 132} \end{array} \right]$$

Stiffness matrices of the regular structure (deck of bridge) are 330×330 and 1410×1410 matrices, respectively. Form of these in [22] are $S = F_{11}(A, B, A)$ and $S = F_{47}(A, B, A)$, respectively, where $A = 2B$ and B is a 30×30 matrix. Thus, by [19] for calculating the inverse of the stiffness matrices of the regular structures, we only need to calculate the eigen-values of 11 and 47 matrices with dimensions 30×30, respectively.

Flexibility matrices of the structure are obtained as follows:

$$\mathbf{F} = \left[\begin{array}{c|c} (\mathbf{S}^{-1})_{330\times 330} & \mathbf{Z} \\ \hline \mathbf{Z} & (\mathbf{F}_e)_{10\times 10} \end{array} \right] \qquad \text{For first part of structure}$$

$$\mathbf{F} = \left[\begin{array}{c|c} (\mathbf{S}^{-1})_{1410\times 1410} & \mathbf{Z} \\ \hline \mathbf{Z} & (\mathbf{F}_e)_{66\times 66} \end{array} \right] \qquad \text{For second part of structure}$$

F_e is the block diagonal flexibility matrix of the irregular elements (cables). The vectors Q is obtained with dimensions 22×1 and 132×1, respectively. Similarly, the vectors $\boldsymbol{\delta}$ is obtained with dimensions 22×1 and 132×1, respectively. Because of the instability of the structure we need to vectors δ in Eq. (5), the nodal displacement vectors are obtained.

Here, the analysis of the irregular structure are performed by calculating the SVD of 24×22 and 132×132 matrices and finding the inverse of matrices of dimensions 2×2 and 4×4, and two 22×22 matrices and finding eigen-values of 58 matrices with dimensions 30×30. While due to instability of structure, direct analysis of structure by inversing their stiffness matrices (340×340 and 1476×1476 matrices for two parts of structure, respectively) is not possible and it requires more complex analysis.

Hence, with this example, we observe the ability of method in reducing the size of the required matrices in the process of analysis. Additionally, the ability of the method in simple analysis of structures for which no simple approach is available becomes apparent.

6. Conclusions

In this article, an efficient method is presented for the analysis of regular structure with additional or lack of nodes and supports and elements. The main assumption is that the regular structure resulted by elimination of the irregularity, has simple analysis.

The main application of the present method is in iterative approaches. As an example, in optimization techniques in each step the variables change according to some pattern and in order to control the effects, the analysis is repeated. Since the changes in each step of optimization is a limited amount, thus using the present method, the analysis can be simplified and great saving can be achieved in the entire process of optimization.

What distinguishes this method from other irregular structure analysis methods is the ability of this method in the analysis of unstable structures (stabilized by certain external forces). In Example 3, one of common and typical applications of this problem is observed.

The other advantage of this method is its ability in the analysis of structures with various types of irregularity. Indeed, this method can handle different types of irregularity simultaneously. Instead of common irregular structure analysis method that are limited to certain types and conditions, this method according to general form of presented matrices, is capable of analyzing a wide variety of irregular structures.

On the other hand, as it can be seen form the practical examples, it is obvious that this method has a positive impact on the process of the analysis of irregular structures, and it clearly decreases the dimensions of the matrices involved.

For instance, if a stable irregular structure has *m* DOFs in the global coordinate system and if by removing or adding the elements or nodes to the structure (for example, adding or removing elements with *f* internal loads, removing the additional nodes with *e* degrees of freedom and adding nodes with e_1 DOFs to structure) it is transformed to a structure with $\mathbf{C}_a \otimes \mathbf{A}_z$ rotational symmetry (this form of regular structure is obtained by rotational repeating of a substructures with z degrees of freedom for each one) and if regular structure is connected by i degrees of freedom to the irregular elements, then instead of finding the inverse of a $m \times m$ matrix for analyzing the irregular structure, via the present method, we can perform the entire analysis merely by finding the eigen-values of a matrices of dimension $z \times z$ and calculating of the SVD of $(i+e+e_1)\times(i+f)$ matrix and inverting a $q \times q$ matrix (where $q = f - e - e_1$).

If a structure is unstable, we should first calculate the SVD of matrix of dimension $(i+e+e_1)\times(i+f)$. if p and q are the numbers of zero rows and columns of singular value matrix of the above decomposition, for completing the process of analysis, we should find the eigen-values of a matrices of dimension $z \times z$ and inverse two matrices of dimensions $p \times p$ and $q \times q$. Since the dimensions of these matrices are very lower than m, the speed of analysis will increase significantly.

The other advantage of this method is its capability to generalizing this method to tackle other possible types of irregularities. For example, we can apply it in the same idea in the separation of structures consisting of multiple regular substructures, each regular structure being composed of more simple ones. This issue shows the necessity of further studies on this method.

Table 1 shows the comparison of computational time for the examples presented in this paper. Here the direct method, method developed by Pellegrino [2], and the present method are compared. From this table it can be seen that by increasing the size of the structure (DOFs), the computational time decreases.

Table 1 Comparison of the storage and computational time for different methods

	Direct Method		**Pellegrino [2] Method**			**Present Method**				**Ratio**
Example No.	**Inv Dim**	**Time (s)**	**Inv Dim**	**Svd Dim**	**Time (s)**	**Inv Dim**	**Svd Dim**	**Eig Dim**	**Time (s)**	$\frac{Direct}{present}$
1	[462x462]	0.0339	[8x8]	[462x470]	0.1741	[14x14]	[57x71]	50x[9x9]	0.0058	**5.79**
2	[12600x12600]	330.2260	[22440x22440]	[12600x35040]	2071.776	[1530x1530]	[1344x2874]	24x[546x546]	27.9592	**11.81**
3	[340x340]	0.7574	[220x220]	[342x562]	10.1358	[2x2] , [4x4]	[24x22]	58x[30x30]	0.0441	**17.15**
	[1476x1476]		[906x906]	[1476x2382]		2x[22x22]	[132x132]			

Acknowledgements: The first author is grateful to the University of Tehran for financial support under grant number 27938/1/09. The second author is thankful to Iran National Science Foundation for the support.

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Appendix A: Notations

m is the number of DOFs of the irregular structure in global coordinate system.

n is the number of internal element forces of the irregular structure.

A is the equilibrium matrix of irregular structure.

P is the external load vector of the structure.

Q is the vector of internal forces of the elements of the irregular structure.

F is the block diagonal flexibility matrix of the irregular structure.

F_e is the block diagonal flexibility matrix of the irregular elements.

δ is the nodal displacement vector of the irregular structure in local coordinate system.

Δ is the nodal displacement vector of the irregular structure in global coordinate system.

S is stiffness matrix of regular structure.

U, V and W are matrices containing of the left and right singular vectors and singular value matrix of the equilibrium matrix, respectively.

A_1 is part of the equilibrium matrix that is related to the DOFs of irregular elements and some DOFs of the regular structure that are related to irregular elements.

N is the matrix containing some columns of the matrix A_1 corresponding to DOFs of the irregular elements in local coordinate system.

t is the number of unrelated DOFs of the regular structure to irregular elements.

i is the number of related DOFs of regular structure to irregular elements.

q and p are numbers of zero columns and rows of the matrix W, respectively.

f is the number of internal forces of the irregular elements.

e is the number of DOFs of the additional nodes.

e_1 is the number of DOFs of node shortage for reaching to the regular structure form in irregular structure.

Appendix B: Description of the main equations of the paper

This section describes the main equations which are utilized in this article. Nodal force equilibrium equation is mentioned in the global coordinates of structure in Eq. (1). In Eq. (2) equilibrium matrix relates the displacement vector in local coordinates system to the global coordinates.

General form of the singular value decomposition of the equilibrium matrix is as follows:

$$A = U.W.V^t \tag{A1}$$

U and V are matrices containing the left and right singular vectors of the equilibrium matrix, respectively. W is the diagonal matrix of the singular values of the equilibrium matrix. In fact, some of the rows and columns in this matrix are zero.

By decomposition of the equilibrium matrix according to Eq. (A1), the pseudo-inverse of this matrix can be written as

$$pinv\,(A) = V_d\,.D^{-1}.U_d{}^t \tag{A2}$$

$Pinv$ is the notation used for the pseudo-inverse and D is a diagonal square matrix containing the nonzero singular values of the equilibrium matrix. By partitioning the matrix W according to its nonzero values, we obtain:

$$W = \left[\begin{array}{c|c} D & Z \\ \hline Z & Z \end{array}\right] \tag{A3}$$

Z is a zero matrix.

Similarly, the matrices V and U correspond to the partitioning of the matrix W as follows:

$$V = [V_d \mid V_z] \quad ; \quad U = [U_d \mid U_z] \tag{A4}$$

The matrices V_d, U_d and V_z, U_z contain the right and left singular vectors of the equilibrium matrix corresponding to nonzero and zero singular values of the equilibrium matrix, respectively.

According to [2], utilizing the Eqs. (1), (A1), (A2) and

(A4), the internal forces of structural elements can be presented by the following relation:

$$Q = pinv(A).P + V_z.\alpha \tag{A5}$$

α is the vector of redundants of the structure.

The compatibility equations in structural nodes can be presented by the following orthogonal relationship:

$$V_z^t.\delta = Z \tag{A6}$$

On the one hand, the governing equilibrium equation in local coordinate system is as follows:

$$\delta = F.Q \tag{A7}$$

F is the block-diagonal flexibility matrix of the structural elements. Here, δ is the nodal displacement vector in the local coordinate system of the structural elements. By substituting Eq. (A5) in (A7) and Eq. (A7) in (A6), and by simplification, we have:

$$\alpha = -(V_z^t.F.V_z)^{-1}.(V_z^t.F.pinv(A).P) \tag{A8}$$

In the following, by substituting α in Eq. (A5), the vector Q is derived as

$$Q = pinv(A).P - V_z.(V_z^t.F.V_z)^{-1}V_z^t.F.pinv(A).P \tag{A9}$$

Similarly, by substituting Q in Eq. (A7), we obtain the vector δ.

Nodal displacement vector Δ is obtained by Eq. (2) similar to the formation of Eq. (A5):

$$A^t.\Delta = \delta \quad \Rightarrow \quad \Delta = pinv(A^t).\delta + U_z.\beta \tag{A10}$$

If the structure is stable, then $U_z = []$. Therefore, the vector of nodal displacements of the stable structure is found as:

$$\begin{aligned} \Delta &= Pinv\ (A^t).\delta \quad \rightarrow \\ &= Pinv\ (A^t).F.(pinv(A).P - V_z.(V_z^t.F.V_z)^{-1}V_z^t.F.pinv(A).P) \end{aligned} \tag{A11}$$