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Research Paper

Capacity Bounds and High-SNR Capacity of the Additive Exponential Noise Channel With Additive Exponential Interference

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Abstract: Communication in the presence of a priori known interference at the encoder has gained great interest because of its many practical applications. In this paper, additive exponential noise channel with additive exponential interference (AENC-AEI) known non-causally at the transmitter is introduced as a new variant of such communication scenarios. First, it is shown that the additive Gaussian channel with a priori known interference at the encoder when the transmitter suffers from a fast-varying phase noise can be modeled by the AENC-AEI. Then, capacity bounds for this channel under a non-negativity constraint as well as a mean value constraint on input are derived. Finally, it is shown both analytically and numerically that the upper and lower bounds coincide at high signal to noise ratios (SNRs), and therefore, the capacity of the AENC-AEI at high SNRs is obtained. Interestingly, this high SNR-capacity has a simple closed-form expression and is independent of the interference mean, analogous to its Gaussian counterpart.

Keywords: Capacity Bounds, Dirty Paper Channel, Exponential Noise, High-SNR Capacity.

1 Introduction

IN the information theory, the Gaussian channel is the most important, popular and widely analyzed continuous alphabet channel. The main reasons are as follows [1-4]:

- The Gaussian channel provides a simple model for several real-world communication channels.
- The Gaussian distribution plays a key role in achieving simple closed-form expressions for the Gaussian channel capacity and some information-theoretic quantities.
- Among variance-constrained random variables (RVs), the Gaussian distribution maximizes differential entropy.
- For an additive channel with the Gaussian input (and a given average power), the Gaussian noise is the worst noise.

As demonstrated in [3, 4], the exponential channel is of interest because of sharing a number of analogous traits:

- The exponential channel has applications in optical communications and non-coherent communications.
- The exponential distribution plays a key role in achieving simple closed-form expressions for the exponential channel capacity and some information-theoretic quantities.
- Among all nonnegative RVs with a given mean, the exponentially distributed RV is the differential entropy-maximizing one.
- For an additive channel with a non-negative additive noise (and a given mean), the exponential noise is the worst noise.

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Specifically, Verdu in [3] obtained the capacity of the additive exponential noise (AEN) channel and the capacity-achieving (optimal) input distribution. He showed that

- The capacity of the AEN channel in terms of the signal-to-noise ratio (SNR) is log(1+SNR). This capacity is completely analogous to the capacity of the complex-valued additive white Gaussian noise (AWGN) channel.
- The capacity-achieving input distribution in the AEN channel is not purely exponential, while the optimal input distribution in the AWGN channel is Gaussian, the same distribution that the noise has.

In spite of a number of notable similarities between the exponential channel and Gaussian channel [3, 4], there are basic and intrinsic differences between them due to a few reasons. One of the main reasons is that the sum of two independent Gaussian RVs is a Gaussian RV, while the sum of two independent exponential RVs is not an exponential RV, especially when the two added RVs have different mean values. Another reason is that in contrast to a Gaussian RV, an exponential RV is distributed only on the right-hand side of the Euclidean two-dimensional space. Therefore, for exponential RVs, we cannot enjoy the flexibilities that a symmetric system brings into the hand.

The "dirty paper channel" is a famous group of communication channels in which, besides noise, the transmitted signal is also corrupted by a priori known interference [5-12]. The study of such channels is of interest because of capability of the transmitter in mitigating the negative effect of the interference and improving data-rate. The multiple-input multiple-output (MIMO) broadcast channels, cooperative networks, intersymbol interference (ISI) precoding, watermarking and storage systems with defective cells are a few application areas of the dirty paper channel [1].

Costa [5] introduced the Gaussian dirty paper channel in which the transmitted signal is corrupted by an additive white Gaussian noise as well as an additive white Gaussian interference (AWGI). This Gaussian interference (dirt) is assumed to be a priori known at the encoder. Here, we denote Costa's dirty paper channel which is indeed a dirty AWGNC with Gaussian dirt by AWGNC-AWGI, for convenience. Costa showed the surprising result that the capacity of the AWGNC-AWGI is the same as the capacity of the interference-free AWGNC (clean AWGNC). This means that the optimal encoder can adapt its signal to a priori known interference and completely eliminate the negative effect of the interference [5].

1.1 Our Motivation and Work

As mentioned in [4, 13, 14], the AEN channel has practical importance in non-coherent communication settings and in optical communication scenarios. One communication scenario in which the AEN channel appears as its natural model is the continuous-time Gaussian channel when the transmitted signal suffers from a fast-varying phase noise. Due to the incoherency between signal and noise components, information can only be sent by modulating the signal energy. In this case, the receiver uses a non-coherent detector, inspired in optical direct detection, to recover the transmitted information. Moreover, as mentioned above, knowing the interference at the transmitter leads to better communications performance.

In this paper, we introduce the AEN channel afflicted by an extra additive exponential interference (AEI) in which full knowledge of the additive interference is given to the transmitter. This channel, denoted here by AENC-AEI, is, in fact, an exponential version of the Costa's dirty paper channel. Similar to [3, 4], the input of the AENC-AEI is assumed to be a non-negative real RV and is also subject to a mean value constraint. By extending the Gelfand-Pinsker capacity Theorem [16], we obtain lower and upper bounds on the capacity of the AENC-AEI. Then, we write the lower bound in terms of an infinite series which coincides with the upper bound at high SNR and hence, the capacity of the channel at high SNRs is derived. Interestingly, this high SNR-capacity, which is independent of the interference mean, has a simple closed-form expression, analogous to its Gaussian counterpart. Some numerical results are also provided to evaluate the proposed bounds.

1.2 Paper Organization

We begin by describing a communication scenario that models the AENC-AEI in Section 2. All main results are presented in Section 3, where we first derive lower and upper bounds on the capacity of the AENC-AEI. Then, we show both analytically and numerically that the upper and lower bounds coincide at high SNRs. In Section 4, the exact evaluation of the lower bound is given. Finally, Section 5 concludes the paper.

2 The AENC-AEI Model and Direct (Non-Coherent) Detection

In this section, we describe a communication scenario that its discrete-time model is the AENC-AEI. In this paper, Gaussian variables are assumed to be zero-mean and complex-valued.

In [4] Martinez has shown how an AWGN channel with an extra phase noise may lead to an AEN channel. Similarly, we briefly show here that an AWGNC-AWGI with an extra fast time-varying phase noise leads to an AENC-AEI. To do this, consider a continuous dirty AWGNC with Gaussian dirt (AWGNC-AWGI) modeled by

$$y(t) = x(t) + s(t) + z(t). \tag{1}$$

In (1), $x(t)$ and $y(t)$ are transmitted and received signals, respectively, $z(t)$ is an AWGN, and $s(t)$ is an additive Gaussian interference known non-causally at the transmitter and is independent of $z(t)$. This additive interference can be considered as a part of noise that the encoder is informed of it or can be an interfering signal sent by the neighboring transmitter which is known a priori to the main transmitter [1, 5].

To obtain the discrete-time variant of the channel (1), we can decompose the signals in (1) using a set of orthonormal functions $\psi(t)$, k = 0, 1, … (like Fourier decomposition). For example, the signal $y(t)$ can be decomposed as

$$y(t)=\sum_k y'_k \psi_k(t). \tag{2}$$

Discretization in (2) can be considered as the Fourier decomposition of $y(t)$ in which y'_k is the k-th Fourier mode (component) and is actually the projection of $y(t)$ onto the $\psi_k(t)$.

In coherent detection, the receiver uses a bank of correlators to calculate the received signal components y'_k (k = 0, 1, …). The output of the k-th correlator is

$$\int_0^T y(t)\psi_k^*(t)dt = y'_k = x'_k + s'_k + z'_k \tag{3}$$

where, $\psi_k^*(t)$ is the complex conjugate of $\psi_k(t)$, x'_k, s'_k, and z'_k are the k-th components of the transmitted signal, interference and noise, respectively. As seen in (3), the desired component x'_k is corrupted by the interference and noise components. Note that y'_k, x'_k, s'_k, and z'_k are complex numbers. In addition, since $s(t)$ and $z(t)$ are Gaussian random processes, s'_k and z'_k are Gaussian RVs. Therefore, expression (3) shows the discrete-time complex-valued AWGNC-AWGI.

Phase noise in oscillators can (severely) degrade the performance of communication systems. Multiple-input multiple-output (MIMO) systems ([17]) and orthogonal frequency division multiplexing (OFDM) systems ([18]) are two important instances of such systems. Optical communication systems are another significant instance in which phase noise is often a serious impairment. To consider such a noise in a communication system, it is sufficient to replace the transmitted signal $x(t)$ by ([4, 19, 20])

$$x(t)=\sum_k x'_k e^{j\theta_k(t)}\psi_k(t). \tag{4}$$

In (4) the phase noise $\theta_k(t)$, which is modeled by a continuous Brownian motion (Wiener) process ([4, 19, 20]), is a Gaussian process with zero mean and Var[$\theta_k(t)$] $2\beta_k t$, $\beta_k \geq 0$ [19, 21, 22]. Therefore, when the received signal is degraded by the phase noise, additive interference and noise, the k-th component of the received signal is

$$y'_k = x'_k e^{j\theta_k(t)} + s'_k + z'_k. \tag{5}$$

In the presence of the fast time-varying phase noise (i.e., when $\beta_k T \to \infty$), coherent detection fails to detect the transmitted signal because in this case, the transmitted (desired) signal $x(t)$ at the k-th correlator appears as

$$\mathbb{E}\left[\frac{1}{T}\int_0^T \left(x'_k e^{j\theta_k(t)}\psi_k(t)\right)\psi_k^*(t)dt\right] = \frac{x'_k}{T}\int_0^T e^{-\beta_k t}dt = \frac{x'_k\left(1-e^{-\beta_k T}\right)}{\beta_k T} \tag{6}$$

This shows that for $\beta_k > 0$, if $T \to \infty$, then $\frac{x'_k\left(1-e^{-\beta_k T}\right)}{\beta_k T} \to 0$, and therefore, the coherent detector cannot detect the desired signal. Note that to obtain the first equality in (6) we have used $\mathbb{E}\left[e^{j\theta_k(t)}\right] = e^{-\beta_k t}$ which can be proved by considering the characteristic function of a Gaussian RV and the point that $\theta_k(t)$ is a Gaussian process with zero mean and Var[$\theta_k(t)$] $2\beta_k t$ [21,22].

A solution to the problem of vanishing the desired signal $x(t)$ at the k-th correlator in the presence of the fast time-varying phase noise (due to coherent detection) is to use a *direct* detection receiver with a bank of Mach-Zehnder interferometers (Fig. 1) [4]. Similar to [4], we can show that the bank of Mach-Zehnder interferometers acts as a demultiplexer and produces a collection of parallel signals $\xi_k(t)$ as

$$\xi_k(t) = \left(x'_k e^{j\theta_k(t)} + s'_k + z'_k\right)\psi_k(t), \qquad k = 1, 2, \cdots \tag{7}$$

As shown in Fig. 1, after passing the signal $\xi_k(t)$ through an envelope detector and integrating over (0, T), the output y'_k is generated as

$$\begin{aligned} y'_k &= \mathbb{E}\left[\frac{1}{T}\int_0^T \left|\xi_k(t)\right|^2 dt\right] \\ &= |x'_k|^2 + |s'_k|^2 + |z'_k|^2 \\ &\quad + \frac{2}{T}\int_0^T \mathrm{Re}\left\{x'_k \mathbb{E}\left[e^{j\theta_k(t)}\right]\left(s'^*_k + z'^*_k\right)\right\}dt \\ &= |x'_k|^2 + |s'_k|^2 + |z'_k|^2 \\ &\quad + \frac{2\left(1-e^{-\beta_k T}\right)}{\beta_k T}\mathrm{Re}\left\{x'\left(s'^*_k + z'^*_k\right)\right\}. \end{aligned} \tag{8}$$

By considering (8), we see that in the presence of the fast time-varying phase noise (i.e., $\beta_k T \to \infty$), the output y'_k tends to the sum of the energies of the transmitted signal, interference and noise, that is

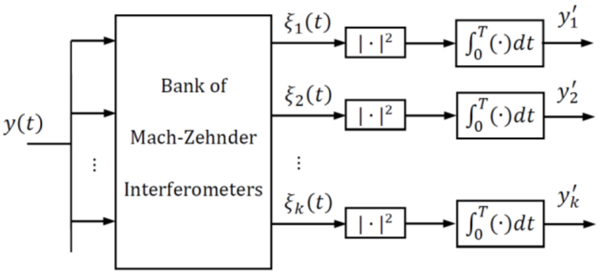


Fig. 1 Direct detection using a bank of Mach-Zehnder interferometers.

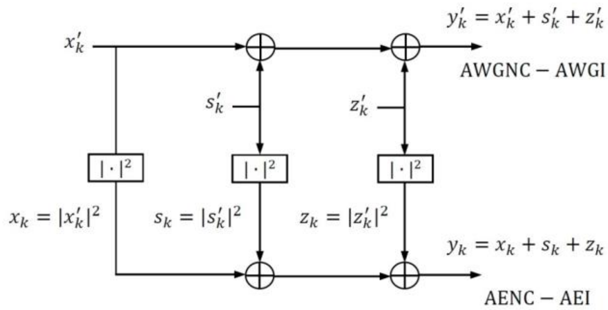


Fig. 2 AENC-AEI model and its AWGNC-AWGI counterpart.

$$\lim_{\beta_k T\to\infty} y_k' = |x_k'|^2 + |s_k'|^2 + |z_k'|^2. \tag{9}$$

Now, let $x_k = |x'_k|^2$, $s_k = |s'_k|^2$ and $z_k = |z'_k|^2$, i.e., unprimed letters denote the signal energies and primed letters denote the corresponding complex amplitudes. In this case, s_k and z_k are exponentially distributed because the energies $|s'_k|^2$ and $|z'_k|^2$ are the squared amplitudes of circularly-symmetric complex Gaussian variables. Moreover, if we define the energy of a variable as its squared amplitude, then the average energy of the interference component s'_k and noise component z'_k are the mean of the interference component s_k and noise component z_k, respectively, i.e., $\mathbb{E}\left[|S_k'|^2\right] = \mathbb{E}[S_k] = \mathrm{E}_s$ and $\mathbb{E}\left[|Z_k'|^2\right] = \mathbb{E}[Z_k] = \mathrm{E}_z$. Therefore, in the presence of the fast time-varying phase noise and using direct detection, the variance-constrained complex-valued AWGNC-AWGI leads to the mean-constrained non-negative real-valued AENC-AEI. In the AENC-AEI, output y_k is an energy and defined as the sum of the energies of x'_k, s'_k and z'_k, that is, $y_k = x_k + s_k + z_k$.

The relationship between the AWGNC-AWGI and its AENC-AEI counterpart is shown in Fig .2. Note that in Fig. 2, the primed letters are AWGNC-AWGI variables and unprimed letters are AENC-AEI variables. Also, $s'_k \sim CN(0, E_s)$ means that s'_k is a complex Gaussian (Normal) RV with mean 0 and variance E_s, and $s_k \sim \exp(E_s)$ means that s'_k is an exponential RV with mean E_s. Similar notation is used for z'_k and z_k. It is worth noting that an average energy constraint (e.g., $\frac{1}{n}\sum_{k=1}^{n}|x_k'|^2 \le E_x$) in the complex-valued AWGNC-AWGI corresponds to a mean constraint (i.e., $\frac{1}{n}\sum_{i=1}^{n}X_i \le E_x$) in its AENC-AEI counterpart. Therefore, the discrete-time AENC-AEI, depicted in Fig. 2, can be modeled by

$$Y_k = X_k + S_k + Z_k\,, \quad k = 1,2,\cdots,n. \tag{10}$$

where z_k is the k-th component of additive exponential noise with mean E_z and s_k is the k-th component of additive exponential interference with mean E_s, known a priori at the encoder and independent of z_k, the k-th components of input X_k and output Y_k are non-negative real numbers. Also, the channel input X is subject to input mean constraint E_x.

Background and Notation Conventions: Fix positive scalars E_x, E_s, and E_z. Let S and Z be two independent exponential RVs with means E_s and E_z, respectively, and X be a non-negative RV subject to a mean constraint E_x. Moreover, let $\bar{X}$ be a non-negative RV, independent of S and Z, and with mean E_x and a mixed distribution as (the capacity-achieving input distribution of the AEN channel [3,4])

$$f_{\bar{X}}(x) = \frac{E_z}{E_x+E_z}\delta(x) + \frac{E_x}{(E_x+E_z)^2}e^{\frac{-x}{E_x+E_z}}u(x) \tag{11}$$

where $\delta(x)$ is the Dirac delta function and $u(x)$ is the unit step function. The Laplace transform of the mixed distribution $f_{\bar{X}}(x)$ is

$$\begin{aligned}\mathcal{L}\{f_{\bar{X}}(x)\} &= \int_0^\infty e^{-sx}f_{\bar{X}}(x)dx \\ &= \frac{E_z}{E_x+E_z} + \frac{E_x}{E_x+E_z}\frac{1}{1+(E_x+E_z)s}.\end{aligned} \tag{12}$$

It is easy to show that the random variable $T = \bar{X} + Z$ is exponentially distributed with mean $E_t = E_x + E_z$. Throughout this paper, $h(.)$ denotes the differential entropy, $Y = X + S + Z$ and $\bar{Y} = \bar{X} + S + Z$.

Gelfand and Pinsker [16] showed that the capacity of a single-user discrete memoryless channel with side information, when side information sequence S^n is non-causally available at the transmitter, is given by

$$C = \max_{p(u,x|s)}\{I(U;Y) - I(U;S)\}. \tag{13}$$

In (13) the maximum is over all joint distributions $p(s, u, x, y)$ that factor as $p(s)p(u, x|s)p(y|x, s)$ and U is an auxiliary RV. Note that the channel is characterized by a conditional probability $p(y|x, s)$ and by the state probability $p(s)$ and $U \to S$, $X \to Y$ form a Markov chain [1]. Also Note that although (13) has been derived for the discrete memoryless case but, as mentioned in [5] and [23], it can be extended to memoryless channels with discrete-time and continuous alphabets (AENC-AEI in this case) by finely quantizing the continuous variables [24]. The Gaussian version of the Gelfand-Pinsker problem was studied by Costa in [5], where he surprisingly showed that the capacity of the AWGNC-AWGI is equal to the capacity of the interference-free AWGNC.

3 Lower and Upper Bounds on the Capacity of the AENC-AEI

We here derive upper and lower bounds on the capacity of the AENC-AEI which coincide with each other at high SNRs and hence the high SNR-capacity of the AENC-AEI is obtained.

3.1 An Upper Bound on the Capacity of the AENC-AEI

Theorem 1. $C_{out} \triangleq \log\left(1+\frac{E_x}{E_z}\right) = \log\left(1+\mathrm{SNR}\right)$ is an upper bound on the capacity of the AENC-AEI.

Proof. By considering a genie which reveals S^n to the receiver, we can reach to C_{out} which is equal to the capacity of the interference-free AEN channel, that is

$$
\begin{aligned}
C &\le \max_{p(x|s)} I\left(X;Y \mid S\right) \\
&= \max_{p(x|s)} \left\{ h\left(X+Z\right) - h\left(Z\right) \right\} && (14)\\
&= h\left(\bar{X}+Z\right) - h\left(Z\right) && (15)\\
&= \log\left(1+\frac{E_x}{E_z}\right) && (16)
\end{aligned}
$$

where (14) follows from the facts that (i) $Y = X + S + Z$, (ii) conditioning does not increase entropy and (iii) Z is independent of X and S, (15) follows from the facts that (i) for a fixed $p(z)$, the maximization in (14) is done only over $h(X + Z)$, (ii) random variable $T \triangleq \bar{X} + Z$ is exponentially distributed ([3,4]) and (iii) the exponential distribution maximizes the entropy for a given mean constraint ([2]), (16) follows from the facts that the differential entropy of the exponential distribution with mean E is equal to $\log(Ee)$ [3]. Note that the upper bound C_{out} holds for all U and X.

3.2 A Lower Bound for the Capacity of the AENC-AEI

Theorem 2. C_{in} defined in (17) is a lower bound on the capacity of the AENC-AEI, where $A = \frac{1}{E_s} - \frac{1}{E_x+E_z}$, $B = \frac{\left(E_s-E_z\right)\left(E_x+E_z\right)}{E_x E_s}$, $F\left(k\right) = \frac{k}{E_s} - \frac{k-1}{E_x+E_z}$, and $G\left(k\right) = \frac{k}{E_x+E_z} - \frac{k-1}{E_s}$.

Proof. First note that by using any distribution for input RV, we can obtain a lower bound for the capacity. To obtain a lower bound, we choose $X = \bar{X}$ and utilize the Costa strategy at high SNR, i.e., define $U = X + S$. Therefore, considering (5) and (13) we can write:

$$
\begin{aligned}
C &= \max_{p(u,x|s)} \left\{ I\left(U;Y\right) - I\left(U;S\right) \right\} \\
&\ge \max_{p^*(u,x|s)} \left\{ I\left(U;Y\right) - I\left(U;S\right) \right\} && (18)\\
&= \max_{p^*(u,x|s)} \left\{ h\left(Y\right) - h\left(Y \mid U\right) - h\left(U\right) + h\left(U \mid S\right) \right\} \\
&= h\left(\bar{X}+S+Z\right) - h\left(Z\right) - h\left(\bar{X}+S\right) + h\left(\bar{X}\right) \triangleq C_{in} && (19)
\end{aligned}
$$

where $p^*(u, x|s)$ is a subset of the set of all distributions $p(u, x|s)$ in which $X = \bar{X}$ and thereupon $U = \bar{U} = \bar{X} + S$. Note that choosing $X = \bar{X}$ makes $\bar{X} + Z$ an exponential RV that maximize $h(X + Z)$ subject to a mean constraint [2]. Before exact evaluation of C_{in} (in Section 4), we show that at high SNRs, term $\{h(\bar{X} + S + Z) - h(\bar{X} + S)\}$ tends to zero and therefore, C_{in} coincides with C_{out} and gives the AENC-AEI capacity at high SNRs.

3.3 The Capacity of the AENC-AEI at High SNRs

Theorem 3. $C_{hsnr} \triangleq \log\left(\frac{E_x}{E_z}\right) = \log\left(\mathrm{SNR}\right)$ is the capacity

$$
C_{in} = \begin{cases}
\left\{ \dfrac{E_z \log\left(\frac{E_x}{E_z^2}\right) + E_x \log\left(\frac{E_x+E_z}{E_z}\right)}{E_x+E_z} + \left(\dfrac{\log e}{E_x+E_z-E_s}\right)\left(\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}\left\{\frac{1-\frac{B^k E_x}{E_x+E_z}}{F\left(k\right)} - \frac{1-\frac{B^k\left(E_s-E_z\right)}{E_s}}{F\left(k+1\right)}\right\}\right)\right\}, & A>0,\ -1\le B<1 \\[2ex]
\left\{ \begin{aligned} &\dfrac{E_x \log\left(\frac{E_x+E_z}{E_x}\right) - E_z \log\left(eE_z\right)}{E_x+E_z} + \dfrac{E_z \log e}{E_s} + \log\left(\dfrac{\left(E_s-E_z\right)\left(E_x+E_z\right)}{E_s E_z}\right) \\ &+\left(\dfrac{\log e}{E_s-E_x-E_z}\right)\left(\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}\left\{\frac{1-\frac{E_s-E_z}{B^k E_s}}{G\left(k\right)} - \frac{1-\frac{E_x}{B^k\left(E_x+E_z\right)}}{G\left(k+1\right)}\right\}\right) \end{aligned} \right\}, & A<0,\ -1\le\dfrac{1}{B}<1
\end{cases} \quad (17)
$$

of the AENC-AEI at high SNRs.

Proof. We can easily see from (11) that at high SNRs, $f_{\bar{X}}(x)$ gets closer to an exponential distribution. In other words, $f_T(t)$ tends to $f_{\bar{X}}(x)$ as SNR increases and gets closer to infinity. Therefore, $h(T + S)$ tends to $h(\bar{X} + S)$ as SNR tends to infinity. Consequently, we can evaluate the lower bound (19) at high SNRs as

$$\lim_{\mathrm{SNR}\to\infty} C_{in} = h\left(\bar{X}\right) - h\left(Z\right) = \log\left(\mathrm{SNR}\right). \tag{20}$$

By considering (16) and (20) we obtain the capacity of AENC-AEI at high SNRs which is equal to the capacity of interference-free AENC.

Remark 1: From the Theorems, we see that there are some similarities between the AENC-AEI and its Gaussian counterpart (i.e., the complex-valued AWGNC-AWGI): (i) the high SNR-capacity of both channels (i.e., C_{hsnr}) is completely analogous (in terms of the SNR) and has a simple closed-form expression which is given by log(SNR); (ii) in the AENC-AEI (resp. in the complex-valued AWGNC-AWGI) the capacity C_{hsnr} is independent of interference mean (resp. interference variance) and is attained by an exponential input (resp. a Gaussian input) that satisfies the mean constraint (resp. the variance constraint).

3.4 Numerical Results

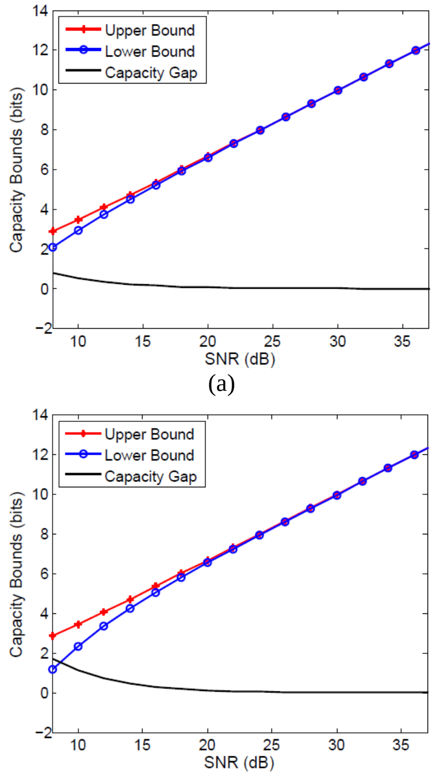
We here compute the upper and lower bounds of the capacity of AENC-AEI shown in (16) and (17), respectively. Fig. 3 depicts the capacity bounds and capacity gap of AENC-AEI for two values of $E_z = 10^2$ and $E_z = 10^4$. It is seen that the lower bound coincides with the upper bound at high SNRs and hence, gives the capacity of AENC-AEI at high SNRs. As seen in Fig. 3, at high SNRs the capacity is linearly increasing in SNR, because C_{hsnr} = log(SNR) and SNR is in dB in this figure.

4 Exact Evaluation of C_{in}

We now calculate each entropy term $h(.)$ in (19) separately. The random variable Z has an exponential density as $f_Z(z) = \frac{1}{E_z} e^{\frac{-z}{E_z}} u(z)$ with differential entropy $h(z) = \log(eE_z)$. Note that $u(z)$ denotes the unit step function. Considering (11), the differential entropy of $\bar{X}$ is

$$h\left(\bar{X}\right) = \frac{-E_z}{E_x+E_z}\log\left(\frac{E_z}{E_x+E_z}\right) + \frac{E_x}{E_x+E_z}\log\left(\frac{e\left(E_x+E_z\right)^2}{E_x}\right) \tag{21}$$

To calculate $h(\bar{Y})$ we need to find the probability density function (PDF) of the random variable $\bar{Y} = \bar{X} + S + Z$. The random variable $T = \bar{X} + Z$ has an exponential PDF with mean $E_x + E_z$. Therefore, the Laplace transform of the PDF of the random variable $\bar{Y} = T + S$ is



(b)

Fig. 3 Capacity bounds and capacity gap of AENC-AEI for two values of a) $E_z = 10^2$ and b) $E_z = 10^4$.

$$\mathcal{L}\left\{f_{\bar{Y}}(y)\right\} = \frac{1}{E_x+E_z-E_s}\left(\frac{E_x+E_z}{1+\left(E_x+E_z\right)s} - \frac{E_s}{1+\left(E_s\right)s}\right) \tag{22}$$

where, (22) follows from the facts (i) the PDF $f_{\bar{Y}}(y)$ is given by the convolution of the PDFs $f_T(t)$ and $f_S(s)$, (ii) the Laplace transform of the convolution of two functions is equal to product of their Laplace transforms, and (iii) the Laplace transform of an exponential PDF with mean μ is $1/(1+\mu s)$. By applying the inverse Laplace transform to (22) we have

$$f_{\bar{Y}}\left(y\right) = \frac{1}{E_x+E_z-E_s}\left(e^{\frac{-y}{E_x+E_z}} - e^{\frac{-y}{E_s}}\right)u\left(y\right). \tag{23}$$

where, (23) follows from the fact $\mathcal{L}\left\{Ke^{\frac{-t}{\mu}}u(t)\right\} = \frac{K}{\frac{1}{\mu}+s} = \frac{K\mu}{1+\mu s}$. Consequently,

$$h\left(\bar{Y}\right) = -\int_{y>0}\left(\frac{e^{\frac{-y}{E_x+E_z}} - e^{\frac{-y}{E_s}}}{E_x+E_z-E_s}\right)\log\left(\frac{e^{\frac{-y}{E_x+E_z}} - e^{\frac{-y}{E_s}}}{E_x+E_z-E_s}\right)dy \tag{24}$$

Before evaluating the integral of (24), we need to

express the Taylor series of the ln(1+x) around $x = 0$.
Remark 2: The Taylor series of the ln(1+x) around $x = 0$ (Maclaurin series) is:

$$\ln\left(1+x\right)=\sum_{k=1}^{\infty}\frac{\left(-1\right)^{k+1}}{k}x^{k},\quad -1<x\leq 1 \tag{25}$$

Note that the series approximation converges to the function only in the region $-1 < x \leq 1$.
By considering the convergence region of ln(1+x) and also knowing that $y > 0$, the term $\ln\left(e^{\frac{-y}{E_x+E_z}}-e^{\frac{-y}{E_s}}\right)$ in (24) can be written as $\ln\left(e^{\frac{-y}{E_x+E_z}}\left(1-e^{-yA}\right)\right)$ for $A > 0$ and as $\ln\left(-e^{\frac{-y}{E_s}}\left(1-e^{yA}\right)\right)$ for $A < 0$, where $A=\frac{1}{E_s}-\frac{1}{E_x+E_z}$. We first calculate $h(\bar{Y})$ for $A > 0$. Utilizing Remark 2, if $A > 0$ (or equivalently $E_x + E_z > E_s$) then ln(1-e^{-ya}) = $-\sum_{k=1}^{\infty}\frac{e^{-kyA}}{k}$. Therefore, for $A > 0$ we have

$$h\left(\bar{Y}\right)=\log\left(E_x+E_z-E_s\right)+\left(\frac{E_x+E_z+E_s}{E_x+E_z}\right)\log e+\left(\frac{\log e}{E_x+E_z-E_s}\right)\left(\sum_{k=1}^{\infty}\frac{1}{k}\left\{\frac{1}{F\left(k\right)}-\frac{1}{F\left(k+1\right)}\right\}\right) \tag{26}$$

where $F(k)=\frac{k}{E_s}-\frac{k-1}{E_x+E_z}$. Similarly, for $A < 0$ (or equivalently $E_x + E_z < E_s$):

$$h\left(\bar{Y}\right)=\log\left(E_s-E_x-E_z\right)+\left(\frac{E_x+E_z+E_s}{E_s}\right)\log e+\left(\frac{\log e}{E_s-E_x-E_z}\right)\left(\sum_{k=1}^{\infty}\frac{1}{k}\left\{\frac{1}{G\left(k\right)}-\frac{1}{G\left(k+1\right)}\right\}\right) \tag{27}$$

where $G\left(k\right)=\frac{k}{E_x+E_z}-\frac{k-1}{E_s}$. Finally, to calculate $h(\bar{U})$, we should find the PDF of the random variable $\bar{U}=\bar{X}+S$. Using Laplace transform we have

$$\mathcal{L}\left\{f_{\bar{U}}\left(u\right)\right\}=\frac{1}{E_x+E_z-E_s}\left(\frac{E_x}{1+\left(E_x+E_z\right)s}-\frac{E_s-E_z}{1+E_s s}\right). \tag{28}$$

By applying the inverse Laplace transform to (28) we obtain

$$f_{\bar{U}}\left(u\right)=\left(\frac{\frac{E_x}{E_x+E_z}e^{\frac{-u}{E_x+E_z}}-\frac{E_s-E_z}{E_s}e^{\frac{-u}{E_s}}}{E_x+E_z-E_s}\right)u\left(u\right) \tag{29}$$

Consequently,

$$h\left(\bar{U}\right)=-\int_{u>0}\left(\frac{\frac{E_x}{E_x+E_z}e^{\frac{-u}{E_x+E_z}}-\frac{E_s-E_z}{E_s}e^{\frac{-u}{E_s}}}{E_x+E_z-E_s}\right)\times\log\left(\frac{\frac{E_x}{E_x+E_z}e^{\frac{-u}{E_x+E_z}}-\frac{E_s-E_z}{E_s}e^{\frac{-u}{E_s}}}{E_x+E_z-E_s}\right)du \tag{30}$$

Similar to the expansion of (24), considering the convergence region of ln(1 + x) and knowing that u > 0, the term $\ln\left(\frac{E_x}{E_x+E_z}e^{\frac{-u}{E_x+E_z}}-\frac{E_s-E_z}{E_s}e^{\frac{-u}{E_s}}\right)$ in (30) can be written as $\ln\left(\frac{E_x}{E_x+E_z}e^{\frac{-u}{E_x+E_z}}\left(1-Be^{-uA}\right)\right)$ for $A > 0$, $-1 \leq B < 1$ or as $\ln\left(\frac{E_z-E_s}{E_s}e^{\frac{-u}{E_s}}\left(1-\frac{1}{B}e^{uA}\right)\right)$ for $A < 0$, $-1 \leq 1/B < 1$ where A is as before and $B=\frac{\left(E_s-E_z\right)\left(E_x+E_z\right)}{E_xE_s}$. Utilizing Remark 2, if $A > 0$ and $-1 \leq B < 1$ then $\ln\left(1-B_1e^{-uA}\right)=-\sum_{k=1}^{\infty}\frac{B^ke^{-kuA}}{k}$.
Therefore, for $A > 0$ and $-1 \leq B < 1$ we have

$$h\left(\bar{U}\right)=\log\left(\frac{\left(E_x+E_z\right)\left(E_x+E_z-E_s\right)}{E_x}\right)+\frac{\left(E_x-\frac{E_s\left(E_s-E_z\right)}{E_x+E_z}\right)\log e}{E_x+E_z-E_s}+\left(\frac{\log e}{E_x+E_z-E_s}\right)\left(\sum_{k=1}^{\infty}\frac{B^k}{k}\left\{\frac{\frac{E_x}{E_x+E_z}}{F\left(k\right)}-\frac{\frac{E_s-E_z}{E_s}}{F\left(k+1\right)}\right\}\right) \tag{31}$$

where $F(k)$ is the same as before. Similarly, for $A < 0$ and $-1 \leq 1/B \leq 1$, $h(\bar{U})$ is

$$h\left(\bar{U}\right)=\log\left(\frac{E_s\left(E_s-E_x-E_z\right)}{E_s-E_z}\right)+\frac{\left(\frac{E_x\left(E_x+E_z\right)}{E_s}-\left(E_s-E_z\right)\right)\log e}{E_x+E_z-E_s}$$

$$+\left(\frac{\log e}{E_x+E_z-E_s}\right)\left(\sum_{k=1}^{\infty}\frac{1}{kB^k}\left\{\frac{\frac{E_s-E_z}{E_s}}{G(k)}-\frac{\frac{E_x}{E_x+E_z}}{G(k+1)}\right\}\right)\quad(32)$$

where $G(k)$ is the same as before. Therefore, the lower bound for the capacity of AENC-AEI is obtained as shown in (17). Note that the lower bound (17) tends to the lower bound (20) as SNR = E_x/E_z tends to infinity. This was also shown numerically in the previous section.

5 Conclusion

We first introduced the exponential version of the Costa's dirty paper channel (denoted by the AENC-AEI) and modeled a noncoherent communication scenario by the AENC-AEI. We then derived upper and lower bounds on the capacity of the AENC-AEI that coincide at high SNRs. Hence the high SNR-capacity of the channel is established which is completely parallel to its well-known Gaussian counterpart.

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