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Research Paper

# Synthesis of Antenna Arrays of Maximum Directivity for a Specified Beamwidth

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**Abstract:** Linear antenna arrays are synthesized to have maximum directivity for a specified beamwidth. The directivity is maximized subject to a given beamwidth such as null to null or half power one. The excitation currents are obtained using a matrix equation obtained from the Lagrange multiplier method. The performance of the proposed method is studied by means of some examples. The synthesized arrays have the pre-specified beamwhidths and their directivity is close to the number of elements.

**Keywords:** Antenna Array Synthesis, Maximum-Directivity Arrays, Specified Beamwidth Arrays.

## 1 Introduction

ARRAY factor characteristics of antenna arrays are important for many applications such as communication systems, radars, and imaging [1]. The sidelobe level (SLL), directivity, and beamwidth are three important features of antenna arrays which depend on the excitation currents of the antennas [2-4]. Dolph-Chebyshev arrays have a minimum beamwidth for a given sidelobe level or have a minimum sidelobe level for a given beamwidth [2, 3, 5]. However, these types of arrays do not have maximum directivity. In [6], the level of sidelobes is set to arbitrary values but the resulted directivity is not necessarily maximum.

Uniformly excited arrays of distances equal to or more than a half wavelength have the maximum possible directivity. However, the sidelobe level of uniformly excited arrays is high and about -13.2 dB. In fact, uniformly excited arrays have fixed sidelobe level and beamwidth which makes them less desirable for many applications [2-3]. It is favorable to synthesize arrays so that they have maximum directivity while either their sidelobe level or their beamwidth are a desired value. In [7], an optimization method is addressed to synthesize arrays of maximum directivity for specified sidelobe levels. In [8], a method is proposed to control the beamwidth of the arrays although does not present maximum directivity. In [9, 10], analytic methods are proposed to design arrays so that they have as low as possible sidelobe level while having directivity as close as to that of uniformly excited arrays.

In this paper, an analytic method is addressed to synthesize arrays of maximum directivity for a specified beamwidth. For this purpose, the Lagrange multiplier method [11] is utilized to reach a square matrix equation. The performance of the proposed method is studied by means of some examples. The paper is organized as follows: In Section 2, an analytic method is presented to synthesize arrays of maximum directivity for a fixed beamwidth. In Section 3, some comprehensive examples are presented to evaluate the performance of the presented method.

## 2 Maximum-Directivity for A Fixed Beamwidth

A linear antenna array consists of $L$ identical antennas of uniform space $d$ on the $z$ axis. The array factor of a linear antenna array having symmetric excitations is given by

$$F(\psi) = \begin{cases} \sum_{n=0}^{N} A_n \cos(n\psi) & ; \; L = 2N+1 \\ \sum_{n=1}^{N} A_n \cos\left((n-0.5)\psi\right) & ; \; L = 2N \end{cases} \quad (1)$$

In Eq. (1), $A_0 = I_0$ and $A_n = 2I_n$ for $n \neq 0$, where $I_n$ is

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the excitation current of the $n$-th and ($-n$)-th elements. Also, $\psi$ is a real variable defined as $\psi = k\,d\cos\theta + \alpha$, where $k = 2\pi/\lambda$ and $\alpha = -k\,d\cos\theta_0$ in which $\theta_0$ is the angle of maximum radiation.

To normalize the array factor, i.e. $|F(0)|=1$, the following condition is assumed.

$$\begin{cases} \sum_{n=0}^{N} A_n = 1 \ ; \ L = 2N+1 \\ \sum_{n=1}^{N} A_n = 1 \ ; \ L = 2N \end{cases} \tag{2}$$

The directivity of linear antenna arrays of symmetric excitations can be obtained using (1) in the following relation [2, 3].

$$D = \frac{2kd\,|F(0)|^2}{\int_{-kd+\alpha}^{kd+\alpha} |F(\psi)|^2\, d\psi} \tag{3}$$

After some mathematical manipulations, the directivity is obtained as follows for odd and even number of elements, respectively, assuming $a = 0$.

$$D = \frac{\left(\sum_{n=0}^{N} A_n\right)^2}{\frac{1}{2}\sum_{n=0}^{N}\sum_{m=0}^{N} A_n A_m \left[\mathrm{sinc}\left(\frac{kd}{\pi}(n-m)\right) + \mathrm{sinc}\left(\frac{kd}{\pi}(n+m)\right)\right]} \tag{4}$$

$$D = \frac{\left(\sum_{n=1}^{N} A_n\right)^2}{\frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N} A_n A_m \left[\mathrm{sinc}\left(\frac{kd}{\pi}(n-m)\right) + \mathrm{sinc}\left(\frac{kd}{\pi}(n+m-1)\right)\right]} \tag{5}$$

The aim is to find the maximum directivity subject to a fixed beamwidth $2\psi_r$ so that $|F(\psi_r)| = r$. The parameter $r$ is less than one for normalized array factors and states the type of beamwidth. For example, $r = 1/\sqrt{2}$ means half power beamwidth (HPBW), $r = 0$ for Null to Null beamwidth and $r = 1/\sqrt{10}$ for $-10$ dB beamwidth. Therefore, the following relation represents the beamwidth constraint.

$$\begin{cases} \sum_{n=0}^{N} A_n \cos(n\psi_r) = r \ ; \ L = 2N+1 \\ \sum_{n=1}^{N} A_n \cos\left((n-0.5)\psi_r\right) = r \ ; \ L = 2N \end{cases} \tag{6}$$

It is worthy to note that writing array factor in terms of real cosine functions, as in (1), instead of complex exponential functions, makes it real so that $F(\psi_r) = \pm|F(\psi_r)|$. The positive sign is for the main lobe and negative sign is for the first sidelobe. Therefore, by (1), the main lobe is differentiated from the sidelobe using $F(\psi_r) = +r$ as in (6) against $F(\psi_r) = -r$.

Now, synthesis of maximum directivity leads to minimizing the denominator of (4) or (5) provided that two equations (2) and (6) are held. This can be done by the Lagrange multiplier method [11]. In this method, the following functions must be minimized for odd and even number of elements, respectively.

$$\frac{1}{2}\sum_{n=0}^{N}\sum_{m=0}^{N} A_n A_m \left[\mathrm{sinc}\left(\frac{kd}{\pi}(n-m)\right) + \mathrm{sinc}\left(\frac{kd}{\pi}(n+m)\right)\right] - \beta_1\left(\sum_{n=0}^{N} A_n - 1\right) - \beta_2\left(\sum_{n=0}^{N} A_n\cos(n\psi_r) - r\right) \to \mathrm{Min.} \tag{7}$$

$$\frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N} A_n A_m \left[\mathrm{sinc}\left(\frac{kd}{\pi}(n-m)\right) + \mathrm{sinc}\left(\frac{kd}{\pi}(n+m-1)\right)\right] - \beta_1\left(\sum_{n=1}^{N} A_n - 1\right) - \beta_2\left(\sum_{n=1}^{N} A_n\cos\left((n-0.5)\psi_r\right) - r\right) \to \mathrm{Min.} \tag{8}$$

where $\beta_1$ and $\beta_2$ are two unknown Lagrangian multipliers. The derivative of (7) and (8) with respect to all $A_n$ must be equated to zero. Therefore, the following sets of equations are obtained for odd and even number of elements, respectively.

$$\sum_{m=0}^{N} A_m \left[\mathrm{sinc}\left(\frac{kd}{\pi}(n-m)\right) + \mathrm{sinc}\left(\frac{kd}{\pi}(n+m)\right)\right] - \beta_1 - \beta_2\cos(n\psi_r) = 0 \qquad \text{for } n = 0,1,\cdots,N \tag{9}$$

$$\sum_{m=1}^{N} A_m \left[\mathrm{sinc}\left(\frac{kd}{\pi}(n-m)\right) + \mathrm{sinc}\left(\frac{kd}{\pi}(n+m-1)\right)\right] - \beta_1 - \beta_2\cos\left((n-0.5)\psi_r\right) = 0 \qquad \text{for } n = 1,\cdots,N \tag{10}$$

Finally, the sets of equations in (9) or (10) along with (2) and (6), make a square matrix equation whose solution gives us all unknown parameters $A_n$ and then the excitation currents $I_n$.

It is worth noting that if $\alpha \neq 0$, the terms $\cos((n-m)\alpha)$, $\cos((n+m)\alpha)$, and $\cos((n+m-1)\alpha)$ must be multiplied into respective sinc functions existing in (4), (5), and (7)–(10). Of course, $\alpha$ has no effect when $d = \lambda/2$.

## 3 Examples and Discussion

To verify the proposed method to synthesize maximum-directivity arrays, some examples are presented. Here, an expanding factor is defined as $s = \psi_r/\psi_{ru}$ in which $2\psi_{ru}$ is the beamwidth of uniformly excited array so that $F(\psi_{ru}) = r$.

First, an array with $L = 11$ elements of space $d = 0.5\lambda$ is designed to have maximum directivity subject to some specified Null to Null beamwidths, i.e. $r = 0$. Here, the first null of patterns with respect to those of uniform array, i.e. $\psi_{ru} = 2\pi/L$, are chosen so that $s$ = 0.9, 1.0, 1.1, 1.15, and 1.2.

Fig. 1 shows the resultant patterns which have exactly the specified null to null beamwidths. It is seen that as $s$ or equivalently the beamwidth increases the value of SLL decreases. However, when expanding factor $s$ increases beyond some value about 1.15, SLL starts to increase due to increasing the level of second sidelobe.

Fig. 2 shows the required excitation currents of synthesized maximum directivity arrays. The current

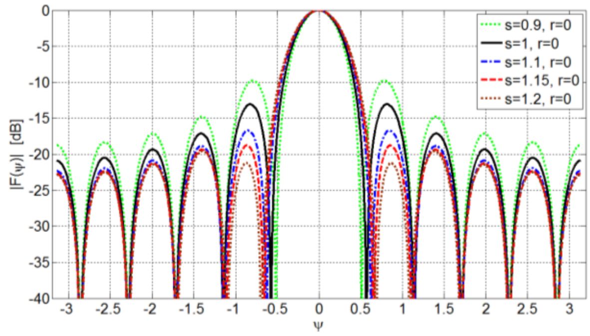


**Fig. 1** Synthesized maximum directivity array factors for some specified Null to Null beamwidths for $L$ = 11 elements of space $d = 0.5\lambda$.

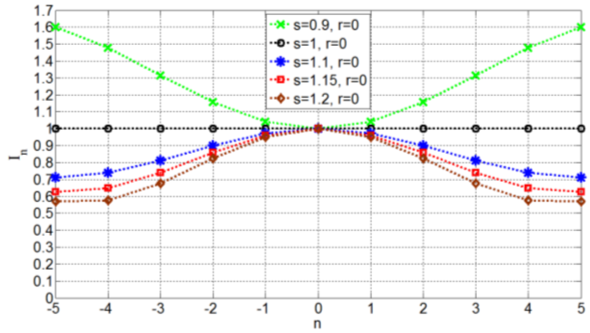


**Fig. 2** Required excitation currents of maximum directivity arrays of specified null to null beamwidths.

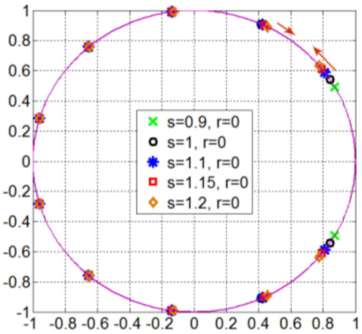


**Fig. 3** Locus of zeros of maximum directivity arrays of specified null to null beamwidths on the Schelkunoff's unit circle.

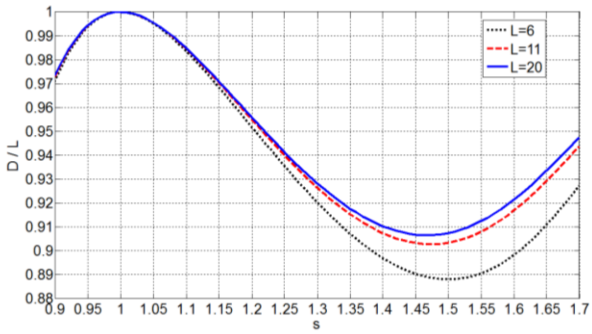


**Fig. 4** Normalized directivity of synthesized arrays of $L$ = 6, 11, and 20 elements of space $d = 0.5\lambda$ for specified null to null beamwidth.

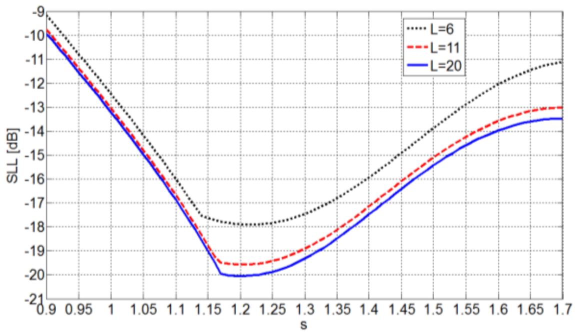


**Fig. 5** Sidelobe level of synthesized arrays of $L$ = 6, 11, and 20 elements of space $d = 0.5\lambda$ for specified null to null beamwidth.

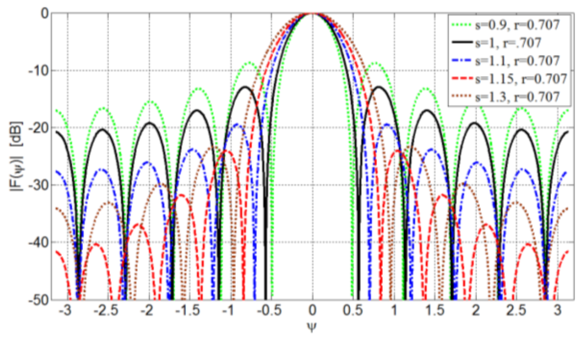


**Fig. 6** Synthesized maximum directivity array factors for some specified half power beamwidths for $L$ = 11 elements of space $d = 0.5\lambda$.

distribution is tapering and dynamic range is less than 1.7. The beamwidths less than that of uniform array, need inverse tapering. Also, Fig. 3 shows the locus of the zeros on the Schelkunoff's unit circle [12]. It is seen that as the null to null beamwidth increases, the first zeros move toward the second zeros while the second zeros move toward the first zeros. This is the cause of decreasing the level of the first sidelobe while growing the level of the second sidelobe.

Figs. 4 and 5 illustrate the resultant normalized directivity, i.e. $D/L$, and SLL of three arrays of $L$ = 6, 11, and 20 elements with space $d = 0.5\lambda$ versus the expanding factor $s$ of null to null beamwidth, respectively. It is seen that the maximum normalized directivity and SLL are almost independent of the number of elements. Also, SLL is decreasing toward about -19 dB as the null to null beamwidth increases toward $s$ = 1.15 times of that of uniformly excited array. In this range of null to null beamwidth, the directivity of synthesized arrays is very near to directivity of uniformly excited arrays, i.e. more than $0.97L$.

Secondly, an array with $L$=11 elements of space $d$=$0.5\lambda$ is designed to have maximum directivity subject to some specified half power beamwidths (HPBW), i.e. $r$ = 0.707. Here, the first half power of patterns with respect to those of uniform array, i.e. $\psi_{ru} = 0.886\pi/L$, are chosen so that $s$ = 0.9, 1.0, 1.1, 1.15, and 1.2.

Fig. 6 shows the resultant patterns which have exactly

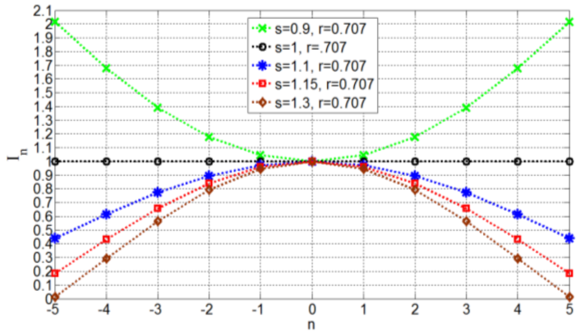


**Fig. 7** Required excitation currents of maximum directivity arrays of specified half power beamwidths.

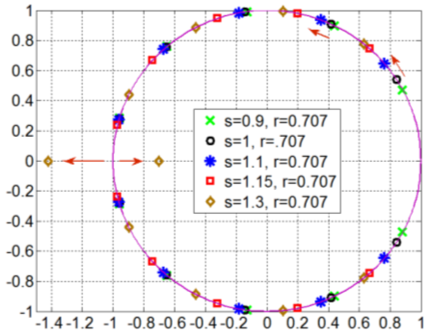


**Fig. 8** Locus of zeros of maximum directivity arrays of specified half power beamwidths on the Schelkunoff's unit circle.

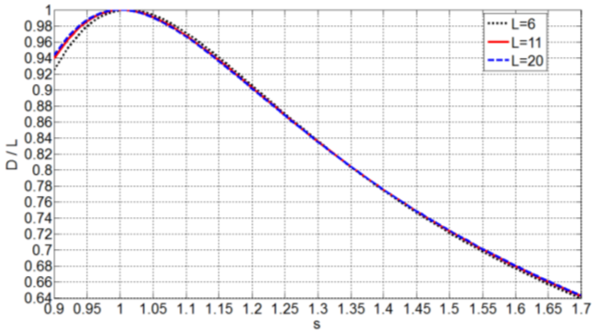


**Fig. 9** Normalized directivity of synthesized arrays of $L$ = 6, 11, and 20 elements of space $d$ = 0.5$\lambda$ for specified half power beamwidth.

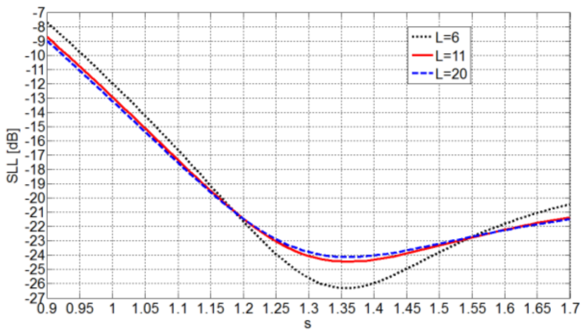


**Fig. 10** Sidelobe level of synthesized arrays of $L$ = 6, 11, and 20 elements of space $d$ = 0.5$\lambda$ for specified half power beamwidth.

the specified half power beamwidths. It is seen that as $s$ or equivalently the beamwidth increases the value of SLL decreases. In the case of fixing half power beamwidth, the first sidelobe is always greater than the second one.

Fig. 7 shows the required excitation currents of synthesized maximum directivity arrays. The current distribution is tapering and dynamic range exceeds 5 for expanding factor $s$ larger than 1.15. Also, Fig. 8 shows the locus of the zeros on the Schelkunoff's unit circle [12]. It is seen that as the HPBW increases, the first zero as well as other zeros move toward $z$= –1 point. For larger HPBWs, two zeros starts to exit from unit circle and move away from $z$ = –1 point so that their product remains equal to one.

Figs. 9 and 10 illustrate the resultant normalized directivity, i.e. $D/L$, and SLL of three arrays of $L$ = 6, 11, and 20 elements of space $d$ = 0.5$\lambda$ versus the expanding factor $s$ of half power beamwidth, respectively. It is seen that the number of elements has no significant effect on the maximum normalized directivity and SLL. Also, SLL is decreasing toward about –24 dB as the HPBW increases toward $s$ = 1.35 times of that of uniformly excited array. In this range of null to null beamwidth, the directivity of synthesized arrays lessens toward 0.8 $L$. If we limit the expansion of HPBW so that $s$ to be less than 1.15, SLL can be lessened toward about –20 dB while directivity is very near to directivity of uniformly excited arrays, i.e. more than 0.94$L$.

It is noteworthy that the beamwidth in $\psi$ domain is related to the beamwidth in $\theta$ domain. This relation is as follows [8].

$$\psi_r = kd\sin\left(\Delta\theta/2\right)\sqrt{1-\frac{\cos^2(\theta_0)}{\cos^2(\Delta\theta/2)}} \tag{11}$$

where $\Delta\theta$ is the beamwidth of the main lobe of at the angle of maximum radiation $\theta_0$.

## 4 Conclusion

Linear antenna arrays are synthesized to have maximum directivity for a specified beamwidth. The directivity is maximized subject to a given beamwidth such as null to null or half power one. The excitation currents are obtained using a matrix equation obtained from the Lagrange multiplier method. The performance of the proposed procedure is studied by means of some examples. It is almost independent of the number of elements. The directivity of the synthesized arrays is close to the number of elements. As the specified beamwidth is less than 1.15 times of that of uniformly excited arrays, SLL and directivity are almost in the same range for both null to null and half power beamwidths.

## Intellectual Property

The authors confirm that they have given due consideration to the protection of intellectual property associated with this work and that there are no impediments to publication, including the timing of publication, with respect to intellectual property.

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**M. Khalaj-Amirhosseini:** Idea & conceptualization, Research & investigation, Data curation, Analysis, Writing - Original draft, Revise & editing.

**Declaration of Competing Interest**

The authors hereby confirm that the submitted manuscript is an original work and has not been published so far, is not under consideration for publication by any other journal and will not be submitted to any other journal until the decision will be made by this journal. All authors have approved the manuscript and agree with its submission to “Iranian Journal of Electrical and Electronic Engineering”.

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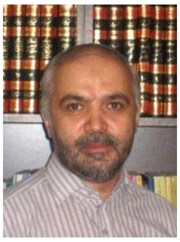

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