A New Theorem Concerning Scattering of Electromagnetic Waves in Multilayered Metamaterial Spherical Structures

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Abstract: The proposed theorem in this paper is indicative of a kind of duality in the propagation of waves in the dual media of ENG↔MNG and DPS↔DNG in the spherical structures. Independent of wave frequency, the number of layers, their thickness, and the type of polarization, this theorem holds true in case of any change in any of these conditions.

Keywords: Double Negative Materials (DNG), Double Positive Materials (DPS), Epsilon Negative Materials (ENG), Metamaterials, Mu Negative Materials (MNG), Radar Cross Section (RCS).

1 Introduction

Metamaterials are the group of materials in which the real part of the permittivity and/or permeability is negative [1-6]. These materials are classified in three groups of ENG (Epsilon Negative) with negative permittivity, MNG (Mu Negative) with negative permeability and DNG (Double Negative) with negative permittivity and permeability. Natural materials are called DPS (Double Positive). The chief purpose of this paper is to establish a duality between two dual planar multilayered spherical structures with interchanges DPS↔DNG or ENG↔MNG. Consider a plane wave incident on a multilayered spherical structure. The core of the structure may be PEC, metamaterial or dielectric. If we apply the interchange DPS↔DNG or ENG↔MNG for the constituting materials of the spherical structure and the surrounding medium, the radar cross section of the structure will not change in any direction. Establishing a duality between different classes of these structures may be used to extend the practical limitations of realization of metamaterials. This kind of duality was shown for planar structures [3-5]. Expanding the application scope of this duality to include spherical media as well is what this paper deals itself with.

The structure that will be studied in this paper is a multilayered spherical structure, which is particularly important owing to the fact that it can be considered as one of the basic and canonical shape of practical objects. In general, electromagnetic waves scattering from multilayered spherical structures are studied by analytical, approximate and numerical methods [6-8]. Here we use the theoretical method of addition theorems, where the fields are expanded in terms of the spherical Eigen functions in various media. In this method the wave in each layer is considered as the sum of the two forward and backward waves [6]. It should be noted that due to employing different materials (conventional materials and metamaterials) the correct sign of the real and imaginary parts of the wave number k and intrinsic impedance η should be selected [9-12].

The formulation used in this paper is valid for any frequency bandwidth, number of layers and constitutive materials, and for both monostatic and bistatic radars.

At first the numerical procedure is discussed and then the theorem is given a mathematical proof. At last some typical examples are represented to validate this theorem.

2 Numerical Procedure

Consider a sphere of radius r_1 coated by several concentric spherical coatings of radii r_i, $i = 2, 3, \ldots$ as shown in Fig. 1. The inner sphere may be composed of a perfectly electric conductor or dielectric material or metamaterial.

In order to calculate the electromagnetic plane waves scattering from this spherical structure, we use a method based on the radial components of electric and magnetic Hertzian potential vectors. The advantage of this method is using the Scalar wave equations instead of its vector state [6]. The field components in spherical coordinates can be expressed in:

$$\vec{\Pi}_e = \Pi_1 \hat{r}\ ,\ \vec{\Pi}_m = \Pi_2 \hat{r} \qquad (1)$$

where Π_1 is responsible for producing all TM modes as $H_r = 0$, and Π_2 produces all TE modes as $E_r = 0$.

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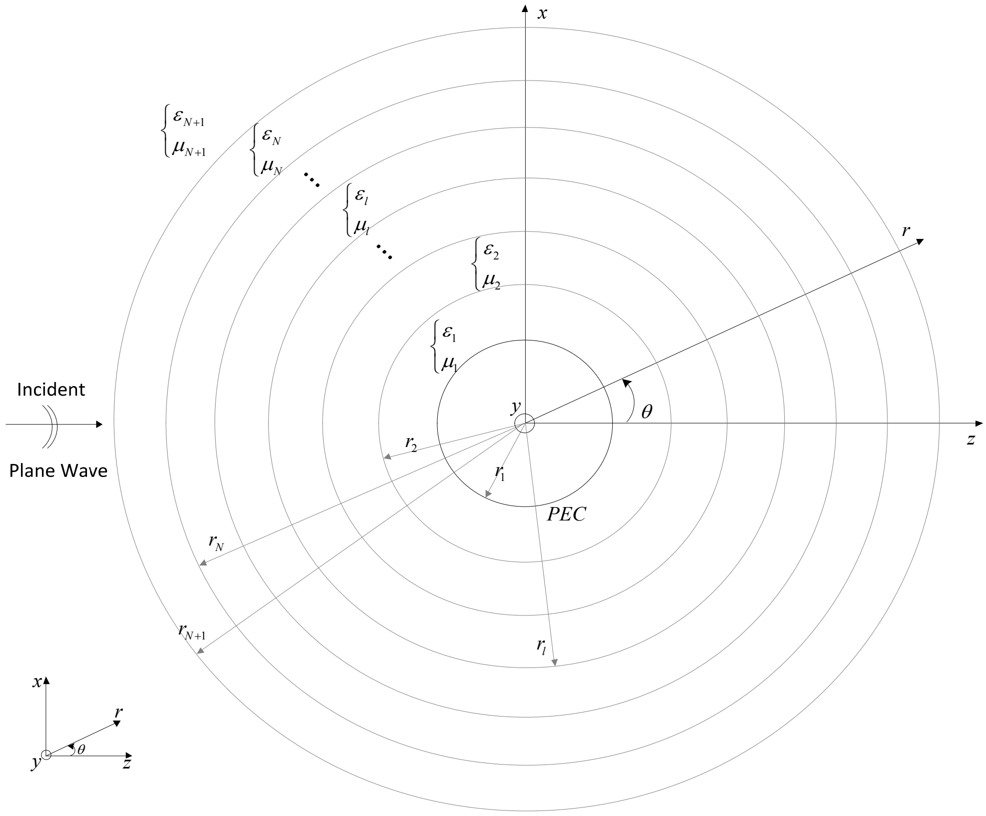

Fig. 1 Geometry of the problem.

Functions Π_1 and Π_2 satisfy the scalar wave equations:

$$(\nabla^2 + k^2)\Pi_i = 0\,, \quad i = 1,2 \tag{2}$$

Electric and magnetic fields can be achieved by the equations below. Here, boundary conditions for Π_1 and Π_2 are different:

$$\begin{cases}\overline{E} = \overline{\nabla}\times\overline{\nabla}\times(r\Pi_1\hat{r}) - j\omega\mu\overline{\nabla}\times(r\Pi_2\hat{r})\\ \overline{H} = j\omega\varepsilon\overline{\nabla}\times(r\Pi_1\hat{r}) + \overline{\nabla}\times\overline{\nabla}\times(r\Pi_2\hat{r})\end{cases} \tag{3}$$

Thus H_Φ, H_θ, H_r, E_Φ, E_θ and E_r in spherical coordinates can be expressed in terms of Π_1 and Π_2:

$$\begin{cases}E_\varphi = \dfrac{1}{r\sin\theta}\dfrac{\partial^2}{\partial r\partial\varphi}(r\,\Pi_1) + j\omega\mu\dfrac{\partial}{\partial\theta}\Pi_2\\ H_\varphi = -j\omega\varepsilon\dfrac{\partial}{\partial\theta}\Pi_1, + \dfrac{1}{r\sin\theta}\dfrac{\partial^2}{\partial r\partial\varphi}(r\Pi_2)\\ E_\theta = \dfrac{1}{r}\dfrac{\partial^2}{\partial r\partial\theta}(r\Pi_1) - j\omega\mu\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\varphi}\Pi_2\\ H_\theta = j\omega\varepsilon\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\varphi}\Pi_1 + \dfrac{1}{r}\dfrac{\partial^2}{\partial r\partial\theta}(r\Pi_2)\\ E_r = \dfrac{\partial^2}{\partial r^2}(r\Pi_1) + k^2 r\Pi_1\\ H_r = \dfrac{\partial^2}{\partial r^2}(r\Pi_2) + k^2 r\Pi_2\end{cases} \tag{4}$$

Now, consider a plane wave incident on the spherical structure:

$$\begin{cases}\overline{E}_{inc} = e^{-jk_{N+1}z}\hat{x}\\ \overline{H}_{inc} = \dfrac{1}{\eta_{N+1}}e^{-jk_{N+1}z}\hat{y}\end{cases} \tag{5}$$

The radial components of the fields are:

$$\begin{cases}E_{ir} = \overline{E}_{inc}.\hat{r} = e^{-jk_{N+1}r\cos\theta}\sin\theta\cos\phi\\ H_{ir} = \overline{H}_{inc}.\hat{r} = \dfrac{1}{\eta_{N+1}}e^{-jk_{N+1}r\cos\theta}\sin\theta\sin\phi\end{cases} \tag{6}$$

We assume the time dependence $e^{j\omega t}$, which should be considered for the selection of right signs of ε and μ [9-12].

The boundary conditions are the continuity of tangential components of electric and magnetic fields, namely:

$$\begin{aligned}E_{i,\theta} &= E_{i+1,\theta} \;\;, \;\; E_{i,\phi} = E_{i+1,\phi}\\ H_{i,\theta} &= H_{i+1,\theta} \;\;, \;\; H_{i,\phi} = H_{i+1,\phi}\end{aligned} \tag{7}$$

The expressions of boundary conditions will be functions of both Π_1 and Π_2, as Eq. (2) shows. However, we prefer the boundary conditions to be functions of only one Hertzian potential, namely Π_1 or Π_2. We may obtain such relations by a linear combination of the fields.

Consequently, the continuity of the following quantities is arrived at the boundaries $((\frac{\partial(r\Pi_1)}{\partial r}, \frac{\partial(r\Pi_2)}{\partial r}, \varepsilon\Pi_1, \mu\Pi_2))$.

We then express the incident plane wave in terms of the expansions of spherical functions for the radial Hertzian potential using the addition theorems [6]:

$$\begin{cases} \Pi^i_{(N+1)1} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \hat{J}_n(k_{N+1}r)P_n^m(\cos\theta)[A_{mn}^{(1)}\cos m\varphi + \\ \qquad\qquad B_{mn}^{(1)}\sin m\varphi] \\ \Pi^i_{(N+1)2} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \hat{J}_n(k_{N+1}r)P_n^m(\cos\theta)[A_{mn}^{(2)}\cos m\varphi + \\ \qquad\qquad B_{mn}^{(2)}\sin m\varphi] \end{cases} \tag{8}$$

where the surrounding medium in free space is denoted by $i = N+1$ and:

$$\begin{cases} r\Pi^i_{(N+1)1} = \frac{1}{k_{N+1}^2}\sum_{n=1}^{\infty} A_n\hat{J}_n(k_{N+1}r)P_n^1(\cos\theta)\cos\varphi \\ r\Pi^i_{(N+1)2} = \frac{1}{\eta_{N+1}k_{N+1}^2}\sum_{n=1}^{\infty} A_n\hat{J}_n(k_{N+1}r)P_n^1(\cos\theta)\sin\varphi \end{cases} \tag{9}$$

where $A_{1n}^{(1)} = \frac{(-j)^{n-1}(2n+1)}{n(n+1)} = A_n$.

The scattered field is given by the following Hertzian potential with superscript s instead of i in Eq. (9).

$$\begin{cases} r\Pi^s_{(N+1)1} = \frac{-1}{k_{N+1}^2}\sum_{n=1}^{\infty} A_n a_n\hat{H}_n^{(2)}(k_{N+1}r)P_n^1(\cos\theta)\cos\varphi \\ r\Pi^s_{(N+1)2} = \frac{-1}{\eta_{N+1}k_{N+1}^2}\sum_{n=1}^{\infty} A_n b_n\hat{H}_n^{(2)}(k_{N+1}r)P_n^1(\cos\theta)\sin\varphi \end{cases} \tag{10}$$

In function Π_{mn}, the first subscript m refers to the layer and the second one refers to the mode n = 1, 2.

We now express the Hertzian potentials in the spherical layers by the sum of spherical Bessel and Hankle functions as:

$$\begin{cases} r\Pi_{l1} = \frac{1}{k_l^2}\sum_{n=1}^{\infty} A_n\left[c_n\hat{J}_n(k_l r) + d_n\hat{Y}_n(k_l r)\right]\times \\ \qquad\qquad P_n^1(\cos\theta)\cos\varphi \\ r\Pi_{l2} = \frac{1}{\eta_l k_l^2}\sum_{n=1}^{\infty} A_n\left[c'_n\hat{J}_n(k_l r) + d'_n\hat{Y}_n(k_l r)\right]\times \\ \qquad\qquad P_n^1(\cos\theta)\sin\varphi \end{cases} \tag{11}$$

Inside the inner sphere, the function $\hat{Y}_n(k_l r)$ is omitted, because it is singular at $r = 0$.

We then consider the case of a PEC sphere. The tangential electric field on the surface of sphere should be zero:

$$E_\theta\Big|_{r=r_l} = E_\varphi\Big|_{r=r_l} = 0 \Rightarrow \begin{cases} \frac{\partial}{\partial r}(r\Pi_{11})\Big|_{r=r_l} = 0 \\ \Pi_{12}\Big|_{r=r_l} = 0 \end{cases} \tag{12}$$

Specifically, for Eq. (11) we have:

$$\begin{cases} c_n\hat{J}'_n(k_1 r_l) + d_n\hat{Y}'_n(k_1 r_l) = 0 \\ c'_n\hat{J}_n(k_1 r_l) + d'_n\hat{Y}_n(k_1 r_l) = 0 \end{cases} \tag{13}$$

We use the following identity:

$$\frac{\partial}{\partial r}\hat{J}_n(kr) = k\hat{J}_{n-1}(kr) - \frac{n}{r}\hat{J}_n(kr) \tag{14}$$

in Eq. (13) to obtain:

$$\begin{cases} c_n\left[k r_l\hat{J}_{n-1}(k_1 r_l) - n\hat{J}_n(k_1 r_l)\right] \\ \quad + d_n\left[k r_l\hat{Y}_{n-1}(k_1 r_l) - n\hat{Y}_n(k_1 r_l)\right] = 0 \\ c'_n\hat{J}_n(k_1 r_l) + d'_n\hat{Y}_n(k_1 r_l) = 0 \end{cases} \tag{15}$$

The continuity of tangential fields at the boundary between layers l and $l+1$ are:

$$\begin{cases} \frac{\partial}{\partial r}\left[r\Pi_{l1}\right]\Big|_{r=r_{l+1}} = \frac{\partial}{\partial r}\left[r\Pi_{(l+1)1}\right]\Big|_{r=r_{l+1}} \\ \varepsilon_l\Pi_{l1}\Big|_{r=r_{l+1}} = \varepsilon_{l+1}\Pi_{(l+1)1}\Big|_{r=r_{l+1}} \end{cases} \tag{16}$$

Combining Eq. (11) and the first relation in Eq. (16), we have:

$$\begin{aligned} &k_{l+1}^{-1}\left[e_n\hat{J}'_n(k_{l+1}r_l) + f_n\hat{Y}'_n(k_{l+1}r_l)\right] \\ &= k_l^{-1}\left[c_n\hat{J}'_n(k_l r_l) + d_n\hat{Y}'_n(k_l r_l)\right] \end{aligned} \tag{17}$$

For the second relation in Eq. (16), we have:

$$\begin{aligned} &\frac{1}{\mu_l}\left[c'_n\hat{J}_n(k_l r_l) + d'_n\hat{Y}_n(k_l r_l)\right] \\ &-\frac{1}{\mu_{l+1}}\left[e_n\hat{J}_n(k_{l+1}r_l) + f_n\hat{Y}_n(k_{l+1}r_l)\right] = 0 \end{aligned} \tag{18}$$

Finally, for the boundary between the outer coating and the free space, we have:

$$\begin{cases} \frac{\partial}{\partial r}\left[r\left(\Pi^i_{(N+1)1} + \Pi^s_{(N+1)1}\right)\right]\Big|_{r=r_{N+1}} = \\ \qquad \frac{\partial}{\partial r}\left[r\Pi_{N1}\right]\Big|_{r=r_{N+1}} \\ \varepsilon_{N+1}\left(\Pi^i_{(N+1)1} + \Pi^s_{(N+1)1}\right)\Big|_{r=r_{N+1}} = \\ \qquad \varepsilon_N\Pi_{N1}\Big|_{r=r_{N+1}} \end{cases} \tag{19}$$

Using Eqs. (9), (10) and (11), we have:

$$
\begin{cases}
\dfrac{1}{k_{N+1}}\left[\hat{J}'_n(k_{N+1}r_{N+1})-a_n\hat{H}^{(2)}_n{}'(k_{N+1}r_{N+1})\right] \\
\quad =\dfrac{1}{k_N}\left[p_n\hat{J}'_n(k_N r_{N+1})+q_n\hat{Y}'_n(k_N r_{N+1})\right] \\
\dfrac{1}{\mu_{N+1}}\left[\hat{J}_n(k_{N+1}r_{N+1})-a_n\hat{H}^{(2)}_n(k_{N+1}r_{N+1})\right] \\
\quad =\dfrac{1}{\mu_N}\left[p_n\hat{J}_n(k_N r_{N+1})+q_n\hat{Y}_n(k_N r_{N+1})\right]
\end{cases}
\tag{20}
$$

Consequently, we collect the boundary conditions for electric Hertzian potentials in a matrix equation as [A][X]=[B] in Eq. (21), where $m_l=\dfrac{k_l}{k_{l+1}}$.

In an exactly similar way a matrix equation may be derived for the magnetic Hertzian potential boundary conditions. In the first line, the derivatives are replaced by the functions, μ is replaced by $\eta\varepsilon$ and $m_l=\dfrac{k_l}{k_{l+1}}\dfrac{\eta_l}{\eta_{l+1}}=\dfrac{\mu_l}{\mu_{l+1}}$.

Also, in the case of dielectric or metamaterial core, the two derived matrix equations will change slightly as we will have 2 more unknowns and by applying the boundary conditions on the surface of the core (matching the tangential components of the fields) we will have 2 more equations too.

With the large argument relations for the spherical Hankle functions and according to Eq. (10), we have:

$$
\begin{cases}
r\Pi^s_{(N+1)1}=\dfrac{e^{-jk_{N+1}r}}{k^2_{N+1}}\displaystyle\sum_{n=1}^{\infty}j^{n-1}A_n a_n P^1_n(\cos\theta)\cos\varphi \\
r\Pi^s_{(N+1)2}=\dfrac{e^{-jk_{N+1}r}}{\eta_{N+1}k^2_{N+1}}\displaystyle\sum_{n=1}^{\infty}j^{n-1}A_n b_n P^1_n(\cos\theta)\sin\varphi
\end{cases}
\tag{22}
$$

and

$$
\frac{\partial}{\partial r}\left(r\Pi^s_{(N+1)i}\right)=-jk_{N+1}\Pi^s_{(N+1)i}\qquad ,\quad i=1,2
\tag{23}
$$

The field components are

$$
\begin{cases}
E^s_\theta=f_\theta(\theta,\varphi)\dfrac{e^{-jK_{N+1}r}}{r} \\
E^s_\varphi=f_\varphi(\theta,\varphi)\dfrac{e^{-jK_{N+1}r}}{r}
\end{cases}
\tag{24}
$$

where

$$
\begin{cases}
f_\theta=-\dfrac{j\cos\varphi S_2(\theta)}{k_{N+1}},\ S_2(\theta)= \\
\displaystyle\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[a_n\frac{d}{d\theta}P^1_n(\cos\theta)+b_n\frac{P^1_n(\cos\theta)}{\sin\theta}\right] \\
f_\varphi=\dfrac{j\sin\varphi S_1(\theta)}{k_{N+1}},\ S_1(\theta)= \\
\displaystyle\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[a_n\frac{P^1_n(\cos\theta)}{\sin\theta}+b_n\frac{d}{d\theta}P^1_n(\cos\theta)\right]
\end{cases}
\tag{25}
$$

in which

$$
\begin{bmatrix}
\hat{J}'_n(k_1r_1) & \hat{Y}'_n(k_1r_1) & 0 & 0\cdots & 0 & 0 & 0 \\
\hat{J}'_n(k_1r_2) & \hat{Y}'_n(k_1r_2) & -m_1\hat{J}'_n(k_2r_2) & -m_1\hat{Y}'_n(k_2r_2)\cdots & 0 & 0 & 0 \\
\frac{1}{\mu_1}\hat{J}_n(k_1r_2) & \frac{1}{\mu_1}\hat{Y}_n(k_1r_2) & -\frac{1}{\mu_2}\hat{J}_n(k_2r_2) & -\frac{1}{\mu_2}\hat{Y}_n(k_2r_2)\cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0\cdots & \hat{J}'_n(k_lr_{l+1}) & \hat{Y}'_n(k_lr_{l+1}) & -m_l\hat{J}'_n(k_{l+1}r_{l+1}) & -m_l\hat{Y}'_n(k_{l+1}r_{l+1})\cdots & 0 \\
0 & 0\cdots & \frac{1}{\mu_l}\hat{J}_n(k_lr_{l+1}) & \frac{1}{\mu_l}\hat{Y}_n(k_lr_{l+1}) & -\frac{1}{\mu_{l+1}}\hat{J}_n(k_{l+1}r_{l+1}) & -\frac{1}{\mu_{l+1}}\hat{Y}_n(k_{l+1}r_{l+1})\cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0\cdots & \hat{J}'_n(k_Nr_{N+1}) & \hat{Y}'_n(k_Nr_{N+1}) & m_N\hat{H}^{(2)'}_n(k_{N+1}r_{N+1}) \\
0 & 0 & 0 & 0\cdots & \frac{1}{\mu_N}\hat{J}_n(k_Nr_{N+1}) & \frac{1}{\mu_N}\hat{Y}_n(k_Nr_{N+1}) & \frac{1}{\mu_{N+1}}\hat{H}^{(2)}_n(k_{N+1}r_{N+1})
\end{bmatrix}
\tag{21}
$$

$$
\times\begin{bmatrix} c_n \\ d_n \\ e_n \\ f_n \\ \vdots \\ p_n \\ q_n \\ a_n \end{bmatrix}
=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ m_N\hat{J}'_n(k_{N+1}r_{N+1}) \\ \frac{1}{\mu_{N+1}}\hat{J}_n(k_{N+1}r_{N+1}) \end{bmatrix}
$$

$$\Lambda_n(\cos\theta) = \frac{P_n^1(\cos\theta)}{\sin\theta},$$
$$\Omega_n(\cos\theta) = \frac{d}{d\theta} P_n^1(\cos\theta) \tag{26}$$

Finally, considering Eq. (24) we can express the total scattered field as:

$$\left|E^s\right|^2 = \left|E^s_\theta\right|^2 + \left|E^s_\varphi\right|^2 = \frac{\left|f_\theta\right|^2 + \left|f_\varphi\right|^2}{r^2} \tag{27}$$

and according to definition of Radar Cross Section (RCS).

$$RCS = \sigma(\theta,\varphi) = \lim 4\pi r^2 \frac{\left|E^s\right|^2}{\left|E_{inc}\right|^2} \tag{28}$$

Finally, we derive the following equation for the RCS.

$$RCS = \sigma(\theta,\varphi) = 4\pi(\left|f_\theta\right|^2 + \left|f_\varphi\right|^2) \tag{29}$$

3 Proof of the Theorem

First, we examine the effects of the interchanges ENG↔MNG and DPS↔DNG on the propagation constants of the layers. Therefore, when the DPS material changes into DNG (or vice versa), we will have the following:

$$DPS \leftrightarrow DNG: \begin{cases} \varepsilon_{DPS} = \varepsilon' - j\varepsilon'' \\ \mu_{DPS} = \mu' - j\mu'' \end{cases}$$
$$\Rightarrow \begin{cases} \varepsilon_{DNG} = -\varepsilon' - j\varepsilon'' = -\varepsilon^*_{DPS} \\ \mu_{DNG} = -\mu' - j\mu'' = -\mu^*_{DPS} \end{cases} \tag{30}$$

The effect of these interchanges on the wave number and on the intrinsic impedance of the layer is thus:

$$\begin{cases} k_{DPS} = k' - jk'' \\ \eta_{DPS} = \eta' \pm j\eta'' \end{cases} \Rightarrow \begin{cases} k_{DNG} = -k' - jk'' = -k^*_{DPS} \\ \eta_{DNG} = \eta' \mp j\eta'' = \eta^*_{DPS} \end{cases} \tag{31}$$

It will be observed that, the effect of this interchange on η and k is the same as the previous one Eq. (31):

$$\begin{cases} k_{ENG} = \pm k' - jk'' \\ \eta_{ENG} = \eta' + j\eta'' \end{cases} \Rightarrow \begin{cases} k_{MNG} = \mp k' - jk'' = -k^*_{ENG} \\ \eta_{MNG} = \eta' - j\eta'' = \eta^*_{ENG} \end{cases} \tag{32}$$

In these relations, $\eta'', \eta', k'', k', \mu'', \mu', \varepsilon''$ and ε' are assumed to be positive. These relations show that as a result of the interchanges ENG↔MNG and DPS↔DNG k, μ, ε become equal to the negative conjugate of their previous case, and η becomes equal to the conjugate of its previous case. In addition, considering the qualities of the extended spherical functions of Bessel, we know that:

$$\hat{J}_n\left(-x^*\right) = (-1)^{n+1}\hat{J}_n^*(x),$$
$$\hat{J}_n'\left(-x^*\right) = (-1)^{n}\hat{J}_n'^*(x),$$
$$\hat{Y}_n\left(-x^*\right) = (-1)^{n}\hat{Y}_n^*(x),$$
$$\hat{Y}_n'\left(-x^*\right) = (-1)^{n+1}\hat{Y}_n'^*(x),$$
$$\hat{H}_n^{(2)}\left(-x^*\right) = (-1)^{n+1}\hat{H}_n^{(2)*}(x),$$
$$\hat{H}_n^{(2)'}\left(-x^*\right) = (-1)^{n}\hat{H}_n^{(2)'*}(x) \tag{33}$$

Now, consider matrix equation Eq. (21). After these interchanges are applied, according to relations Eq. (30-34), the new matrix equation will be thus:

$$(-1)^n [A]^* [X]_{new} = (-1)^n [B]^* \tag{34}$$

By comparing the above relations, it is easily observable that:

$$\begin{bmatrix} c_n \\ d_n \\ e_n \\ f_n \\ \vdots \\ p_n \\ q_n \\ a_n \end{bmatrix}_{new} = \begin{bmatrix} c_n \\ d_n \\ e_n \\ f_n \\ \vdots \\ p_n \\ q_n \\ a_n \end{bmatrix}^*_{old} \Rightarrow (a_n)_{new} = (a_n)^*_{old} \tag{35}$$

In a similar way, it can be seen from boundary conditions for magnetic Hertzian Potentials:

$$\begin{bmatrix} c'_n \\ d'_n \\ e'_n \\ f'_n \\ \vdots \\ p'_n \\ q'_n \\ b_n \end{bmatrix}_{new} = \begin{bmatrix} c'_n \\ d'_n \\ e'_n \\ f'_n \\ \vdots \\ p'_n \\ q'_n \\ b'_n \end{bmatrix}^*_{old} \Rightarrow (b_n)_{new} = (b_n)^*_{old} \tag{36}$$

Now, considering the fact that the functions of $\Lambda_n(\cos\theta)$ and $\Omega_n(\cos\theta)$ are real, according to relations mentioned Eq. (25), for all $0 \le \theta \le \pi$, we will have the following:

$$\left[S_i(\theta)\right]_{new} = \left[S_i(\theta)\right]^*_{old}; i = 1,2 \tag{37}$$

Therefore, taking the relations given in Eq. (25) into account, for f_φ and f_θ we will have:

$$(f_\theta)_{new} = -(f_\theta)^*_{old}, (f_\varphi)_{new} = -(f_\varphi)^*_{old} \tag{38}$$

Therefore:

$$\left|f_\theta\right|^2_{new} = \left|f_\theta\right|^2_{old}, \left|f_\varphi\right|^2_{new} = \left|f_\varphi\right|^2_{old} \tag{39}$$

Finally, according to relation Eq. (29), the theorem is proved, due to the fact that:

$$RCS_{new} = RCS_{old} \tag{40}$$

This conclusion is independent of the direction of the transmitter and the receiver.

4 Numerical Results and Discussion

In this section several typical examples are provided to verify the validity of the theorem for non-dispersive and dispersive metamaterials.

4.1 Non-Dispersive Media

Example 1) Lossy Multilayered Spherical Structure and its Metamaterial Dual

Consider a DPS sphere of radius 10 cm with parameters $\varepsilon = 5 - j0.1$, $\mu = 3 - j0.2$ coated with a layer of lossy DPS materials of thickness 2 cm and parameters $\varepsilon = 1 - j0.3$, $\mu = 7 - j0.05$ located in free space with parameters $\varepsilon = 1$, $\mu = 1$. We calculate the radar cross section at $\theta = \pi$, $\varphi = \pi/2$, in $\Delta f = [2, 15]$ GHz ultra wide frequency band. Then we apply the interchange DPS↔DNG so that the parameters of the central sphere change to $\varepsilon = -5 - j0.1$, $\mu = -3 - j0.2$ and the parameters of the coating to $\varepsilon = -1 - j0.3$, $\mu = -7 - j0.05$, and the free space becomes a metamaterial medium with the parameters $\varepsilon = 1$, $\mu = 1$. We calculate the radar cross section at the same coordinates again, and draw the corresponding curves in Fig. 2. It is observed that the RCSs do not change.

We repeat the same procedure again at $\theta = \pi/4$, $\varphi = \pi/5$, and calculate and draw the corresponding RCS curves in the two dual cases of DPS and DNG. The equality of the RCSs confirms the validity of the theorem proposed in this paper.

Example 2) Conductor Sphere with ENG or MNG Coating

A PEC sphere of radius 8 cm, coated with a layer of ENG materials of thickness 3 cm with parameters $\varepsilon = -5$, $\mu = 3$, receives an incidence of plane wave of frequency f=7 GHz. We calculate the radar cross section for $0 \leq \theta \leq \pi$ in the half planes $\varphi = \pi/3$ and π. Now, we replace the ENG coating with its MNG dual, i.e. $\varepsilon = +5$, $\mu = -3$, and the DPS free space with its metamaterial dual, i.e. $\varepsilon = -1$, $\mu = -1$, and calculate the RCS again and draw it in Fig. 3. It will be observed that the interchanges DPS↔DNG and ENG↔MNG do not have any effect on the RCS. Moreover the theorem is independent of the direction of the transmitter and the receiver.

4.2 Dispersive Media

In the examples above, the considered materials were non-dispersive and had great compatibility with the theorems proposed in this section. Although such examples provide us with a clear idea, the problem with them lies in the fact they cannot have physical realization. In practice, there are various dispersion relations for conventional materials and metamaterials, the most important of which are given in Table 1 [13].

Now, to examine the validity of the above-mentioned theorem for dispersive materials, we calculate the parameters of the dispersion relations in the arbitrary frequency bandwidth of Δf in a way that they realize the DPS↔DNG or ENG↔MNG interchange. For this purpose, we divide the Δf bandwidth to n mid-frequencies, and through the minimum squares method [14], derive parameters of dispersion functions (according to Table 1) in a way make the interchanges of Eq. (28) and (30) possible. Below, we have included examples for various dispersive materials in different structures.

Example 3) Lossy Dispersive and Magnetic Material Sphere or Dispersive Metamaterial Sphere

Consider a sphere of radius 6 cm composed of lossy magnetic materials with parameters $\varepsilon_r = 1$, $\mu_r = 1.74$, $\mu_i = 7.06$ and $\alpha = \beta = 1$. Consider another sphere, in another condition, composed of metamaterials with the dispersion relations of Drude and Lorentz and parameters $f_{ep} = 40.16$, $\gamma_e = 0.001$, $f_{mp} = 30.05$, $f_{mo} = 0.9$ and $\gamma_m = 6.65$.

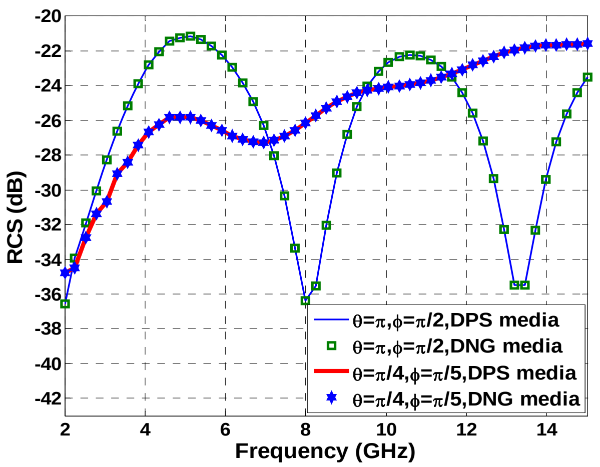

Fig. 2 The radar cross section of the multilayered spherical structure composed of DPS materials and of DNG materials (the dual case) at different angles versus frequency.

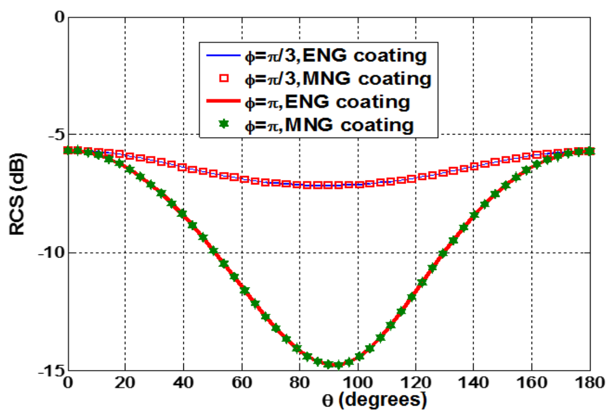

Fig. 3 The radar cross section of the conductor sphere coated with ENG materials and with MNG materials (the dual case) at frequency f = 7 GHz in the half planes $\varphi = \pi/3$ and π versus different angles of observation θ.

Table 1 Dispersion relations for common material and metamaterial media.

Type of Material	Permittivity Model	Permeability Model	Class of Material
DPS	$\varepsilon = \frac{\varepsilon_r}{f^\alpha} - j\frac{\varepsilon_i}{f^\beta}$	$\mu = \mu_r$	Lossy dielectric
DPS	$\varepsilon = \varepsilon_r$	$\mu = \frac{\mu_r}{f^\alpha} - j\frac{\mu_i}{f^b}$	Lossy Magnetic
DPS	$\varepsilon = \varepsilon_r$	$\mu = \frac{\mu_m(f_m^2 - jf_m f)}{f^2 + f_m^2}$	Relaxation-type magnetic
DPS, ENG	$\varepsilon = 1 - \frac{f_{ep}^2}{f^2 - j f \gamma_e}$	$\mu = 1$	Rods Only
DPS, MNG	$\varepsilon = 1$	$\mu = 1 - \frac{f_{mp}^2 - f_{mo}^2}{f^2 - f_{mo}^2 - j f \gamma_m}$	Rings Only
DPS, ENG, MNG, DNG	$\varepsilon = 1 - \frac{f_{ep}^2}{f^2 - j f \gamma_e}$	$\mu = 1 - \frac{f_{mp}^2 - f_{mo}^2}{f^2 - f_{mo}^2 - j f \gamma_m}$	Rods & Rings

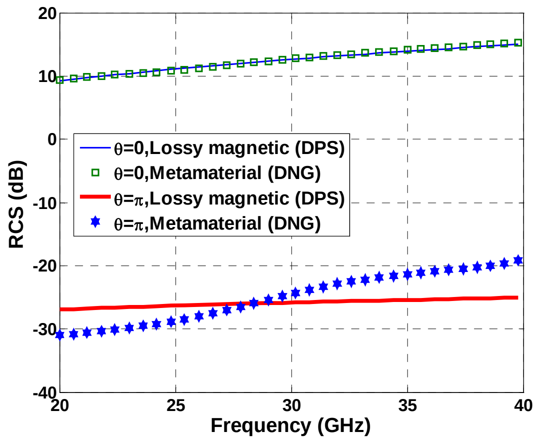

Fig. 4 The radar cross section of the sphere composed of dispersive DPS and DNG (the dual case) materials in directions ($\theta = 0$ and π) versus frequency.

It is noteworthy that these parameters are calculated, through the minimum squares method, in a way that makes the $\text{DPS} \leftrightarrow \text{DNG}$ interchange possible in the assumed Δf = [28, 29] GHz bandwidth. Now, we calculate the forward and backward radar cross sections (θ = 0, π) of this sphere, in Δf = [20, 40] GHz bandwidth, in the two dual states and draw them in Fig. 4. It should be noted that in the first case the DPS sphere is located in free space, and in the second case the DNG sphere is located in the dual metamaterial; that is $\varepsilon = -1$, $\mu = -1$. It will be observed that in the vicinity of $\Delta f = 28.5$ GHz frequency, at which the DPS↔DNG interchange only approximately holds, the calculated radar cross sections are very close to one another, and this confirms the proposed theorem. Now, to show that the theorem is independent of the direction of the transmitter and the receiver, we calculate the radar cross section, in both of the dual states, at frequency f = 28.5 GHz in all directions $(0 \le \varphi \le 2\pi, 0 \le \theta \le \pi)$, and then draw their difference, i.e. $RCS_{DPS} - RCS_{DNG}$, in Fig. 5. It will be observed that $RCS_{DPS} - RCS_{DNG}$ is very close to zero for all θ and φ. That is:

$$RCS_{DPS} \cong RCS_{DNG} \tag{41}$$

This verifies the validity of the proposed theorem even for the situations in which dispersive conventional materials or metamaterials are used.

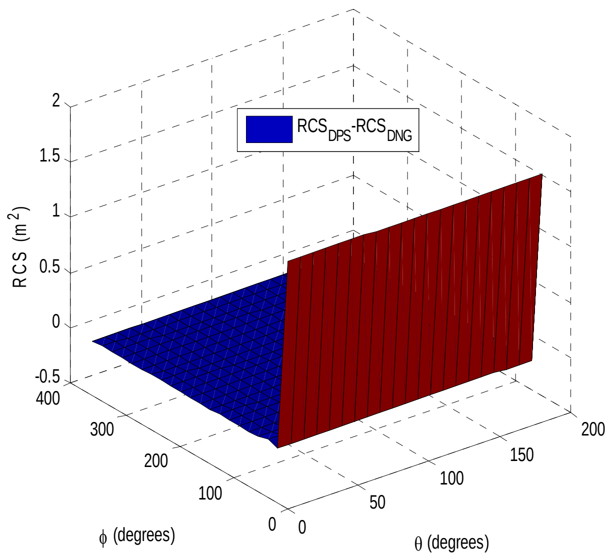

Fig. 5 Three dimensional graph of difference of the radar cross sections of the sphere composed of dispersive DSP and DNG materials (the dual case) at frequency f = 28.5 GHz versus the angles $0^\circ < \varphi < 360^\circ$, $0^\circ < \theta < 180^\circ$.

Example 4) Conductor Sphere with Dispersive ENG or MNG Coating

A conductor sphere of radius 2 cm, coated with a layer of ENG materials of thickness 2 cm with Drude dispersion model and parameters $f_{ep} = 42.9$, $\gamma_e = 0.001$, $\mu_r = 1$, is located in free space. It receives an incidence of plane wave of frequency $\Delta f = [29, 32]$ GHz. We calculate the radar cross section in the direction $\theta = \pi$ and also $\theta = \pi/4$, $\varphi = \pi/2$ and $\theta = \pi/2$, $\varphi = 0$. Now, according to ENG↔MNG and DPS↔DNG interchanges, we assume that the MNG coating follows the Lorentz dispersion model and has the parameters $\varepsilon_r = 1$, $f_{mp} = 42.9$, $f_{mo} = 0.5$ and $\gamma_m = 0.001$, and that the surrounding medium is a metamaterial with parameters $\varepsilon = -1$, $\mu = -1$. Then we calculate the RCS again and draw them all in Fig. 6. The results indicate the validity of the proposed theorems.

The theorem proved in this paper, may be used to extend the practical limitations of realization of metamaterials. Thin wires are used for the realization of negative permittivity and split-ring resonators are used for the realization of negative permeability. This theorem establishes a duality between electric and magnetic resonators. However, a medium with negative constitutive parameters doesn't exist and this limits the application of the theorem.

5 Conclusion

A new theorem has been proved in this paper, for incidence of plane waves on a multilayer spherical structure with PEC, metamaterial or dielectric core coated with metamaterials. The interchanges DPS↔DNG and ENG↔MNG does not change RCS in any direction. Then a duality is established for the radar cross section between two dual structures with substitutions DPS↔DNG and ENG↔MNG.

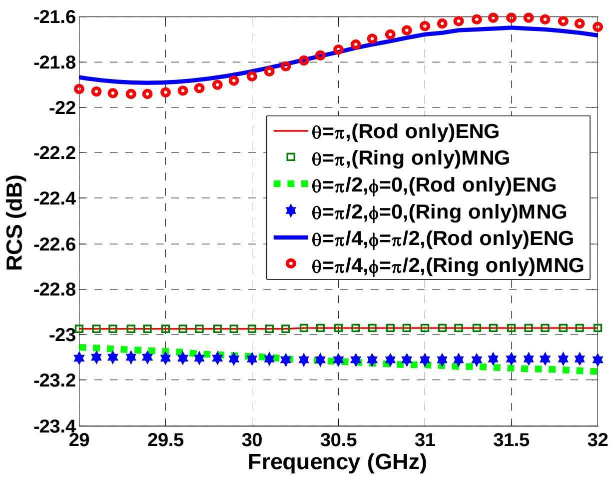

Fig. 6 The radar cross section of the conductor sphere coated with dispersive ENG materials and dispersive MNG materials (the dual case) versus frequency in different directions.

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