A New Switching Strategy for Exponential Stabilization of Uncertain Discrete-Time Switched Linear Systems in Guaranteed Cost Control Problem

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Abstract: Uncertain switched linear systems are known as an important class of control systems. Performance of these systems is affected by uncertainties and their stabilization is a main concern of recent studies. Existing work on stabilization of these systems only provides asymptotical stabilization via designing switching strategy and state-feedback controller. In this paper, considering a given infinite-horizon cost function, a new switching strategy and a state-feedback control laws are designed to exponentially stabilize Uncertain Discrete-Time Switched Linear Systems (UDSLS). Our design procedure consists of two steps. First, we generalize the exponential stabilization theorem of nonlinear systems into UDSLS. Second, a new stabilizing switching strategy based on the Common Lyapunov Function technique presented. Hence, a sufficient condition on the existence of state-feedback controller is provided in the form of Linear Matrix Inequality. Besides, convergence rate of the states is obtained and the upper bound of the cost is calculated. Finally, effectiveness of the proposed method is verified via numerical example.

Keywords: Exponential Stabilization, Guaranteed Cost Control (GCC), Linear Matrix Inequality (LMI), Uncertain Switched Linear Systems.

1 Introduction

In recent years, researches on the switched systems have been widely increased [1]-[4]. The switched linear systems are one of the most important classes of hybrid systems with a switching signal that allows selection of one subsystem at each time to reach some control objectives. Many real-world processes such as switched circuits, switching power converters, computer controlled systems, communication networks and chemical process can be modeled as switched systems [5]-[7].

The issue of stabilization analysis and control of switched systems have been extensively studied in the literature [3], [4], [8]-[14]. Stabilization strategies for switched systems are mainly divided into two categories [1]. First, one problem is to design a suitable state-

feedback control law to stabilize the closed-loop system under any arbitrary switching law given in an admissible set. Second, when switching signal is a design variable, the stabilization problem is to design switching signal and control law to stabilize the switched system.

Stabilization of non-autonomous switched linear systems through switching and continuous control laws has also been studied in the recent years [8], [9], [12]. Most of these results are the extension of the stabilization for the autonomous switched systems [1]-[7]. Robustness of switched linear systems is a main concern and behavior of these systems is affected by uncertainties. Hence, one of the problems associated with this study is how to design switching and control laws to overcome these uncertainties in order to guarantee stability of the system [11]. One way to deal with this uncertainty is called Guaranteed Cost Control (GCC) approach, proposed first in [15]. The advantage of this approach is providing an upper bound on the given performance index to guarantee the system performance dealing with uncertainties. Based on this idea, some results have been reported in the uncertain linear systems [16]-[17]. Also, there are some studies on the robust control of uncertain switched systems [18]-[28]. Especially, in [18], [21] and [23] some significant

results have been reported on the designing state feedback controller and switching laws in GCC problem for uncertain discrete-time switched linear systems with infinite horizon LQR cost function.

In these studies, the switching strategy and the state-feedback controller are designed by using Common Lyapunov Function (CLF) and Multiple Lyapunov Functions (MLFs) techniques. Switching strategy is designed based on selecting a subsystem which has the lowest Lyapunov function value. Furthermore, these switching strategies provide asymptotic stability of the overall switched closed-loop system. Especially, when the CLF technique is used to design switching strategy, some complex Linear Matrix Inequalities (LMIs) are constructed which can be solved. Moreover, none of these studies can guarantee the boundedness of the guaranteed cost.

Recently, exponential stabilization of discrete-time switched linear systems containing an infinite-horizon LQR cost function has been studied [29]-[31]. Most of these studies are based on dynamic programming, control Lyapunov function and piecewise quadratic Lyapunov function approaches for finding switching and control laws.

The main goal of this paper is to exponentially stabilize the uncertain discrete-time switched linear systems and obtain the upper bound of an infinite-horizon LQR cost function. To this end, based on the proposed CLF function, a new stabilizing switching strategy is designed. Also, it is proved that the proposed CLF satisfies the conditions of the exponential stability theorem. Besides, guaranteed cost controller is designed and the upper bound of the cost is obtained. The contributions of the paper are as follows:

(i) Presenting a new switching strategy to guarantee the exponential stability of the uncertain discrete-time switched linear systems,

(ii) Proving that the proposed CLF satisfies the conditions of exponential stability theorem.

(iii) Obtaining convergence rate of states.

The present paper is organized as follows: In the next section, problem formulation and preliminaries are given. In Section 3, exponential stability theorem of nonlinear systems is generalized into UDSLS and a new policy is presented, which guarantees the exponential stability of the overall switched closed-loop system. Moreover, sufficient condition on the existence of guaranteed cost controller is obtained. Section 4 and Section 5 are dedicated for illustrative example and conclusion, respectively.

Also, the following notations will be used throughout this paper. m is some arbitrary positive integer, Z^+ denotes the set of non-negative integers, $\underline{m} \triangleq \{1, 2, ..., m\}$ is the set of subsystem indices, $\|\cdot\|$ denotes the standard Euclidean norm in R^n, $\lambda(A)$ stands for eigenvalues of matrix A and the variable z denotes a generic initial state $x(0)$ of the system in Eq. (1).

2 Problem Formulation and Preliminaries

In this section, problem formulation, necessary Assumption, Definitions, Lemmas and exponential stability concept are presented.

Consider the uncertain discrete-time switched linear systems described by:

$$x(t+1) = (A_{r(x,t)} + \Delta A_{r(x,t)})x(t) +$$
$$(B_{r(x,t)} + \Delta B_{r(x,t)})u_{r(x,t)}(t), \quad t \in Z^+ \quad (1)$$
$$x(t_0) = x(0) \in R^n$$

where, $x(t) \in R^n$ is the state vector, $u_{r(x,t)}(t) \in R^q$ is the control input vector and $r(x,t) \in \underline{m}$ is the piecewise constant discrete switching signal that determines the discrete mode $i \in \underline{m}$. Moreover, $A_i \in R^{n \times n}$ and $B_i \in R^{n \times q}$, $i \in \underline{m}$, are the dynamics of each subsystem with appropriate dimensions and $\Delta A_i, \Delta B_i$, $i \in \underline{m}$ are uncertainties satisfying the following assumption.

Assumption 1 ([11]). The time-varying parameter uncertainties $\Delta A_i, \Delta B_i$ of the system have the following form:

$$[\Delta A_i, \Delta B_i] = N_i F_i(t)[C_i, D_i], \quad i \in \underline{m} \quad (2)$$

where C_i, D_i and N_i are known matrices with appropriate dimensions and $F_i(t)$, $i \in \underline{m}$, are unknown matrices that satisfy:

$$F_i^T(t)F_i(t) \leq I, \quad i \in \underline{m}. \quad (3)$$

We define the following cost function as a total cost, starting from $x(0) = z$ for the uncertain switched system Eq. (1) as follows:

$$J(z) = \sum_{t=0}^{\infty} \left(x(t)^T Q_{r(x,t)} x(t) + u_{r(x,t)}^T(t) R_{r(x,t)} u_{r(x,t)}(t) \right) \quad (4)$$

where $Q_{r(x,t)} \in R^{n \times n}$ and $R_{r(x,t)} \in R^{q \times q}$ are symmetric positive definite weighted matrices. It is clear that, the cost $J(.)$ depends to initial state z.

The main aim is to find switching signal $r(x,t) \in \underline{m}$ and the stat-feedback controller $u_{r(x,t)}(t) = K_i x(t)$, where $K_i \in R^{q \times n}$, $i \in \underline{m}$, such that, the uncertain switched system (1) to be exponentially stable and the cost function (4) satisfies $J(z) \leq J^*, \forall z \in R^n$, where, J^* is defined as a guaranteed cost in Definition 1.

Definition 1 ([19]). For the uncertain switched system (1), if there exist a state feedback control $u_{r(x,t)}^*(t)$ for each subsystem, a switching law $r^*(x,t)$ and a positive scalar J^* such that for all admissible uncertainties, the closed loop system is stable and the value of the cost function Eq. (4) satisfies $J(z) \leq J^*$, then, J^* and $u_{r(x,t)}^*(t)$ are said to be Guaranteed Cost Value (GCV) and guaranteed cost control law (GCCL).

Definition 2 (Policy π). The sequence of pairs $\left\{ r(x,t), u_{r(x,t)}(t) \right\}_{t=0}^{t=\infty}$ is called a switching-control sequence. A sequence of switching–control laws constitutes a feedback policy as the following:

$$\pi \triangleq \left\{ \left(r(x,t_0), u(t_0) \right), \left(r(x,t_1), u(t_1) \right), \ldots \right\} \tag{5}$$

Lemma 1 ([32]). Let H, M and L be real matrices with appropriate dimensions. If P is a positive definite matrix and ε is a positive scalar, such that $\varepsilon^{-1} I - M^T P M > 0$, then, the following inequality holds for the any norm-bounded time varying uncertainty $F(t)$ satisfying $F^T(t) F(t) \le I$.

$$\left(H + M F(t) L \right)^T P \left(H + M F(t) L \right) \le$$
$$H^T \left(P^{-1} - \varepsilon M M^T \right)^{-1} H + \varepsilon^{-1} L^T L \tag{6}$$

Definition 3 (Exponential Stability [29]). The exponential stabilization problem of the system Eq. (1) is to find a sequence of switching-control laws Eq. (5) under which the trajectory $x(t)$ starting from any initial state $x(0) = z$ satisfies:

$$\|x(t)\|^2 \le a c^t \|z\|^2, \quad \forall t \in Z^+ \tag{7}$$

For some constants $a \ge 1$ and convergence rate $c \in (0,1)$. If there exists a stabilizing policy satisfying Eq. (5), then the system Eq. (1) is called exponentially stabilizable.

3 Main Results

In this section, first we extend exponential stability theorem in discrete nonlinear systems. Then we state and prove Lemma 2, Lemma 3 and Theorem 2 to design a new stabilizing switching and control laws in uncertain discrete-time switched linear systems and we show that proposed common Lyapunov function satisfies the conditions of exponential stability theorem.

Theorem 1. Suppose that there exist a policy π Eq. (5) and a function $V : R^n \to R^+$ satisfying conditions (i) and (ii) for all $(t, x(t))$ where, $x(t)$ is the trajectory of the system (1) under the policy π.

(i) $\kappa_1 \|z\|^2 \le V(z) \le \kappa_2 \|z\|^2$ for any $z \in R^n$ and some finite positive constants κ_1 and κ_2;

(ii) $V(x(t+1)) - V(x(t)) \le -\kappa_3 \|x(t)\|^2$ for any $t \in Z^+$ and some constants $\kappa_3 > 0$,

Then system Eq. (1) is exponentially stable under policy π.

Proof. This theorem is a discrete-time version of exponential theorem in [34]. To prove, it must be shown that there exist two positive constants $a \ge 1$ and $c \in (0,1)$ such that Eq. (7) holds. To this end, from (i) and (ii) we have:

$$V(x(t+1)) - V(x(t)) \le -\kappa_3 \|x(t)\|^2 \le -\frac{\kappa_3}{\kappa_2} V(x(t)) \tag{8}$$

Then,

$$V(x(t+1)) \le \left(1 - \frac{\kappa_3}{\kappa_2} \right) V(x(t)) \tag{9}$$

By substitution $t = 0, 1, 2, \ldots$ in Eq. (9) we conclude:

$$V(x(1)) \le \left(1 - \frac{\kappa_3}{\kappa_2} \right) V(x(0)),$$

$$V(x(2)) \le \left(1 - \frac{\kappa_3}{\kappa_2} \right) V(x(1)) \le \left(1 - \frac{\kappa_3}{\kappa_2} \right)^2 V(x(0)),$$

$$\vdots \tag{10}$$

$$V(x(t)) \le \left(1 - \frac{\kappa_3}{\kappa_2} \right)^t V(x(0)), \quad \forall t \in Z^+$$

In order to show that $1 - \frac{\kappa_3}{\kappa_2} < 1$, we use the inequalities (i) and (ii), which yields:

$$0 \le V(x(t+1)) \le V(x(t)) - \kappa_3 \|x(t)\|^2 \le$$
$$\left(\kappa_2 - \kappa_3 \right) \|x(t)\|^2 \tag{11}$$

since $\kappa_2 \ge \kappa_3 > 0$, we define $c = 1 - \frac{\kappa_3}{\kappa_2} \in (0,1)$. Therefore, using (i) and Eq. (10) results

$$\kappa_1 \|x(t)\|^2 \le V(x(t)) \le c^t V(x(0)) \le$$
$$\kappa_2 c^t \|x(0)\|^2 = \kappa_2 c^t \|z\|^2 \tag{12}$$

And:

$$\|x(t)\|^2 \le \frac{\kappa_2}{\kappa_1} c^t \|z\|^2 \tag{13}$$

Obviously $a = \frac{\kappa_2}{\kappa_1} \ge 1$ and thus Eq. (7) holds. Then the proof is completed.

Now, according to the main goals of this paper, we will find the non-negative function V satisfying Theorem 1, a switching law $r(x,t)$ and GCC $u(t)$ as well as stabilizing policy π for the uncertain system Eq. (1) with cost function Eq. (4).

In this section we propose a CLF $V(x) = x^T P x$ for system Eq. (1) and by designing a new stabilizing switching strategy as a policy π in Lemma 2, we will show that this CLF satisfies condition (ii) of Theorem 1. Therefore, based on Theorem 1 under this switching strategy, system Eq. (1) will be exponential stable. Finally, in Lemma 3, the proposed switching strategy is implemented and the guaranteed cost controller is designed via solving a set of LMIs.

Lemma 2. For given positive scalars $\delta_i, i \in \underline{m}$, If there exist a symmetric positive definite matrix P, positive scalars ε_i and matrices $K_i, i \in \underline{m}$, with proper dimensions, such that $\varepsilon_i I - N_i^T P N_i > 0$ and:

$$\sum_{i=1}^{m} \delta_i \begin{bmatrix} Ac_i^T \left(P^{-1} - \varepsilon_i N_i N_i^T \right)^{-1} Ac_i + \\ \varepsilon_i^{-1} E_i^T E_i - P + Q_i + K_i^T R_i K_i \end{bmatrix} < 0, \quad (14)$$

where, $Ac_i = \left(A_i + B_i K_i \right)$ and $E_i = \left(C_i + D_i K_i \right), i \in \underline{m},$, then the systems Eq. (1) is to be exponentially stable under the following stabilizing switching and state feedback control laws.

$$u_{r(x,t)}(t) = K_i x(t), \quad r(x,t) = \arg\min_{i \in \underline{m}} \left\{ x^T S_i x \right\}, \quad (15)$$

where,

$$S_i = Ac_i^T \left(P^{-1} - \varepsilon_i^{-1} N_i N_i^T \right)^{-1} Ac_i + \varepsilon_i^{-1} E_i^T E_i - P + Q_i + K_i^T R_i K_i. \quad (16)$$

Moreover, the GCV is given by $J^* = V(z) = x^T(0) P x(0)$.

Proof. Firstly, let us define the following sectoral sets of R^n.

$$L(S_i) = \left\{ x \in R^n : x^T S_i x < 0 \right\}, \quad i \in \underline{m}. \quad (17)$$

From Eq. (14) and Eq. (16), we have $\sum_{i=1}^{m} \delta_i S_i < 0$ which yields

$$\bigcup_{i=1}^{m} L(S_i) = R^n \setminus \{0\}. \quad (18)$$

Thus, for any $\forall x \in R^n, x \neq 0$, there exist an index $i \in \underline{m}$ such that $x^T S_i x < 0$.

Secondly, we propose $V(x(t)) = x^T(t) P x(t)$ as the CLF for uncertain switched linear system Eq. (1) and we show that for any $t \in Z^+$, there exist an index $i \in \underline{m}$ such that:

$$\Delta V(x(t)) + x^T \left(Q_i + K_i^T R_i K_i \right) x \leq x^T S_i x < 0. \quad (19)$$

For this, by substituting $u(t) = K_i x(t)$ into Eq. (1) combining with Eq. (2), we have:

$$x(t+1) = \left(Ac_i + N_i F_i E_i \right) x(t) \quad (20)$$

and,

$$\Delta V(x(t)) = V(x(t+1)) - V(x(t)) = x^T \left[\left(Ac_i + N_i F_i E_i \right)^T P \left(Ac_i + N_i F_i E_i \right) - P \right] x \quad (21)$$

By using Lemma 1, we have:

$$\left(Ac_i + N_i F_i E_i \right)^T P \left(Ac_i + N_i F_i E_i \right) \leq Ac_i^T \left(P^{-1} - \varepsilon_i N_i N_i^T \right)^{-1} Ac_i + \varepsilon_i^{-1} E_i^T E_i, \quad i \in \underline{m} \quad (22)$$

Now, by adding $-P + Q_i + K_i^T R_i K_i$ to the both sides

of Eq. (22) results

$$x^T(t) \begin{bmatrix} \left(Ac_i + N_i F_i E_i \right)^T P \left(Ac_i + N_i F_i E_i \right) - \\ P + Q_i + K_i^T R_i K_i \end{bmatrix} x(t) \leq x^T(t) \begin{bmatrix} Ac_i^T \left(P^{-1} - \varepsilon_i N_i N_i^T \right)^{-1} Ac_i + \\ \varepsilon_i^{-1} E_i^T E_i - P + Q_i + K_i^T R_i K_i \end{bmatrix} x(t). \quad (23)$$

It follows from Eq. (21) and Eq. (23) that:

$$\Delta V(x(t)) + x^T(t) \left(Q_i + K_i^T R_i K_i \right) x(t) \leq x^T(t) S_i x(t) < 0. \quad (24)$$

and thus, for any $t \in Z^+$, there exist an index $i \in \underline{m}$ such that:

$$\Delta V(x(t)) + x^T(t) \left(Q_i + K_i^T R_i K_i \right) x(t) \leq 0 \quad (25)$$

Since Q_i and R_i are positive definite matrices, then $G_i = Q_i + K_i^T R_i K_i$ is a positive definite matrix for any $i \in \underline{m}$. Using Rayleigh inequality [35], the following inequality holds:

$$-\lambda_{\max} G_i \|x\|^2 \leq -x^T G_i x \leq -\lambda_{\min} G_i \|x\|^2, \quad \forall x \in R^n, \ i \in \underline{m} \quad (26)$$

Then, by choosing:

$$\kappa_3 = \lambda_{\min}(G) = \min_{i \in \underline{m}} \left(\lambda_{\min}(G_i) \right) \quad (27)$$

it is concluded that $\Delta V(x(t) \leq -\kappa_3 \|x(t)\|^2$ for all $t \in Z^+$. Therefore, condition (ii) in Theorem 1 is satisfied. Also for quadratic Lyapunov functional, condition (i) is obviously satisfied. In other words, from Lemma 5 (in the Appendix) we have:

$$\lambda_{\min}(P) \|z\|^2 \leq V(z) = z^T P z \leq \lambda_{\max}(P) \|z\|^2, \quad \forall z \in R^n \quad (28)$$

Then by choosing $\kappa_1 = \lambda_{\min}(P)$ and $\kappa_2 = \lambda_{\max}(P)$, condition (i) in Theorem 1 holds. Therefore, based on Theorem 1, system Eq. (1) is exponentially stable.

To find the GCV, Let T be an arbitrary positive integer, taking the sum both side of Eq. (25) from $t = 0$ to $t = T$, we obtain:

$$\sum_{t=0}^{T} \left(x^T(t) Q_i x(t) + x^T(t) K_i^T R_i K_i x(t) \right) \leq V(x(0)) - V(x(T) \leq V(x(0)). \quad (29)$$

Let $T \to \infty$ then:

$$\sum_{t=0}^{t=\infty} \left(x^T Q_i x + x^T K_i^T R_i K_i x \right) \leq V(x(0)) = V(z), \quad \forall i \in \underline{m}. \quad (30)$$

It follows from Eq. (30) that the cost function $J(z) \leq J^* = x(0)^T P x(0) = V(z)$. The proof is completed.

Remark 1. It is noted that for the implementation of switching Eq. (15) we need to find the matrix P, the

state feedback gains K_i and the scalars $\varepsilon_i > 0$. In the following theorem, by applying Schur complement (Lemma 4 in the Appendix) we show that Eq. (14) is equivalent to the LMIs given by Eq. (31). Therefore, solving Eq. (31) results unknown matrices P, K_i, and positive scalars ε_i and then, stabilizing switching strategy Eq. (15) can be implemented.

Lemma 3. Given positive scalars $\delta_i, i \in \underline{m}$, assume that there exist positive scalars ε_i, invertible symmetric positive definite matrix X and matrices $Y_i, i \in \underline{m}$, such that $X - \varepsilon_i N_i^T N_i > 0$, $i \in \underline{m}$ and the following inequality holds:

$$\begin{bmatrix} -\sum_{i=1}^{m} \delta_i X & A^T & \Xi^T \\ * & -\bar{X} + \varepsilon_i N N^T & 0 \\ * & * & -\Gamma_i \end{bmatrix} < 0, \qquad (31)$$

where

$$A = \begin{bmatrix} \sqrt{\delta_1} A_1 X + B_1 Y_1 \\ \vdots \\ \sqrt{\delta_m} A_m X + B_m Y_m \end{bmatrix}, \quad C = \begin{bmatrix} \sqrt{\delta_1} C_1 X + D_1 Y_1 \\ \vdots \\ \sqrt{\delta_m} C_m X + D_m Y_m \end{bmatrix},$$

$$Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix}, \quad Q = \begin{bmatrix} \sqrt{\delta_1} Q_1 & \sqrt{\delta_2} Q_2 & \cdots & \sqrt{\delta_m} Q_m \end{bmatrix},$$

$$\bar{X} = diag\left(\underbrace{X, X, ..., X}_{m} \right), \quad R = diag\left(R_1, R_2, ..., R_m \right), \qquad (32)$$

$$N = diag\left(N_1, N_2, ..., N_m \right),$$

$$\Xi^T = \begin{bmatrix} Y^T & C^T & X \end{bmatrix}, \quad \Gamma_i = diag\left(R^{-1}, \varepsilon_i I, \left(Q Q^T \right)^{-1} \right).$$

Then condition Eq. (14) holds and the switching strategy Eq. (15) can be implemented.

Proof. Let $P = X^{-1}$ and $K_i = \dfrac{1}{\sqrt{\delta_i}} Y_i X$, $i \in \underline{m}$. By

Schur complement Lemma 4, Eq. (31) is equivalent to:

$$\sum_{i=1}^{m} \delta_i \begin{bmatrix} \left(A_i + B_i K_i \right)^T \left(P^{-1} - \varepsilon_i N_i N_i^T \right)^{-1} \left(A_i + B_i K_i \right) + \\ \varepsilon_i^{-1} \left(C_i + D_i K_i \right)^T \left(C_i + D_i K_i \right) - P + Q_i + K_i^T R_i K_i \end{bmatrix} (33)$$
$$< 0$$

Consequently, condition Eq. (14) of Lemma 2 holds. Thus, by solving Eq. (31), the switching strategy (15) can be implemented and then system Eq. (1) is exponential stable under the switching strategy Eq. (15) as concluded in Lemma 2. The proof is now completed.

Theorem 2. For given positive scalars δ_i, assume that there exist positive scalars ε_i symmetric positive definite matrix X and matrices $Y_i, i \in \underline{m}$, such that

LMIs Eq. (31) hold. Then the system Eq. (1) to be exponentially stable under the switching strategy Eq. (15), where, $P = X^{-1}$ and $K_i = \dfrac{1}{\sqrt{\delta_i}} Y_i X$, $i \in \underline{m}$.

Moreover, the cost function Eq. (4) is bounded by $J^* = V(z) = x^T(0) P x(0)$.

Proof. The proof of Theorem 2 is straight forward from Lemma 2 and Lemma 3.

Remark 2. In summary, to find the guaranteed cost value (GCV) J^*, the guaranteed cost control law (GCCL) $u(t)$ and the switching rule (SR) $r(x,t)$, for given positive scalars δ_i, we can use a two-parameters searching methods with Matlab to find positive scalars $\varepsilon_i, i \in \underline{m}$ subject to LMIs Eq. (31) to obtain symmetric positive matrix X and matrices $Y_i, i \in \underline{m}$. Then the SR, GCCL and GCV can be defined as:

$$r(x,t) = \arg\min_{i \in \underline{m}} \left\{ x^T S_i x \right\},$$

$$u_{r(x,t)}^*(t) = K_i x(t) = \frac{1}{\sqrt{\delta_i}} Y_i X x(t), \qquad (34)$$

$$J^* = x^T(0) X^{-1} x(0).$$

where $S_i, i \in \underline{m}$, are defied in Eq. (16).

4 Numerical Example

Consider the uncertain switched linear system in Eq. (1) with two subsystems [23].

$$\dot{x} = (A_{r(x,t)} + \Delta A_{r(x,t)}) x(t) + (B_{r(x,t)} + \Delta B_{r(x,t)}) u(t),$$

$$r(x,t) \in \underline{m} := \{1,2\}, \quad t \in Z^+, \qquad (35)$$

$$x_0 = x(0) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T.$$

The system parameters are described as follows:

$$A_1 = \begin{bmatrix} 0.75 & -0.6 & 0 \\ 0 & -1.65 & 0 \\ 0.2 & 0 & -1.65 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & -1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.1 & 0.2 & 0 \\ 0.5 & 0 & 0.3 \\ 0 & 0.4 & 0 \end{bmatrix}, D_1 = \begin{bmatrix} 0.1 & 3 \\ 3 & -1 \\ 0 & 1 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0 & 0.3 & 0.5 \\ 0.5 & 0 & 0.4 \\ 0.2 & 0 & 0.6 \end{bmatrix}, A_2 = \begin{bmatrix} 0.25 & -4.25 & -1 \\ 0 & -0.5 & -2.5 \\ 0 & -0.75 & -0.25 \end{bmatrix}, \quad (36)$$

$$B_2 = \begin{bmatrix} 1 & 3 \\ 3 & -1 \\ 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} -0.1 & -0.3 & -0.6 \\ 0 & -0.2 & 0 \\ -0.4 & 0 & -0.3 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.24 \\ 0.4 & 0.16 \end{bmatrix}, N_2 = \begin{bmatrix} 0.5 & 0.3 & 0 \\ 0 & 0.5 & 0.4 \\ 0.2 & 0.7 & 0 \end{bmatrix},$$

$$F_i = \delta_i I, \quad |\delta_i| \le 1, \quad i \in \{1,2\}.$$

Also, we consider the following symmetric positive definite weighted matrices

$$Q_1 = Q_2 = 0.3I, \ R_1 = R_2 = I \qquad (37)$$

Note that the two subsystems of system Eq. (35) are unstable. The aim is to find guaranteed cost controller $u(t) = K_{r(x,t)}x(t)$, switching signal $r(x,t)$ and GCV $J^* = x_0^T P x_0$ for the switched system Eq. (35) with the cost function weighted matrices given in Eq. (37). This problem was discussed in [23]. Solving the obtained LMIs problem, they found an upper bound of the GCV as $J^* = 20.3628$. Here, the results of our proposed method are compared with the results in [23]. We perform the following steps for designing the switching signal and guaranteed state-feedback controller in the presented method.

Step 1. δ_1 and δ_2 are selected as $\delta_1 = 0.5$, $\delta_2 = 0.5$.

Step 2. Solving LMIs Eq. (31) we obtain:

$$\varepsilon_1 = 0.7, \ \varepsilon_2 = 0.3,$$

$$X = \begin{bmatrix} 0.2195 & 0.0376 & 0.0704 \\ 0.0376 & 0.1818 & 0.0444 \\ 0.0704 & 0.0444 & 0.1488 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 0.0129 & 0.1023 & 0.0562 \\ -0.0542 & -0.0229 & -0.0554 \end{bmatrix}, \qquad (38)$$

$$Y_2 = \begin{bmatrix} 0.0536 & 0.0936 & 0.1064 \\ 0.0237 & 0.1473 & 0.0409 \end{bmatrix}.$$

and thus,

$$P = X^{-1} = \begin{bmatrix} 5.4200 & -0.5330 & -2.4066 \\ -0.5330 & 5.9864 & -1.5332 \\ -2.4066 & -1.5332 & 8.3161 \end{bmatrix}, \qquad (39)$$

Step 3. Guaranteed cost controller gains are by:

$$K_1 = \frac{1}{\sqrt{0.5}} Y_1 X^{-1} = \begin{bmatrix} -0.1697 & 0.7345 & 0.3956 \\ -0.2094 & -0.0328 & -0.4177 \end{bmatrix},$$

$$K_2 = \frac{1}{\sqrt{0.5}} Y_2 X^{-1} = \begin{bmatrix} -0.0221 & 0.5215 & 0.8656 \\ -0.0683 & 1.1407 & 0.0807 \end{bmatrix}. \qquad (40)$$

Step 4. The GCV $J^* = x_0^T P x_0 = 18.5494$.

Step 5. Obtain the switching strategy as Eq. (15) for known scalars and matrices in Step 2.

Figs. 1 and 2 depict the states x_1, x_2, and x_3 and switching signal of proposed method in this paper and method [23], starting from an initial condition $x_0 = [1 \quad 0 \quad -1]^T$. Comparing states in Fig. 1 and Fig. 2 show that, the states in proposed method are smoother than states in [23] and tend to zero faster. Comparing the guaranteed cost obtaining from two methods shows that, GCV J^* proposed in this paper is less than that obtained by [23]. Figs. 3 and 4 show that the number of switching in two methods.
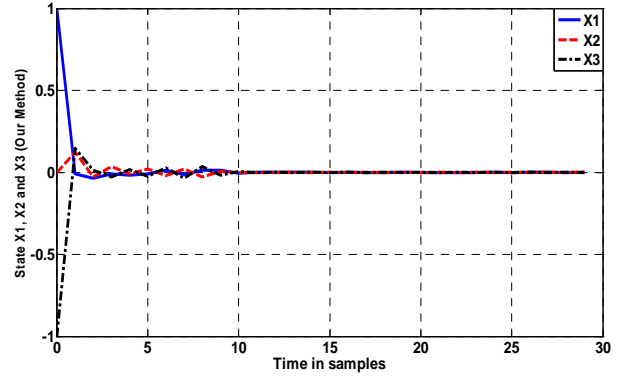

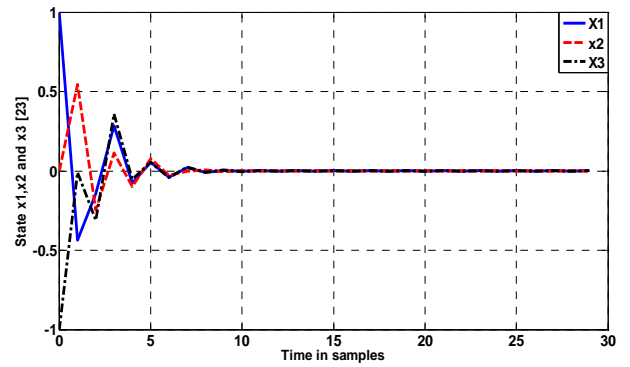**Fig. 1** State trajectory of the closed-loop system of Eq. (35) (our method).

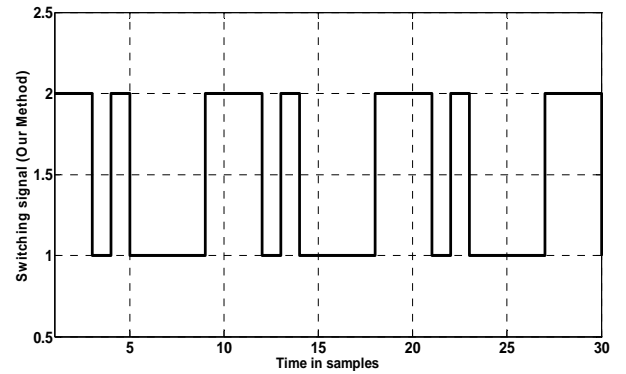**Fig. 2** State trajectory of the system Eq. (35) (Method [23]).

Fig. 3 Switching signal for the system Eq. (35) (our method).
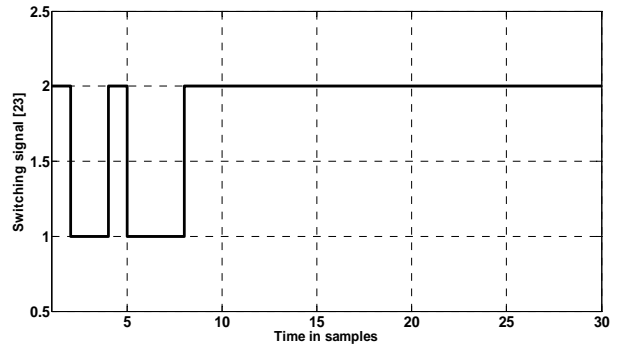

Fig. 4 State trajectory of the system Eq. (35) (Method [23]).

Based on a proposed switching law in [23], the closed-loop system is quadratic D-stable and the closed-loop cost function value is not more than a specified upper bound. This specific upper bound strictly depends on the center (α) and the radius (r) of disk $D(\alpha, r)$ (i.e. $J \le x_0^T \dfrac{P}{r^2} x_0$, note that P depends on α), where, the closed loop system under our proposed switching law is exponentially stable and the upper bound on the given cost function only depends on P.

5 Conclusion

Switching strategy must be chosen consciously, because it may lead to asymptotic or exponential stability. The main objective of this paper was to propose a new switching strategy to provide exponential stability in GCC design for a class of discrete-time uncertain switched linear systems. Our switching design method was based on CLF technique and applying exponential stability theorem. Furthermore, the sufficient condition on the existence of guaranteed state-feedback control was presented in the form of LMIs. Finally, convergence rate of states was determined. Numerical example and comprehensive comparison show the practicality and validity of the proposed method.

Appendix

Lemma 4. (Schur complement [32]). For a given symmetric matrix $W = \begin{bmatrix} W_1 & W_2 \\ W_2^T & W_4 \end{bmatrix}$, where, W_1, W_4 are square matrices, the following three conditions are equivalent:

(i) $W < 0$

(ii) $W_1 < 0$ and $W_4 - W_2^T W_1^{-1} W_2 < 0$ (A1)

(iii) $W_4 < 0$ and $W_1 - W_2 W_4^{-1} W_2^T < 0$

Lemma 5. (Rayleigh's inequality [34]). Suppose that $A \in R^{n \times n}$ is a symmetric positive definite matrix, then:

$$\lambda_{\min}(A)\|x\|^2 \le x^T A x \le \lambda_{\max}(A)\|x\|^2, \ \forall x \in R^n \quad (A2)$$

where $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest positive eigenvalues of matrix A, respectively.

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