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NORMAL 6-VALENT CAYLEY GRAPHS OF ABELIAN GROUPS

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Abstract*: We call a Cayley graph Γ = Cay (G, S) normal for G, if the right regular representation R(G) of G is normal in the full automorphism group of Aut(Γ). In this paper, a classification of all non-normal Cayley graphs of finite abelian group with valency 6 was presented.*

Keywords*: Cayley graph, normal Cayley graph, automorphism group***.**

1. Introduction

Let G be a finite group, and S be a subset of G not containing the identity element 1_G. The Cayley digraph Γ=Cay(G,S) of G relative to S is defined as the graph with vertex set V(Γ) = G and edge set E(Γ) consisting of those ordered pairs (x, y) from G for which $yx^{-1} \in S$. Immediately from the definition we find that, there are three obvious facts: (1) Aut(Γ) contains the right regular representation R(G) of G and so Γ is vertex-transitive.
(2) Γ is connected if and only if $G = \langle S\rangle$. (3) Γ is an undirected if and only if $S^{-1}= S$.
A Cayley (di)graph Γ=Cay(G,S) is called normal if the right regular representation R(G) of G is a normal subgroup of the automorphism group of Γ.
The concept of normality of Cayley (di)graphs is known to be important for the study of arc-transitive graphs and half-transitive graphs (see[1,2]). Given a finite group G, a natural problem is to determine all normal or non-normal Cayley (di)graphs of G. This problem is very difficult and is solved only for the cyclic groups of prime order by Alspach [3] and the groups of order twice a prime by Du et al. [4], while some partial answers for other groups to this problem can be found in [5-8]. Wang et al. [8] characterized all normal disconnected Cayley's graphs of finite groups. Therefore the main work to determine the normality of Cayley graphs is to determine the normality of connected Cayley graphs. In [5, 6], all non-normal Cayley graphs of abelian groups with valency at most 5 were classified. The purpose of this paper is the following main theorem.

Theorem 1.1 Let Γ = Cay (G, S) be a connected undirected Cayley graph of a finite abelian group G on S with valency 6. Then Γ is normal except when one of the following cases happens:

(1): $G = Z_2^5 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle$,
$S = \{a, b, c, abc, d, e\}$.

(2): $G = Z_2^3 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$ ($m \geq 3$),
$S = \{a, b, c, abc, d, d^{-1}\}$.

(3): $G = Z_2^2 \times Z_4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$,

$S = \{a, b, ab, c^2, c, c^{-1}\}$.

(4): $G = Z_2^4 \times Z_4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle$,
$S = \{a, b, c, d, e, e^{-1}\}$.

(5): $G = Z_2^3 \times Z_4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$
$S_1 = \{a, b, c, d^2, d, d^{-1}\}$,
$S_2 = \{a, b, ab, c, d, d^{-1}\}$, $S_3 = \{a, b, c, ad^2, d, d^{-1}\}$.

(6): $G = Z_2^2 \times Z_6 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$,
$S = \{a, b, ab, c^3, c, c^{-1}\}$.

(7): $G = Z_2^3 \times Z_6 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$,
$S = \{a, b, c, d^3, d, d^{-1}\}$.

(8): $G = Z_6 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$ ($m \geq 2$),
$S = \{a^3, b^m, a, a^{-1}, b, b^{-1}\}$.

(9): $G = Z_2 \times Z_6 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m \geq 3$),
$S = \{a, b^3, b, b^{-1}, c, c^{-1}\}$.

(10): $G = Z_4 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$ ($m \geq 2$),
$S = \{a, a^{-1}, a^2, b, b^{-1}, b^m\}$.

(11): $G = Z_2 \times Z_4 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m \geq 3$),
$S_1 = \{a, b, b^{-1}, b^2, c, c^{-1}\}$, $S_2 = \{a, b, b^{-1}, ab^2, c, c^{-1}\}$.

(12): $G = Z_2 \times Z_4 \times Z_{2m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m \geq 2$),
$S = \{a, b, b^{-1}, c, c^{-1}, c^m\}$.

(13): $G = Z_2^2 \times Z_4 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$ ($m \geq 3$), $S = \{a, b, c, c^{-1}, d, d^{-1}\}$.

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(14): $G = Z_2^3 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$ ($m\geq 3$), $S = \{a, b, cd, cd^{-1}, d, d^{-1}\}$.

(15): $G = Z_2^2 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m = 5, 10$), $S = \{a, b, c, c^{-1}, c^3, c^{-3}\}$.

(16): $G = Z_2^2 \times Z_{4m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m \geq 2$), $S = \{a, b, c, c^{-1}, c^{2m+1}, c^{2m-1}\}$.

(17): $G = Z_4\times Z_{2m} = \langle a\rangle \times \langle b\rangle$ ($m\geq 3$, m is odd), $S = \{a, a^3, b, b^{-1}, b^{m+1}, b^{m-1}\}$.

(18): $G = Z_4^2 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m\geq 3$), $S = \{a, a^3, b, b^3, c, c^{-1}\}$.

(19): $G = Z_{4m} \times Z_n = \langle a\rangle \times \langle b\rangle$ ($m\geq 2$, $n\geq 3$), $S = \{a, a^{-1}, a^{2m+1}, a^{2m-1}, b, b^{-1}\}$.

(20): $G = Z_2\times Z_m \times Z_n = \langle a\rangle\times\langle b\rangle\times\langle c\rangle$ ($m\geq 3$, $n\geq 3$), $S =\{ab, ab^{-1}, b, b^{-1}, c, c^{-1}\}$.

(21): $G = Z_m \times Z_n = \langle a\rangle \times \langle b\rangle$ ($m = 5, 10$, $n\geq 3$), $S = \{a, a^{-1}, a^3, a^{-3}, b, b^{-1}\}$.

(22): $G = Z_2^4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$, $S = \{a, b, ab, c, abc, d\}$.

(23): $G = Z_2^2 \times Z_4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, $S = \{a, b, ac^2, c, c^{-1}, c^2\}$.

(24): $G = Z_2^3 \times Z_4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$, $S = \{a, b, c, d, d^{-1}, abd^2\}$.

(25): $G = Z_2^2 \times Z_{3m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m\geq 1$), $S = \{a, b, ac^m, ac^{2m}, c, c^{-1}\}$.

(26): $G = Z_2\times Z_{10} = \langle a\rangle \times \langle b\rangle$, $S= \{a, b, b^3, b^5, b^7, b^9\}$.

(27): $G = Z_2^2 \times Z_{2m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m\geq 2$), $S = \{ac, ac^{-1}, b, c^m, c, c^{-1}\}$.

(28): $G = Z_2\times Z_4\times Z_{2m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m\geq 2$), $S = \{a, b^2c^m, b, b^{-1}, c, c^{-1}\}$.

(29): $G = Z_2\times Z_{2m} = \langle a\rangle \times \langle b\rangle$ ($m\geq 3$), $S = \{a, b^m, b, b^{-1}, b^{m+1}, b^{m-1}\}$.

(30): $G = Z_2^2 \times Z_{2m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m\geq 2$), $S = \{a, b, ac, ac^{-1}, c, c^{-1}\}$.

(31): $G = Z_2\times Z_{6m} = \langle a\rangle \times \langle b\rangle$ ($m \geq 3$, m is odd), $S = \{a, b^2, b^{-2}, b^m, b^{5m}, b^{3m}\}$.

(32): $G = Z_2^2 \times Z_{6m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m\geq 2$), $S = \{a, bc^m, bc^{3m}, bc^{5m}, c, c^{-1}\}$.

(33): $G = Z_2^3 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, $S = \{a, b, c, ab, ac, abc\}$.

(34): $G = Z_2^4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$, $S = \{a, b, c, d, abc, abd\}$.

(35): $G = Z_2^2 \times Z_{2m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m\geq 2$), $S = \{a, b, ac^m, bc^m, c, c^{-1}\}$.

(36): $G = Z_2^2 \times Z_4 = \langle a\rangle\times\langle b\rangle\times\langle c\rangle$,
$S_1= \{a, b, ab, ac^2, c, c^{-1}\}$,
$S_2= \{a, b, ac^2, abc^2, c, c^{-1}\}$.

(37): $G = Z_2^3 \times Z_4= \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$, $S = \{a, b, c, abcd^2, d, d^{-1}\}$.

(38): $G = Z_2\times Z_{6m} = \langle a\rangle \times \langle b\rangle$ ($m\geq 2$), $S = \{a, b^{3m}, ab^{2m}, ab^{4m}, b, b^{-1}\}$.

(39): $G = Z_2\times Z_{4m} = \langle a\rangle \times \langle b\rangle$ ($m\geq 1$), $S = \{a, ab^m, ab^{2m}, ab^{3m}, b, b^{-1}\}$.

(40): $G = Z_4\times Z_{2m} = \langle a\rangle \times \langle b\rangle$ ($m\geq 2$), $S = \{a, a^{-1}, b^m, a^2b^m, b, b^{-1}\}$.

(41): $G = Z_2^2 \times Z_{4m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m\geq 1$), $S = \{a, ac^{2m}, bc^m, bc^{3m}, c, c^{-1}\}$.

(42): $G = Z_2 \times Z_{10} = \langle a\rangle \times \langle b\rangle$, $S = \{a, ab^5, b, b^9, b^3, b^7\}$.

(43): $G = Z_2 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$,
$S_1= \{a, b, b^{-1}, b^m, ab, ab^{-1}\}$, $m\geq 2$,
$S_2=\{a, ab^m, b, b^{-1}, ab, ab^{-1}\}$, $m\geq 2$,
$S_3 = \{ab^m, b^m, b, b^{-1}, ab, ab^{-1}\}$, $m\geq 2$, $S_4 =\{a, ab^m, b, b^{-1}, b^{m+1}, b^{m-1}\}$, $m\geq 3$, $S_5 = \{a, b, b^{-1}, b^m, ab^{m+1}, ab^{m-1}\}$, $m\geq 3$, $S_6 =\{a, ab^m, b, b^{-1}, ab^{m+1}, ab^{m-1}\}$, $m\geq 3$
$S_7=\{ab^m, b, b^{-1}, b^m, ab^{m+1}, ab^{m-1}\}$, $m\geq 3$

(44): $G = Z_2^2 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, $S_1= \{a, b, c, c^{-1}, abc, abc^{-1}\}$, $m\geq 3$, $S_2=\{a, b, c, c^{-1}, ac^{k+1}, ac^{k-1}\}$, $m = 2k$, $k\geq 3$, $S_3= \{a, b, c, c^{-1}, abc^{k+1}, abc^{k-1}\}$, $m =2k$, $k \geq 3$,
$S_4= \{a, bc, bc^{-1}, ack, c, c^{-1}\}$, $m = 2k$, $k \geq 2$,
$S_5= \{a, bc^{k+1}, bc^{k-1}, c^k, c, c^{-1}\}$, $m = 2k$, $k \geq 3$,
$S_6= \{a, bc^{k+1}, bc^{k-1}, ac^k, c, c^{-1}\}$, $m = 2k$, $k \geq 3$,
$S_7= \{a, b, c, c^{-1}, ac, ac^{-1}\}$, $m = 2k - 1$, $k \geq 2$.

(45): $G = Z_{4m} = \langle a\rangle$ ($m\geq 2$), $S = \{a, a^{-1}, a^m, a^{-m}, a^{2m+1}, a^{2m-1}\}$.

(46): $G = Z_{2m} = \langle a\rangle$ ($m\geq 4$), $S = \{a, a^{-1}, a^{m+1}, a^{m-1}, a^k, a^{-k}\}$ ($2 \leq k \leq m - 2$), $(m, k) = l$, if $l > 2$ or $l = 2$ for $m = 4i + 2$; ($k = 2i$, with i odd or $k = 2i + 2$, with i even).

(47): $G = Z_2 \times Z_m = \langle a\rangle \times \langle b\rangle$ ($m\geq 5$),
$S_1 = \{ab, ab^{-1}, b, b^{-1}, b^j, b^{-j}\}$ ($2 \leq j < \frac{m}{2}$), $(m, j) = p > 2$; $m = (t + 1)p$,

$S_2 = \{ab, ab\, b^{-1}, b, b\, b^{-1}, ab^j, ab^{-j}\}$, $(2 \le j < \frac{m}{2})$, $(m, j) = p > 2$; $m = (t+1)p$.

(48): $G = Z_2 \times Z_8 = \langle a\rangle \times \langle b\rangle$,
$S_1 = \{ab, ab^{-1}, b, b^{-1}, b^3, b^{-3}\}$,
$S_2 = \{ab, ab^{-1}, b, b^{-1}, ab^3, ab^{-3}\}$.

(49): $G = Z_{2m} \times Z_n = \langle a\rangle \times \langle b\rangle$ $(m \ge 2, n \ge 3)$,
$S = \{a, a^{-1}, a^m b, a^m b^{-1}, b, b^{-1}\}$.

(50): $G = Z_{2m} \times Z_{2n} = \langle a\rangle \times \langle b\rangle$ $(m \ge 3, n \ge 2)$,
$S = \{a, a^{-1}, a^{m+1}b^n, a^{m-1}b^n, b, b^{-1}\}$.

(51): $G = Z_{6m} = \langle a\rangle$ $(m \ge 2)$, $S_1 = \{a, a^{-1}, a^3, a^{-3}, a^{3m+1}, a^{3m-1}\}$,
$S_2 = \{a, a^{-1}, a^{3m+1}, a^{3m-1}, a^{3m+3}, a^{3m-3}\}$.

(52): $G = Z_m = \langle a\rangle$ $(m = 7, 14)$, $S = \{a, a^{-1}, a^3, a^{-3}, a^5, a^{-5}\}$.

(53): $G = Z_{3m} = \langle a\rangle$ $(m \ge 3)$,
$S = \{a, a^{-1}, a^{m-1}, a^{m+1}, a^{2m-1}, a^{2m+1}\}$.

(54): $G = Z_{16m-4} = \langle a\rangle$ $(m \ge 1)$,
$S = \{a, a^{-1}, a^{4m-2}, a^{12m-2}, a^{8m-3}, a^{8m-1}\}$.

(55): $G = Z_{16m+4} = \langle a\rangle$ $(m \ge 1)$,
$S = \{a, a^{-1}, a^{4m+2}, a^{12m+2}, a^{8m+1}, a^{8m+3}\}$.

(56): $G = Z_3 \times Z_3 = \langle a\rangle \times \langle b\rangle$,
$S = \{a, a^2, b, b^2, a^2b, ab^2\}$.

(57): $G = Z_2 \times Z_4 \times Z_4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$,
$S = \{a, b, b^{-1}, c, c^{-1}, ab^2c^2\}$.

2. Primary Analysis

Proposition 2.1 [9, Proposition 1.5] Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph of G over S, and $A = \mathrm{Aut}(\Gamma)$. Let A_1 be the stabilizer of the identity element 1 in A.
Then Γ is normal if and only if every element of A_1 is an automorphism of G.

Proposition 2.2 [6, Theorem 1.1] Let G be a finite abelian group and S be a generating subset of $G - 1_G$. Assume S satisfies the condition that, if $s, t, u, v \in S$ with $1 \ne st = uv$, implies $\{s, t\} = \{u, v\}$. Then the Cayley graph Cay (G, S) is normal.

Let X and Y be two graphs. The direct product $X \times Y$ is defined as the graph with vertex set $V(X \times Y) = V(X) \times V(Y)$ such that for any two vertices $u = [x_1, y_1]$ and $v = [x_2, y_2]$ in $V(X \times Y)$, $[u, v]$ is an edge in $X \times Y$, whenever $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$ or $y_1 = y_2$ and $[x_1, x_2] \in E(X)$. Two graphs are called relatively prime if they have no nontrivial common direct factor. The lexicographic product $X[Y]$ is defined as the graph vertex set $V(X[Y]) = V(X) \times V(Y)$ such that for any two vertices $u = [x_1, y_1]$ and $v = [x_2, y_2]$ in $V(X[Y])$, $[u, v]$ is an edge in $X[Y]$ whenever $[x_1, x_2] \in E(X)$ or $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$.

Let $V(Y) = \{y_1, y_2, ..., y_n\}$. Then there is a natural embedding nX in $X[Y]$, where for $1 \le i \le n$, the ith copy of X is the subgraph induced on the vertex subset $\{(x, y_i) | x \in V(X)\}$ in $X[Y]$. The deleted lexicographic product $X[Y] - nX$ is the graph obtained by deleting all the edges of (this natural embedding of) nX from $X[Y]$. Let Γ be a graph and α a permutation $V(\Gamma)$ and C_n a circuit of length n. The twisted product $\Gamma \times_\alpha C_n$ of Γ by C_n with respect to α is defined by;

$V(\Gamma \times_\alpha C_n) = V(\Gamma) \times V(C_n) = \{(x, i) \mid x \in V(\Gamma), i = 0, 1, \dots, n-1\}$,
$E(\Gamma \times_\alpha C_n) = \{[(x, i), (x, i+1)] \mid x \in V(\Gamma), i = 0, 1, \dots, n-2\} \bigcup \{[(x, n-1), (x^\alpha, 0)] \mid x \in V(\Gamma)\}$ [$\{[(x, i), (y, i)] \mid [x, y] \in E(\Gamma), i = 0, 1, \dots, n-1\}$.

The graph Q_4^d denotes the graph obtained by connecting all long diagonals of 4-cube Q_4, that is, connecting all vertices u and v in Q_4 such that $d(u, v) = 4$. The graph $K_{m,m} \times_c C_n$ is the twisted product of $K_{m,m}$ by C_n such that c is a cycle permutation on each part of the complete bipartite graph $K_{m,m}$. The graph $Q_3 \times_d C_n$ is the twisted product of Q_3 by C_n such that d transposes each pair of elements on long diagonals of Q_3. The graph $C_{2m}^d[2K_1]$ is defined by:

$V(C_{2m}^d[2K_1]) = V(C_{2m}[2K_1])$,

$E(C_{2m}^d[2K_1]) = E(C_{2m}[2K_1]) \bigcup \{[(x_i, y_j), (x_{i+m}, y_j)] \mid i = 0, 1, \dots, m-1, j = 1, 2\}$, where $V(C_{2m}) = \{x_0, x_1, \dots, x_{2m-1}\}$ and $V(2K_1) = \{y_1, y_2\}$.

Let $G = G_1 \times G_2$ be the direct product of two finite groups G_1 and G_2, let S_1 and S_2 be subsets of G_1 and G_2, respectively, and let $S = S_1 \bigcup S_2$ be the disjoint union of two subsets S_1 and S_2. Then we have,

Lemma 2.3
(1) $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G_1, S_1) \times \mathrm{Cay}(G_2, S_2)$.
(2) If Cay (G, S) is normal, then Cay (G_1, S_1) is also normal.
(3) If both of Cay (G_1, S_1) and Cay (G_2, S_2) are normal and relatively prime, then Cay (G, S) is normal.

3. Proof of the Main Theorem

In this section, Γ always denotes the Cayley graph Cay(G, S) of an abelian group G on S with valency 6. Let $A = \mathrm{Aut}(\Gamma)$. Then A_1 and A_1^* denote the stabilizer of 1 in A and the subgroup of A which fixes $\{1\} \bigcup S$, pointwise, respectively. In order to prove Theorem 1.1 we need several lemmas.

Lemma 3.1 Let $G = Z_{2m} = \langle a\rangle$, $(m \ge 5)$, and $S = \{a^i, a^{-i}, a^{m+i}, a^{m-i}, a, a^{-1}\}$ $2 \le i < \frac{m}{2}$. Then $\Gamma = \mathrm{Cay}(G, S)$ is normal.

Proof Let $\Gamma_2(1)$ be the subgraph of Γ with vertex set $\{1\}\bigcup S\bigcup S^2$ and edge set $\{[1,s], [s, st] \mid s,t\in S\}$. By observing the subgraph $\Gamma_2(1)$, it is easy to prove that A_1^* fixes S^2 pointwise, which implies that $A_1^* = 1$. Thus A_1 acts faithfully on S. Observing the subgraph $\Gamma_2(1)$ again, A_1, as a permutation group on S, is generated by $(a, a^{-1})(a^{m+i}, a^{m-i})$. So $|A_1| = 2$ and $\Gamma = Cay(G, S)$ is normal.

Lemma 3.2: Let $G = Z_2^2 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, $m = 4k$, $k \geq 2$ and $S = \{a, b, c^k, c^{3k}, c, c^{-1}\}$. Then $\Gamma = Cay(G, S)$ is normal.

Proof Set $G_1 = \langle a,b\rangle$, $G_2 = \langle c\rangle$, $S_1= \{a, b\}$, $S_2= \{c^k, c^{3k}, c, c^{-1}\}$. Then $\Gamma_1 = Cay(G_1, S_1) \cong K_2\times K_2$. Note that Γ_1 and $\Gamma_2 = Cay(G_2, S_2)$ are relatively prime. By [5, Theorem 1.1] and [6, Theorem 1.2], Γ_1 and Γ_2 are normal and by Lemma 2.3, $\Gamma = Cay(G, S)$ is normal.

With similar arguments as in Lemmas 3.1 and 3.2, we have the following lemma.

Lemma 3.3 Let G and S be as the following. Then the Cayley graphs $\Gamma = Cay(G, S)$ are normal.

(1): $G = Z_2^4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$,
$S = \{a, b, c, d, ad, abc\}$.

(2): $G = Z_2^2 \times Z_6 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$,
$S = \{a, b, ac^3, c^3, c, c^{-1}\}$.

(3): $G = Z_2^2 \times Z_{2m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m \geq 2)$,
$S = \{a, b, abc^m, c^m, c, c^{-1}\}$.

(4): $G = Z_2 \times Z_{6m} = \langle a\rangle \times \langle b\rangle$ $(m \geq 2)$,
$S = \{a, ab^m, ab^{3m}, ab^{5m}, b, b^{-1}\}$.

(5): $G = Z_2 \times Z_{6m} = \langle a\rangle \times \langle b\rangle$ $(m \geq 2)$,
$S = \{a, b^m, b^{3m}, b^{5m}, b, b^{-1}\}$.

(6): $G = Z_6 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$ $(m \geq 3)$,
$S = \{a, a^{-1}, a^3, a^3b^m, b, b^{-1}\}$.

(7): $G = Z_2 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$,
$S_1= \{a, ab^2, ab^{-2}, b^m, b, b^{-1}\}$, $(m \geq 4)$,
$S_2= \{a, ab^{m+2}, ab^{m-2}, ab^m, b, b^{-1}\}$, $(m \geq 5)$,
$S_3 = \{a, b^{m+2}, b^{m-2}, b^m, b, b^{-1}\}$, $(m = 4, m \geq 6)$.

(8): $G = Z_2 \times Z_{4m+2} = \langle a\rangle \times \langle b\rangle$,
$S_1= \{a, ab^m, ab^{3m+2}, b^{2m+1}, b, b^{-1}\}$,
$(m \geq 2)$, $S_2=\{a,b, b^{-1}, b^m, b^{3m+2}, b^{2m+1}\}$, $(m \geq 2)$,
$S_3 = \{a, ab^{m+1}, ab^{3m+1}, b^{2m+1}, b, b^{-1}\}$, $(m \geq 2)$,
$S_4= \{a, b, b^{-1}, b^{m+1}, b^{3m+1}, b^{2m+1}\}$, $(m \geq 3)$.

(9): $G = Z_4 \times Z_{4m+2} = \langle a\rangle \times \langle b\rangle$ $(m \geq 1)$,
$S_1= \{a^2b^{2m+1}, b^{2m+1}, ab^m, a^3b^{3m+2}, b, b^{-1}\}$,
$S_2= \{a^2, a^2b^{2m+1}, ab^m, a^3b^{3m+2}, b, b^{-1}\}$.

(10): $G = Z_2^2 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, $(m \geq 4, m \neq 6)$,
$S = \{a, b, c, c^{-1}, ac^2, ac^{-2}\}$.

(11): $G = Z_2 \times Z_{4m} = \langle a\rangle \times \langle b\rangle$ $(m \geq 2)$,
$S_1= \{a, ab^m, ab^{3m}, b^{2m}, b, b^{-1}\}$,
$S_2= \{a, b, b^{-1}, b^m, b^{3m}, b^{2m}\}$,
$S_3 = \{a, ab^{2m}, b^m, b^{3m}, b, b^{-1}\}$.

(12): $G = Z_4 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$ $(m \geq 3)$,
$S = \{a^2, a^2b^m, a, a^{-1}, b, b^{-1}\}$.

(13): $G = Z_2^2 \times Z_{4m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m \geq 2)$,
$S_1= \{a, b, abc^m, abc^{3m}, c, c^{-1}\}$, $S_2= \{a, b, ac^m, ac^{3m}, c, c^{-1}\}$,
$S_3 = \{a, b, c^m, c^{3m}, c, c^{-1}\}$,
$S_4 = \{a, c^{2m}, bc^m, bc^{3m}, c, c^{-1}\}$.

(14): $G = Z_2^3 \times Z_{4m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$ $(m \geq 2)$,
$S = \{a, b, cd^m, cd^{3m}, d, d^{-1}\}$.

(15): $G = Z_2^2 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m = 7, 9, m \geq 11)$,
$S = \{a, b, c, c^{-1}, c^3, c^{-3}\}$.

(16): $G = Z_2\times Z_4\times Z_{4m+2} = \langle a\rangle\times\langle b\rangle\times\langle c\rangle$ $(m \geq 1)$,
$S = \{a, b^2c^{2m+1}, bc^m, b^3c^{3m+2}, c, c^{-1}\}$.

(17): $G = Z_2^2 \times Z_{4m+2} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m \geq 2)$,
$S = \{a, c^{2m+1}, bc^m, bc^{3m+2}, c, c^{-1}\}$.

(18): $G = Z_2\times Z_4 \times Z_{2m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m \geq 3)$,
$S = \{a, ac^m, b, b^{-1}, c, c^{-1}\}$.

(19): $G = Z_2 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$ $(m \geq 6)$,
$S_1= \{a, b^m, b, b^{-1}, b^3, b^{-3}\}$,
$S_2=\{a, ab^m, b, b^{-1}, b^3, b^{-3}\}$.

(20): $G = Z_{4m} \times Z_n = \langle a\rangle \times \langle b\rangle$ $(m \geq 2, n \geq 3)$,
$S = \{a, a^{-1}, a^m, a^{3m}, b, b^{-1}\}$.

(21): $G = Z_{4m} \times Z_{4n} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m, n \neq 4)$,
$S = \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$.

(22): $G = Z_4 \times Z_m \times Z_n = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m, n \neq 3)$,
$S = \{a, a^3, b, b^{-1}, c, c^{-1}\}$.

(23): $G = Z_{2m}$ $(m \geq 5)$,
$S = \{a, a^{-1}, a^j, a^{-j}, a^{m+j}, a^{m-j}\}$ $(2 \leq j < \frac{m}{2})$.

(24): $G = Z_m\times Z_n = \langle a\rangle\times\langle b\rangle$ $(m = 7, 9, m \geq 11, n \geq 3)$,
$S = \{a, a^{-1}, a^3, a^{-3}, b, b^{-1}\}$.

(25): $G = Z_{3m-1}\times Z_{3n} = \langle a\rangle\times\langle b\rangle$ $(m \geq 2, n \geq 1)$,
$S = \{a, a^{-1}, b, b^{-1}, a^mb^n, a^{2m-1}b^{2n}\}$.

(26): $G = Z_{3m+1} \times Z_{3n} = \langle a\rangle \times \langle b\rangle$ $(m, n \geq 1)$,
$S = \{a, a^{-1}, b, b^{-1}, a^mb^{2n}, a^{2m+1}b^n\}$.

(27): $G = Z_m \times Z_n = \langle a\rangle \times \langle b\rangle$ $(m \geq 5, n \geq 3)$,
$S = \{a, a^{-1}, b, b^{-1}, a^2b, a^{-2}b^{-1}\}$.

(28): $G = Z_{2m+1} \times Z_n = \langle a\rangle \times \langle b\rangle$ $(m, n \geq 3)$,
$S = \{a, a^{-1}, a^m, a^{m+1}, b, b^{-1}\}$.

(29): $G = Z_{2m+1} \times Z_{2n+1} = \langle a\rangle\times\langle b\rangle$ $(m, n \geq 2)$,
$S = \{a, a^{-1}, b, b^{-1}, a^mb^{n+1}, a^{m+1}b^n\}$.

(30): $G = Z_2\times Z_{2n+1}\times Z_{2m+1} = <a>\times<b>\times<c>$ $(m, n \geq 1)$,
$S = \{ab^mc^{n+1}, ab^{m+1}c^n, b, b^{-1}, c, c^{-1}\}$.

(31): $G = Z_{4m} = <a>$ $(m \geq 2)$,
$S = \{a, a^{-1}, a^k, a^{-k}, a^m, a^{-m}\}$, $(1 < k < 2m, k \neq m, 2m-1$.

(32): $G = Z_4\times Z_m = <a>\times<b>$ $(m \geq 3)$,
$S = \{a, a^{-1}, b, b^{-1}, ab^j, a^{-1}b^{-j}\}$, $1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor$,
(When $m \neq 2k$ for every j or $m = 2k$, $j \neq k$).

(33): $G = Z_4 \times Z_{2m} = <a> \times <b>$ $(m \geq 2)$,
$S = \{a, a^{-1}, b, b^{-1}, a^2b^j, a^2b^{-j}\}$ $1 \leq j < m$
(for every $j \neq 1, m-1$).

(34): $G = Z_4 \times Z_{2m-1} = <a> \times <b>$ $(m \geq 2)$,
$S = \{a, a^{-1}, b, b^{-1}, a^2b^j, a^2b^{-j}\}$ $(1 < j < \frac{2m-1}{2})$.

(35): $G = Z_4 \times Z_m = <a> \times <b>$ $(m \geq 5)$,
$S = \{a, a^{-1}, b, b^{-1}, b^j, b^{-j}\}$ $(1 < j < \frac{m}{2})$,
when $m \neq 2k, 5$ or $m = 2k$ $(k \geq 3, k \neq 5)$, $j \neq k-1$ or $m = 10$, $j \neq 3$.

(36): $G = Z_{2m} = <a>$ $(m \geq 4)$,
$S = \{a, a^{-1}, a^j, a^{-j}, a^{m+1}, a^{m-1}\}$ $(2 \leq j \leq m-2)$,
when $(m, j) = 1$ or $(m, j) = 2, m \neq 4i + 2$ $(i \geq 1)$.

(37): $G = Z_2 \times Z_m = <a> \times <b>$ $(m \geq 5, m \neq 8)$,
$S_1 = \{ab, ab^{-1}, b, b^{-1}, b^j, b^{-j}\}$,
$S_2 = \{ab, ab^{-1}, b, b^{-1}, ab^j, ab^{-j}\}$ $(2 \leq j < \frac{m}{2})$, when $(m, j) = p \leq 2$.

(38): $G = Z_2\times Z_8 = <a>\times<b>$,
$S_1 = \{ab, ab^7, b, b^7, b^2, b^6\}$,
$S_2 = \{ab, ab^7, b, b^7, ab^2, ab^6\}$.

(39): $G = Z_m = <a>$ $(m \geq 9, m \neq 14)$,
$S = \{a, a^{-1}, a^3, a^{-3}, a^j, a^{-j}\}$ $j \neq 3, 2 \leq j < \frac{m}{2}$) when $m \neq 6k$, $\forall$ j or $m = 6k$, $j \neq 3k-1$.

(40): $G = Z_{14} = <a>$,
$S = \{a, a^{-1}, a^3, a^{-3}, a^j, a^{-j}\}$ for $j = 2, 4, 6$.

(41): $G = Z_m = <a>$ $(m \geq 7)$,
$S = \{a, a^{-1}, a^{3j}, a^{-3j}, a^j, a^{-j}\}$, $(2 \leq j < \frac{m}{2}$, $3j \not\equiv 0, 1, m-1, j, m-j, \frac{m}{2} \pmod m$) , when $m \neq 7, 14, 6k$ $(k \geq 2)$ and $m = 7$; $j = 2$ or $m = 14$; $j = 2, 3, 4, 6$ or $m = 6k$; $j \neq 3k-1$.

(42): $G = Z_m = <a>$ $(m \geq 8, m \neq 14)$,
$S = \{a, a^{-1}, a^{2+j}, a^{-2-j}, a^j, a^{-j}\}$ (if $m = 2k$ then $2 \leq j \leq \frac{m}{2} - 3$ and if $m = 2k+1$ then $2 \leq j \leq \frac{m}{2} - 1$). When $m \neq 3k$ for every j and when $m = 3k$, for k odd ; $j \neq k-1$ and for k even ; $j \neq k-1$, $3\frac{k}{2} - 3$.

(43): $G = Z_{14} = <a>$,
$S = \{a, a^{-1}, a^{2+j}, a^{-2-j}, a^j, a^{-j}\}$ for $j = 2, 4$.

(44): $G = Z_2 \times Z_4 \times Z_{2m} = <a> \times <b> \times <c>$ $(m \geq 3)$,
$S = \{a, ab^2c^m, b, b^{-1}, c, c^{-1}\}$.

Now we are in a position to prove Theorem 1.1. Immediately from Lemma 2.3, [5, Theorem 1.1] and [6, Theorem 1.2], we have the Cases (1)-(32) of Theorem 1.1. Assume that Γ is not normal. In view of Proposition 2.2, we have the following assumption: $\exists$ s, t, u, $v \in S$ such that $st = ub \neq 1$ but $\{s, t\} \neq \{u, v\}$. (*).
We divide S into four cases:

Case 1: $S = \{a, b, c, d, e, f\}$, where a, b, c, d, e, f are involutions. In this case G is an elementary abelian 2-group and a, b, c, d, e, f are not independent by the assumption (*). Consequently $G = Z_2^3$ or $G = Z_2^4$ or $G = Z_2^5$. If $G = Z_2^3 = <a>\times<b>\times<c>$ by the assumption (*) we can let $S = \{a, b, c, ab, ac, abc\}$. We have $\sigma = (a, abc) \in A_1$, but $\sigma \notin Aut(G, S)$; and by Proposition 2.1 , $\Gamma = Cay(G, S)$ is not normal , the Case (33) of Theorem 1.1. If $G = Z_4^2 = <a> \times <b> \times <c> \times <d>$ by the assumption (*) we see that S is one of the following cases:
(i) $S_1 = \{a, b, c, d, abc, abd\}$, (ii) $S_2 = \{a, b, c, d, ab, abc\}$,
(iii) $S_3 = \{a, b, c, d, abc, abc\}$.

When $S = S_1$, $\sigma = (a, b) \in A_1$, but $\sigma \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (34) of Theorem 1.1. When $S = S_2$, we have the Case (22) of the main theorem. Also when $S = S_3$, Γ is normal by Lemma 3.3. If $G = Z_2^5 = <a> \times <b> \times <c> \times <d> \times <e>$ we can let $S = \{a, b, c, d, e, abc\}$ and hence $\Gamma = Cay (G, S)$ is non-normal, the Case (1) of Theorem 1.1.

Case 2: $S = \{a, b, c, d, e, e^{-1}\}$, where a, b, c, d are involutions but e is not. In this case, $S^2 - 1 = \{ab, ac, ad, ae, ae^{-1}, bc, bd, be, be^{-1}, cd, ce, ce^{-1}, de, de^{-1}, e^2, e^{-2}\}$. By the assumption (*) $d = abc$, $o(e) = 4$ or $d = e^3$. Suppose $d = abc$. Then $G = Z_2^2 \times Z_{2m}$,$(m \geq 2)$ or $G = Z_2^3 \times Z_m$, $(m \geq 3)$.

If $G = Z_2^2 \times Z_{2m} = <a>\times<b>\times<c>$, $(m \geq 2)$, we can let
$S = \{a, b, ac^m, bc^m, c, c^{-1}\}$ or
$S = \{a, b, c^m, abc^m, c, c^{-1}\}$.
When $S = \{a, b, ac^m, bc^m, c, c^{-1}\}$,
$\sigma = (ab, abc^m)(abc, abc^{m+1})...(abc^{m-1}, abc^{2m-1}) \in A_1$, but $\sigma \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (35) of the main theorem.
When $S = \{a, b, c^m, abc^m, c, c^{-1}\}$, $\Gamma = Cay(G, S)$ is normal by Lemma 3.3(3). If $G = Z_2^3 \times Z_m = <a> \times <b> \times <c> \times <d>$, $(m \geq 3)$, $S = \{a, b, c, abc, d, d^{-1}\}$, the Case (2) of Theorem 1.1. Suppose $o(e) = 4$. Then G =

$\mathbf{Z}_2^2 \times Z_4$, $\mathbf{Z}_2^3 \times Z_4$ or $\mathbf{Z}_2^4 \times Z_4$. If $G = \mathbf{Z}_2^2 \times Z_4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, we have S is one of the following cases:
$S_1 = \{a, b, ab, ac^2, c, c^{-1}\}$, $S_2 = \{a, b, ae^2, bc^2, c, c^{-1}\}$,
$S_3 = \{a, b, ac^2, abc^2, c, c^{-1}\}$.
$S_4 = \{a, b, ab, c^2, c, c^{-1}\}$,
$S_5 = \{a, b, ac^2, c^2, c, c^{-1}\}$,
$S_6 = \{a, b, abc^2, c^2, c, c^{-1}\}$.

When $S = S_1$, $\sigma = (ac^2, c)(ac, c^2)(bc, abc^2)(abc, bc^2) \in A_1$, but $\sigma \notin \mathrm{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \mathrm{Cay}(G, S)$ is not normal, the Case $(36 - S_1)$ of Theorem 1.1 . When $S = S_2$, by Proposition 2.1, $\Gamma = \mathrm{Cay}(G, S)$ is not normal, the Case (35, m = 2) of Theorem 1.1. When $S = S_3$, $\sigma = (a, c)(ab, bc)(c^2, ac^3)(bc^3, abc^3) \in A_1$, but $\sigma \notin \mathrm{Aut}(G, S)$; by Proposition 2.4, $\Gamma = \mathrm{Cay}(G, S)$ is not normal the Case $(36 - S_2)$ of Theorem 1.1. When $S = S_4$, we have the Case (3) of Theorem 1.1. When $S = S_5$, we have the Case (23) of Theorem 1.1. When $S = S_6$, Γ is normal by Lemma 3.3 (3, m=2) .If $G = \mathbf{Z}_2^3 \times Z_4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$, we have $S = \{a, b, c, d, d^{-1}, u\}$, where $u = d^2, ab, ad^2, abc, abd^2$ or $abcd^2$. When $u = d^2$, we have the Case $(5 - S_1)$ of Theorem 1.1. When $u = ab$, we have the Case $(5 - S_2)$ of Theorem 1.1. When $u = ad^2$, we have the Case $(5 - S_3)$ of Theorem 1.1. When $u = abc$, we have the Case (2) of Theorem 1.1. When $u = abd^2$, we have the Case (24) of Theorem 1.1. When $u = abcd^2$, $\sigma = (abcd^2, d)(bcd^2, ad)(acd^2, bd)(abd^2, cd)\ (abcd, d^2)(cd^2, abd)(bd^2, acd)$ and $(bcd, ad^2) \in A_1$, but $\sigma \notin \mathrm{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \mathrm{Cay}(G, S)$ is not normal, the Case (37) of Theorem 1.1. If $G = \mathbf{Z}_2^4 = \langle a\rangle\times\langle b\rangle\times\langle c\rangle\times\langle d\rangle\times\langle e\rangle$, $S = \{a, b, c, d, e, e^{-1}\}$, we have the Case (4) of Theorem 1.1. Now suppose $d = e^3$. Then $G = \mathbf{Z}_2^2 \times Z_6$ or $G = \mathbf{Z}_2^3 \times Z_6$. If $G = \mathbf{Z}_2^2 \times Z_6 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, we see that S is one of the following cases: $S_1 = \{a, b, ab, c^3, c, c^{-1}\}$, $S_2 = \{a, b, ac^3, c^3, c, c^{-1}\}$, $S_3 = \{a, b, abc^3, c^3, c, c^{-1}\}$.
When $S = S_1$, we have the Case (6) of Theorem 1.1. For S_2 and S_3, we have the Cases (2) and (3, m = 3) of Lemma 3.3 respectively. If $G = \mathbf{Z}_2^3 \times Z_6 = \langle a\rangle\times\langle b\rangle\times\langle c\rangle\times\langle d\rangle$, then $S = \{a, b, c, d^3, d, d^{-1}\}$, the Case (7) of Theorem 1.1.

Case 3: $S = \{a, b, c, c^{-1}, d, d^{-1}\}$, where a, b are involutions but c, d are not. By the assumption (*) and the symmetry of c, c^{-1}, d and d^{-1}, we have five sub cases (I) $a = c^3$, (II) $a = c^2d$, (III) $o(c) = 4$, (IV) $c^3 = d$ and (V) $c^2 = d^2$. Suppose $a = c^3$, then G is isomorphic to one of the following: $Z_2\times Z_{6m}$ $(m\geq 2)$, $Z_2\times Z_6$, $Z_6\times Z_{2m}$ $(m\geq 2)$, $\mathbf{Z}_2^2 \times Z_{3m}$ $(m\geq 1)$, $Z_2\times Z_6\times Z_m$ $(m\geq 3)$. If $Z_2\times Z_{6m} = \langle a\rangle\times\langle b\rangle$, $(m\geq 2)$, we see that S is one of the following cases:
$S_1 = \{a, b^{3m}, ab^{2m}, ab^{4m}, b, b^{-1}\}$, $S_2 = \{a, ab^{3m}, ab^m, ab^{5m}, b, b^{-1}\}$, $S_3 = \{a, b^{3m}, b^m, b^{5m}, b, b^{-1}\}$. When $S = S_1$, $\sigma = (a, ab^{2m}, ab^{4m})(ab, ab^{2m+1}, ab^{4m+1})\ldots(ab^{2m-1}, ab^{4m-1}, ab^{6m-1}) \in A_1$, but $\sigma \notin \mathrm{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \mathrm{Cay}(G, S)$ is not normal, the Case (38) of the main theorem. For the Cases $S = S_2$ and $S = S_3$, we have the Cases (4) and (5) of Lemma 3.3. If $G = Z_2\times Z_6 = \langle a\rangle \times \langle b\rangle$, we see that S is one of the following cases:

$S_1 = \{a, b^3, ab^2, ab^4, b, b^{-1}\}$, $S_2 = \{a, b^3, b, b^{-1}, b^2, b^4\}$,
$S_3 = \{a, b^3, b, b^{-1}, ab, ab^{-1}\}$.
When $S = S_1$, $\sigma = (a, ab^2, ab^4)(ab, ab^3, ab^5) \in A_1$, but $\sigma \notin \mathrm{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \mathrm{Cay}(G, S)$ is not normal, the Case $(43 - S_5)$ of Theorem 1.1. When $S = S_2$, we have the Case (29, m=3) of Theorem 1.1. When $S = S_3$, $\sigma = (b^5, ab^5)(b^2, ab^2) \in A_1$, but $\sigma \notin \mathrm{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \mathrm{Cay}(G, S)$ is not normal, the Case $(43 - S_1)$ of Theorem 1.1. If $G = Z_6\times Z_{2m} = \langle a\rangle \times \langle b\rangle$, we see that S is one of the following cases:

$S_1 = \{a^3, b^m, a, a^{-1}, b, b^{-1}\}$, $S_2 = \{a^3, a^3b^m, a, a^{-1}, b, b^{-1}\}$.
When S =, by Proposition 2.1, $\Gamma = \mathrm{Cay}(G, S)$ is not normal, the Case (8) of Theorem 1.1.
For $S = S_2$, when m = 2, $\sigma = (b^2, a^3b)(ab^2, a^4b)(a^2b^2, a^5b)(a^3b^2, b)\ (a^4b^2, ab)(a^5b^2, a^2b) \in A_1$, but $\sigma \notin \mathrm{Aut}(G, S)$; $\Gamma = \mathrm{Cay}(G, S)$ is not normal, the Case (40, m=3) of Theorem 1.1, and when $m\geq 3$, $\Gamma = \mathrm{Cay}(G, S)$ is normal by Lemma 3.3(6). If $G = \mathbf{Z}_2^2 \times Z_{3m} = \langle a\rangle\times\langle b\rangle\times\langle c\rangle$ $(m\geq 1)$, $S = \{a, b, ac^m, ac^{2m}, c, c^{-}1\}$. Then we obtain the Case (25) of Theorem 1.1. If $G = Z_2\times Z_6 \times Z_m = \langle a\rangle\times\langle b\rangle\times\langle c\rangle$ $(m\geq 3)$, $S = \{b^3, a, b, b^{-1}, c, c^{-1}\}$. Then we obtain the Case (9) of Theorem 1.1. Suppose $a = c^2d$. Then we have one of the following cases:
(1): $G = Z_2\times Z_{2m} = \langle a\rangle \times \langle b\rangle$ $(m\geq 3)$,
$S = \{a, b^m, b, b^{-1}, ab^{-2}, ab^2\}$.

(2): $G = Z_2\times Z_{2m} = \langle a\rangle \times \langle b\rangle$,
$S_1 = \{ab^m, a, b, b^{-1}, ab^{m-2}, ab^{m+2}\}$ $(m\geq 3)$,
$S_2 = \{b^m, a, b, b^{-1}, b^{m-2}, b^{m+2}\}$, $m\geq 4$,

(3): $G = Z_2\times Z_{4m+2} = \langle a\rangle \times \langle b\rangle$,
$S_1 = \{a, b, b^{-1}, b^{2m+1}, ab^m, ab^{3m+2}\}$ $(m\geq 1)$,
$S_2 = \{a, b, b^{-1}, b^{2m+1}, b^m, b^{3m+2}\}$, $m\geq 2$
$S_3 = \{a, b, b^{-1}, b^{2m+1}, b^{3m+1}, b^{m+1}\}$ $(m\geq 1)$,
$S_4 = \{a, b^{2m+1}, ab^{3m+1}, ab^{m+1}, b, b^{-1}\}$, $m\geq 1$,

(4): $G = Z_4 \times Z_{4m+2} = \langle a\rangle \times \langle b\rangle$,
$S_1 = \{a^2b^{2m+1}, b^{2m+1}, ab^m, a^3b^{3m+2}, b, b^{-1}\}$, $m\geq 1$
$S_2 = \{^{a2b2m+1}, a^2, ab^m, a^3b^{3m+2}, b, b^{-1}\}$, $m\geq 1$.

(5): $G = \mathbf{Z}_2^2 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m\geq 3)$,
$S = \{a, b, c, c^{-1}, ac^{-2}, ac^2\}$.

(6): $G = Z_2\times Z_4\times Z_{4m+2} = \langle a\rangle\times\langle b\rangle\times\langle c\rangle$ $(m\geq 1)$,
$S = \{a, b^2c^{2m+1}, bc^m, b^{-1}c^{-m}, c, c^{-1}\}$.

(7): $G = \mathbf{Z}_2^2 \times Z_{4m+2} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m\geq 1)$,
$S = \{a, c^{2m+1}, bc^m, bc^{-m}, c, c^{-1}\}$.

In the Case (1), when m = 3, $\sigma = (b^2, b^4) \in A_1$, but $\sigma \notin \mathrm{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \mathrm{Cay}(G, S)$ is not

normal, the Case (43− S_5, m = 3) of Theorem 1.1. When m≥ 4, Γ is normal by Lemma 3.3(7− S_1).
In the Case (2), S = S_1 when m = 3, σ = (b^2, ab^2)(b^5, ab^5) ∈ A_1, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43− S_2) of Theorem 1.1.

When m = 4, σ = (b, b^7)(b^2, b^6)(b^3, b^7) ∈ A_1, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case 39 (m = 2) of Theorem 1.1. When m≥ 5, Γ = Cay(G, S) is normal by Lemma 3.3 (7− S_2). In the Case (2), S = S_2, when m = 5, we have the Case (26) of Theorem 1.1. When m≥ 6, Γ is normal by Lemma 3.3 (7− S_3).

In the Case (3), S = S_1, when m = 1, we have the Case (43 − S_1) of Theorem 1.1. When m≥ 2, Γ is normal by Lemma 3.3 (8 − S_1). In the Case (3), S = S_2, Γ is normal by Lemma 3.3 (8 − S_2). In the Case (3), S = S_3, when m = 1, 2, we have the Cases (29,m = 3, 5) of Theorem 1.1 respectively. When m≥ 3, Γ is normal by Lemma 3.3(8 − S_4). In the Case (3), S = S_4, when m= 1, σ = (ab, ab^5) ∈ A_1, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (29,m = 3) of Theorem 1.1. When m≥ 2, Γ = Cay (G, S) is normal by Lemma 3.3(8 − S_3). In the Case (4), Γ = Cay (G, S) is normal by Lemma 3.3(9). In the Case (5), when m = 3, 6, by Proposition 2.1, Γ is not normal, the Case (25, m = 1, 2) of Theorem 1.1. Otherwise Γ is normal by Lemma 3.3(10). In the Case (6), Γ is normal by Lemma 3.3(16). In the Case (7), when m = 1, by Proposition 2.1, Γ is not normal, the Case 27 (m = 1) of Theorem 1.1. When m≥ 2, Γ is normal by Lemma 3.3 (17). Suppose o(c) = 4. Then we have one of the following cases:
(I) G = Z_2× Z_4 = <a> × <b>, S_1= {a, b^2, b, b^{-1}, ab, ab^{-1}},

(II) G = Z_2 × Z_{4m} = <a> × <b>, S_1= {a, b^{2m}, ab^m, ab^{3m}, b, b^{-1}}, (m≥ 2), S_2={a, ab^{2m}, ab^m, ab^{3m}, b, b^{-1}}, (m≥ 1),
S_3= {a, b^{2m}, b^m, b^{3m}, b, b^{-1}}, (m≥ 2),
S_4= {a, ab^{2m}, b^m, b^{3m}, b, b^{-1}}, (m≥ 2).

(III) G = Z_4 × Z_{2m} = <a> × <b> (m≥ 2),
S_1= {a^2, b^m, a, a^{-1}, b, b^{-1}}, S_2=
{a^2, a^2b^m, a, a^{-1}, b, b^{-}1}, S_3 = {a^2b^m, b^m, a, a^{-1}, b, b^{-1}}.

(IV): G = Z_2^2 ×Z_4 = <a>×<b>×<c>,
S_1= {a, b, c, c^{-1}, ac, ac^{-1}}, S_2= {a, b, c, c^{-1}, abc, abc^{-1}}.
(V): G = Z_2^2 × Z_{4m} = <a> × <b> × <c> (m≥ 2),
S_1= {a, b, abc^m, abc^{3m}, c, c^{-1}}, S_2= {a, b, ac^m, ac^{3m}, c, c^{-1}}, S_3= {a, b, c^m, c^{3m}, c, c^{-1}}.

(VI): G = Z_2× Z_4 × Z_m = <a> × <b> × <c> (m≥ 3),
S_1= {a, b^2, b, b^{-1}, c, c^{-1}},
S_2 = {a, ab^2, b, b^{-1}, c, c^{-1}}.

(VII): G = Z_2×Z_4× Z_{2m} = <a>×<b>×<c> (m≥ 2),
S_1= {a, c^m, b, b^{-1}, c, c^{-1}}, S_2 = {a, ac^m, b, b^{-1}, c, c^{-1}},
S_3= {a, b^2c^m, b, b^{-1}, c, c^{-1}}, S_4= {a, ab^2c^m, b, b^{-1}, c, c^{-1}}.

(VIII): G = Z_2^2 × Z_{4m} = <a> × <b> × <c> (m≥ 1),
S_1= {a, c^{2m}, bc^m, bc^{3m}, c, c^{-1}},
S_2 = {a, ac^{2m}, bc^m, bc^{3m}, c, c^1}.

(IX): G = Z_2^2 × Z_4 × = <a> × <b> × <c> × <d> (m≥ 3),
S = {a, b, c, c^{-1}, d, d^{-1}}.

(X): G = Z_2^3 × Z_{4m} = <a> × <b> × <c> × <d> (m≥ 1),
S = {a, b, cd^m, cd^{3m}, d, d^{-1}}.

In the Case (I), σ = (ab, b^3) ∈ A_1, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43 − S_1) of Theorem 1.1. In the Case (II), S = S_1, Γ = Cay(G, S) is normal by Lemma 3.3(11 − S_1). In the Case (II), S = S_2, σ = (b, b^{-1})(b^2, b^{-2})...(b^{2m-1}, b^{2m+1})(a, ab^m)...(ab^{2m+1}, $ab^{-(m+1)}$) ∈ A_1, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal , the Case (39) of Theorem 1.1. In the Case (II), S = S_3, and S = S_4, Γ is normal by Lemma 3.3, the Case (11 − S_2, S_3). In the Case (III), when S = S_1, we have the Case (10) of Theorem 1.1. When S = S_2, m = 2, σ = (a^2b^2, b)(a^3b^2, ab)(ab^2, a^3b)(b^2, a^2b) ∈ A_1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (40, m = 2) of Theorem 1.1. When S = S_2, m≥ 3, Γ = Cay(G, S) is normal by Lemma 3.3(12). When S = S_3, σ = (a^2, ab^m)(a^2b, ab^{m+1})...(a^2b^{2m-1}, $ab^{m+(2m-1)}$) ∈ A_1 but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal , the Case (40) of Theorem 1.1.

In the Case (IV), when S = S_1, σ = (c^2, ac^2)(bc^2, abc^2) ∈ A_1,but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44− S_2) of Theorem 1.1. When S = S_2, σ = (ac^2, bc^2) ∈ A_1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44− S_3) of Theorem 1.1. In the Case (V), S = S_1, when m = 1, with an argument similar to the Case (IV − S_2) we obtain the same result. When m≥ 2, Γ is normal by Lemma 3.3 (13− S_1). In the Case (V), S = S_2, when m = 1, with an argument similar to the Case (IV−S_1), we obtain the same result.

When m≥ 2, Γ is normal by Lemma 3.3 (13 − S_2). In the Case (V), S = S_3, Γ is normal by Lemma 3.3(13− S_3). In the Case (VI), we have the Case (11) of Theorem 1.1. In the Case (VII), S = S_1, S = S_3 and S = S_2 (m = 2), we have the Cases (12), (28) and (11 − S_2, m = 4) of Theorem 1.1 respectively. In the Case (VII), S = S_2, m≥ 3, Γ is normal by Lemma 3.3(18). In the Case (VII), S = S_4, for m = 2, σ = (b^3, c)(ab^3, ac)(abc^2, ab^2c^3)(b^2, bc)(b^3c^3, c^2)(b^2c, b^2c^3)(ab^2, abc)(ab^3c^3, ac^2) ∈ A_1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (57) of Theorem 1.1, and for m≥ 3, Γ is normal by Lemma 3.3(44). In the Case (VIII), S = S_1 when m = 1, we have the Case (21, m = 2) of Theorem 1.1. If m≥ 2, Γ is normal by Lemma 3.3 (13 − S_4). In the Case (VIII), S = S_2, σ = (ab, abc^{2m})(abc, abc^{2m+1})...(abc^{2m-1}, abc^{4m-1}) ∈ A_1, but σ ∉ Aut(G, S); by Proposition 2.1 ,Γ = Cay(G, S) is not normal, the Case (41) of Theorem 1.1. In the Case

(IX), we have the Case (13) of Theorem 1.1. In the Case (X), m = 1, we have the Case (14) of Theorem 1.1, and for $m\geq 2$, Γ = Cay (G, S) is normal by Lemma 3.3(14). Suppose $c^3 = d$, then G= $\mathbf{Z}_2^2 \times Z_{2m}$, ($m\geq 4$) or G = $\mathbf{Z}_2^2 \times Z_m$ ($m\geq 5$, $m \neq 6$). If G = $Z_2 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$ ($m\geq 4$), we can let S to be

S_1= {a, b^m, b, $b^{-}1$, b^3, b^{-3}} or S_2= {a, ab^m, b, b^{-1}, b^3, b^{-3}}. Let S = S_1, for m = 4, 5 we have the Cases (29), (26) of Theorem1.1 respectively, and for $m\geq 6$, Γ is normal by Lemma 3.3(19 − S_1). Let S = S_2. When m = 4, $\sigma = (ab^2, ab^6) \in A_1$, but $\sigma \notin$ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43− S_4), m = 4) of Theorem 1.1. When m = 5, $\sigma = (b^3, b^7)(ab^3, ab^7)(b^2, b^8)(ab^2, ab^8) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (42) of Theorem 1.1. When $m\geq 6$, Γ = Cay (G, S) is normal by Lemma 3.3(19 − S_2). If G = $\mathbf{Z}_2^2 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m\geq 5$, m 6= 6), S = {a, b, c, c^{-1}, c^3, c^{-3}}. When m = 5, 10 and m = 8 we have the Cases (15), and (16) of Theorem 1.1 respectively. When m = 7, 9, $m\geq 11$, Γ = Cay (G, S) is normal by Lemma 3.3(15). Suppose $c^2 = d^2$, then G = $Z_2 \times Z_{2m}$, G = $\mathbf{Z}_2^2 \times Z_{2m}$ ($m\geq 3$) G = $\mathbf{Z}_2^2 \times Z_{2m-1}$ ($m\geq 2$) or G = $\mathbf{Z}_2^2 \times Z_m$ ($m\geq 3$). If G= $Z_2 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$ we see that S is one of the following cases:

1) S_1= {a, b^m, b, b^{-1}, ab, ab^{-1}}, $m\geq 2$,

2) S_2= {a, ab^m, b, b^{-1}, ab, ab^{-1}}, $m\geq 2$,

3)S_3= {a, b^m, b, b^{-1}, b^{m+1}, b^{m-1}}, $m\geq 3$,

4) S_4= {a, ab^m, b, b^{-1}, b^{m+1}, b^{m-1}}, $m\geq 3$,

5) S_5={a, b^m, b, b^{-1}, ab^{m+1}, ab^{m-1}}, $m\geq 3$,

6) S_6= {a, ab^m, b, b^{-1}, ab^{m+1}, ab^{m-1}}, $m\geq 3$,

7) S_7= {ab^m, b^m, b, b^{-1}, ab, ab^{-1}}, $m\geq 2$,

8) S_8 = {ab^m, b^m, b, b^{-1}, ab^{m+1}, ab^{m-1}}, $m\geq 2$.

In the Case (1), $m\geq 2$, when m = 2i, $\sigma = (b^i, ab^i)(b^{3i}, ab^{3i}) \in A_1$, but $\sigma \notin$ Aut(G, S) and when m = 2i + 1, $\sigma = (b^{i+1}, ab^{i+1})(b^{3i+2}, ab^{3i+2}) \in A_1$, but $\sigma \notin$ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43 − S_1) of Theorem 1.1. In the Case (2), similarly Case (1), Γ = Cay(G, S) is not normal, the Case (43− S_2) of Theorem 1.1. In the Case (3), we have the Case (29) of Theorem 1.1. In the Case (4), when m = 2i, $\sigma = (ab^i, ab^{3i}) \in A_1$, but $\sigma \notin$ Aut(G, S) and when m = 2i + 1, $\sigma = (ab^{i+1}, ab^{3i+2}) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43 − S_4) of Theorem 1.1. In the Case (5), when m = 2i,$\sigma = (b^{3i}, ab^i) \in A_1$, but $\sigma \notin$ Aut(G, S) and when m = 2i+1, $\sigma = (b^{i+1}, ab^{3i+2}) \in A_1$,but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43− S_5) of Theorem 1.1. In the Case (6), when m = 2i, $\sigma = (b^i, ab^{3i})(b^{3i}, ab^i) \in A_1$,but $\sigma \notin$ Aut(G, S) and when m = 2i + 1, $\sigma = (b^{i+1}, ab^{3i+2})(b^{3i+2}, ab^{i+1}) \in A_1$, but $\sigma \notin$ Aut(G, S). Hence by Proposition 2.1, Γ = Cay (G, S) is not normal, the Case (43 − S_6) of Theorem 1.1.

In the Case (7), for m = 2i and m = 2i + 1, $\sigma = (b^{i+1}, ab^{i+1}) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay (G, S) is not normal, the Case (43 − S_3) of Theorem 1.1. In the Case (8), for m = 2i and m = 2i − 1, $\sigma = (b^i, ab^{i+m})(b^{m+i}, ab^i) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43 − S_1) of Theorem 1.1. If G = $\mathbf{Z}_2^2 \times Z_{2m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, we can let S to be one of the following cases:

(1): S_1= {a, b, c, c^{-1}, ac, ac^{-1}}, $m\geq 2$,
(2): S_2= {a, b, c, c^{-1}, abc, abc^{-1}}, $m\geq 2$,
(3): S_3= {a, b, c, c^{-1}, c^{m+1}, c^{m-1}}, $m\geq 3$,
(4): S_4= {a, b, c, c^{-1}, ac^{m+1}, ac^{m-1}}, $m\geq 2$,
(5): S_5= {a, b, c, c^{-1}, abc^{m+1}, abc^{m-1}}, $m\geq 2$,
(6): S_6= {a, cm, c, c^{-1}, bc, b c^{-1}}, $m\geq 2$,
(7): S_7 = {a, ac^m, c, c^{-1}, bc, bc^{-1}}, $m\geq 2$,
(8): S_8 = {a, c^m, c, c^{-1}, bc^{m+1}, bc^{m-1}}, $m\geq 2$,
(9): S_9 = {a, ac^m, c, c^{-1}, bc^{m+1}, bc^{m-1}}, $m\geq 2$.

In the Case (1), Γ is not normal, the Case (30) of Theorem 1.1. In the Case (2),$\sigma = (ac^{m-1}, bc^{m-1}) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1 , Γ=Cay(G, S) is not normal, the Case (44 − S_1) of Theorem 1.1. In the Case (3), when m = 2i, Γ = Cay (G, S) is not normal, the Case (16) of Theorem 1.1.

When m = 2i+1, Γ =Cay (G, S) is not normal, we have the Case 14 (with m odd) of Theorem 1.1. In the Case (4), when m = 2i, i ≥ 2, $\sigma = (c^i, ac^{3i})(ac^i, c^{3i})(bc^i, abc^{3i})(abc^i, bc^{3i}) \in A_1$, but $\sigma \notin$ Aut(G, S), and when m =2i+1, $\sigma = (c^{i+1}, ac^{3i+2})(ac^{i+1}, c^{3i+2})(bc^{i+1}, abc^{3i+2})(abc^{i+1}, bc^{3i+2}) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44 − S_2) of Theorem 1.1. In the Case (5), when m = 2i, i ≥ 2, $\sigma = (c^{3i}, abc^i)(ac^{3i}, bc^i)(bc^{3i}, ac^i)(abc^{3i}, c^i) \in A_1$, but $\sigma \notin$ Aut(G, S) and when m = 2i + 1, $\sigma = (c^{3i+2}, abc^{i+1})(ac^{3i+2}, bc^{i+1})(bc^{3i+2}, ac^{i+1}) (abc^{3i+2}, c^{i+1}) \in A_1$, but $\sigma \notin$ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44 − S_3) of Theorem 1.1. In the Case (6), $m\geq 2$, Γ is not normal, we have the Case (27) of Theorem 1.1. In the Case (7), if $m\geq 3$, for m = 2i and m = 2i − 1, σ = (ci, bci)(aci, abci)(ci+m, bci+m) (aci+m, abci+m) $\in A_1$, but $\sigma \notin$ Aut(G, S), and if m = 2, $\sigma = (b, bc^2)(ab, abc^2) \in A_1$, but $\sigma \notin$ Aut(G, S). Then by Proposition 2.1, Γ = Cay (G, S) is not normal , the Case (44 − S_4) of Theorem 1.1. In the Case (8), for m = 2i and m = 2i−1, $\sigma = (c^i, bc^{i+m})(ac^i, abc^{i+m})(c^{i+m}, bc^i)(ac^{i+m}, abc^i) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44 − S_5) of Theorem 1.1. In the Case (9), similarly Case (8), Γ = Cay(G, S) is not normal . We have the Case (44 − S_6) of Theorem 1.1.

If G = $\mathbf{Z}_2^2 \times Z_{2m-1} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, ($m\geq 2$), then S is S_1= {a, b, c, c^{-1}, ac, ac^{-1}} or S_2= {a, b, c, c^{-1}, abc, abc^{-1}}. When S = S_1, σ = (cm, acm)(bcm, abcm) $\in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal , the Case (44− S_7) of the main theorem .

When $S = S_2$, $\sigma = (ac^{m-1}, bc^{m-1}) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44− S_1) of Theorem 1.1. If $G = Z_2^2 \times Z_m$= <a>×<b>×<c>×<d>, we can consider m≥ 3, S = {a, b, d, d^{-1}, cd, cd^{-1}}. In this case for m = 2i and m = 2i−1, (i≥2) $\sigma = (d^i, cd^i)(ad^i, acd^i)(bd^ibcd^i)(abd^i, abcd^i) \in A_1$, but $\sigma \notin$ Aut(G, S) and by Proposition 2.1, Γ = Cay(G, S) is not normal the Case (14) of Theorem 1.1.

Case 4: S = {a, a^{-1}, b, b^{-1}, c, c^{-1}}, where the elements of the set S are not involution By the assumption (*), o(a) = 4, $a^2 = b^2$, $a^3 = b$ or $c = a^2b$. Suppose o(a) = 4, then G is isomorphic to one of the following: Z_{4m} (m≥ 2), $Z_4 \times Z_m$, $Z_{4m} \times Z_n$ (m≥ 2, n≥3), $Z_{4m} \times Z_{4n}$ (m≥ 1, n≥1) , $Z_4 \times Z_m \times Z_n$ (m, n≥3). If G = Z_{4m} = <a> (m≥ 2), we can let S = {a^m, a^{-m}, a, a^{-1}, a^j , a^{-j}}, where 1 <j <2m, j ≠ m. When j = 2m − 1, $\sigma = (a^m, a^{-m}) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (45) of Theorem 1.1. When j ≠ 2m − 1, Γ = Cay (G, S) is normal by Lemma 3.3(31). If G = $Z_4 \times Z_m$ = <a> × <b>, we can let S to be one of the following cases:

(1): S_1= {a, a^3, b, b^{-1}, ab^j , a^3b^{-j}}, m≥ 3, 1 ≤ j ≤[m/2] ,

(2): S_2= {a, a^3, b, b^{-1}, a^2b^j , a^2b^{-j}},
m≥ 2, 1 ≤ j ≤ (m/2) ,

(3): S_3= {a, a^3, b, b^{-1}, b^j , b^{-j}}, m≥ 5, 1 < j< (m/2).

When S = S_1, for m = 2j, $\sigma = (a^2, a^2b^j)(a^2b, a^2b^{j+1})...(a^2b^{j-1}, a^2b^{2j-1}) \in A_1$, but$\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (49) of the main theorem. Otherwise, Γ is normal by Lemma 3.3(32). When S = S_2, j = 1 for m = 2k and m = 2k − 1, k ≥ 2, $\sigma = (ab^k, a^3b^k) \in A_1$, but $\sigma \notin$ Aut(G, S), and when j = k − 1,m = 2k (k ≥ 3), $\sigma = (b^{k-1}, a^2b^{-1})(ab^{k-1}, a^3b^{-1})(a^2b^{k-1}, b^{-1})(a3b^{k-1}, ab^{-1}) \in A_1$, but $\sigma \notin$ Aut(G, S), then these graphs are non-normal and we have the Cases (49, 50) of Theorem 1.1. Otherwise, Γ = Cay (G, S) is normal by Lemma 3.3 (33, 34). When S = S_3, for j = k − 1, m = 2k, if k is odd we have the Case (17) of Theorem 1.1 and if k is even we have the Case 19 (m = 4) of the main theorem. For m = 5; j = 2 and m = 10; j = 3 we have the Case 21(m = 4) of the main theorem.
Otherwise, Γ = Cay(G, S) is normal by Lemma 3.3 (35). If G = $Z_{4m} \times Z_n$ = <a> × <b> (m≥ 2, n≥3), S = {a^m, a^{-m}, a, a^{-1}, b, b^{-1}}, then Γ = Cay (G, S) is normal by Lemma 3.3(20). If G = $Z_{4m} \times Z_{4n}$ = <a> × <b> (m≥ 1, n≥1), S = {a^mb^n, $a^{-m}b^{-n}$, a, a^{-1}, b, b^{-1}}, then Γ = Cay(G, S) is normal by Lemma 3.3(21).If G = $Z_4\times Z_m \times Z_n$ = <a>×<b>×<c> (m, n≥3), we can consider S = {a, a^3, b, b^{-1}, c, c^{-1}}. In this case, for m = 4, Γ = Cay (G, S) is not normal, the Case (18) of Theorem 1.1, and for m, n ≠ 4, Γ = Cay(G, S) is normal by Lemma 3.3(22). Suppose $a^2 = b^2$. Then G is isomorphic to one of the following: Z_{2m}, $Z_2\times Z_m$ (m≥ 5), $Z_{2m} \times Z_{2n+1}$, $Z_{2m} \times Z_{2n}$ (m≥ 3, n≥2) , $Z_2 \times Z_n$ (m≥ 3, n≥3). If G = Z_{2m} = <a>, we can let S to be S_1= {a^j , a^{-j} , a^{m+j} , a^{m-j} , a, a^{-1}}, 2 ≤ j ≤ m/2, m≥ 5, or S_2= {a, a^{-1}, a^{m+1}, a^{m-1}, a^j , a^{-j}}, 2≤ j ≤ m − 2, m≥ 4. When S = S_1, Γ = Cay(G, S) is normal by Lemma 3.3(23). When S = S_2, (m, j) =2, for m = 4i + 2, j = 2i (with i odd) and j = 2i + 2 (with i even), $\sigma = (a^2, a^{2+m/2})(a^6, a^{6+m/2})...(a^{2m-2}, a^{m/2-2}) \in A_1$, but $\sigma \notin$ Aut (G, S), and when (m, j) = 1 >2, then $\sigma = (a^2, a^{m+2})(a^{2+1}, a^{m+2+1})...(a^{m+2-1}, a^{2-1}) \in A_1$, but $\sigma \notin$ Aut (G, S), then by Proposition 2.1 these graphs are non-normal, and we have the Case (46) of the main theorem. Otherwise, Γ = Cay(G, S) is normal by Lemma 3.3 (36). If G = $Z_2 \times Z_m$ = <a> × <b> m≥ 5, we can let S to be S_1= {b, b^{-1}, ab, ab^{-1}, b^j , b^{-j}},
2 ≥ j > m/2 or S_2 = {b, b^{-1}, ab, ab^{-1}, ab^j , ab^{-j}}, 2 ≥ j > m/2. Let S = S_1. When (m, j) = p > 2; m = (t + 1)p, $\sigma = (b, ab)(b^{p+1}, ab^{p+1})...(bt^{p+1}, abt^{p+1}) \in A_1$, but $\sigma\notin$ Aut (G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (47− S_1) of the main theorem.
When m = 8, j = 3, $\sigma = (b^2, b^6)(ab, a b^7)(a b^3, a b^5) \in A_1$, but $\sigma \notin$ Aut (G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (48− S_1) of Theorem 1.1. Otherwise, Γ = Cay(G, S) is normal by Lemma 3.3(37, 38− S_1). Let S = S_2. When (m, j) = p > 2; m = (t + 1)p, $\sigma = (b, ab)(b^{p+1}, ab^{p+1})\dots(b^{tp+1}, ab^{tp+1}) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (47 − S_2) of Theorem 1.1. When m = 8, j = 3, $\sigma = (b^2, b^6)(b^3, b^5)(b, b^7) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case(48−S_2) of main theorem. Otherwise, Γ = Cay(G, S) is normal by Lemma 3.3(37, 38 − S_2). If G = $Z_{2m} \times Z_n$ = <a> × <b>, we can let S to be one of the following cases:
(1): S_1= {a, a^{-1}, a^{m-1}, a^{m-1}, b, b^{-1}}, m≥ 3,
(2): S_2 = {b,b^{-1}, a^mb, a^mb^{-1}, a, a^{-1}}, m≥ 2,
(3): S_3= {b, b^{-1}, $a^{m+1}b^l$, $a^{m-1}b^l$, a, a^{-1}}, n = 2l, l ≥ 2.

Let S = S_l. When m = 2i, Γ = Cay(G, S) is not normal , the Case (19) of Theorem 1.1. When m = 2i + 1, $\sigma = (a^{m-1}, a^{2m-1})(a^{m-1}b, a^{2m-1}b)\dots(a^{m-1}b^{n-1}, a^{2m-1}b^{n-1}) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.4, Γ = Cay(G, S) is not normal, the Case 20 (with m odd) of Theorem 1.1. Let S = S_2. When n = 2j, 2j − 1 (j ≥ 2), $\sigma = (b^j, a^mb^j)(ab^j, a^{m+1}b^j)\dots(a^{m-1}b^j, a^{2m-1}b^j) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (49) of Theorem 1.1. When S = S_3, $\sigma = (a^{m-1}, a^{-1}b^l)(a^{m-1}b, a^{-1}b^{l+1})\dots(a^{m-1}b^{2l-1}, a^{-1}b^{l-1}) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (50) of Theorem 1.1. If G =$Z_2 \times Z_m \times Z_n$ = <a> × <b> × <c>, m≥ 3, n≥3, S = {b, b^{-1}, ab, ab^{-1}, c, c^{-1}}, we have the Case (20) of the main theorem. Suppose $a^3 = b$, then we have one of the following cases :

(1): G = Z_m = <a>, m≥ 7, S_1= {a, a^{-1}, a^3, a^{-3}, a^j , a^{-j}}, (j≠3, 2 ≤j ≤ m/2),
S_2 = { a^j , a^{-j} , a^{3j} , a^{-3j} , a, a^{-1}}, (2 ≤ j ≤ m/2, 3j ≠ 0, 1,m − 1, j, m − j ,m/2(mod m)).

(2): G = $Z_m \times Z_n$ = <a> × <b>, (n≥3, m≥ 5, m ≠ 6),
S = {a, a^{-1}, a^3, a^{-3}, b, b^{-1}}.

(3): G = $Z_{3m-1}\times Z_{3n}$ = <a>×<b>, (m≥ 2,n≥1),

$S = \{a^m b^n, a^{2m-1}b^{2n}, a^3, a, a^{-1}, b, b^{-1}\}$.

(4): $G = Z_{3m+1}\times Z_{3n} = \langle a\rangle \times \langle b\rangle$, (m, n≥1), $S = \{a^{2m+1}b^n, a^m b^{2n}, a, a^{-1}, b, b^{-1}\}$.

In the Case (1), when m = 6k, j = 3k−1, k ≥ 2, $\sigma = (a, a^{3k+1})(a^4, a^{3k+4})\ldots(a^{3k-2}, a^{6k-2}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (51) of Theorem 1.1. In this case for S_1, when m= 7, j = 2, $\sigma = (a^2, a^5) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (52) of Theorem 1.1. When m = 8, j = 2, $\sigma = (a^2, a^6) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (45) of the main theorem.
When m = 14; j = 5, $\sigma = (a^2, a^{12})(a^5, a^9) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (52) of Theorem 1.1. Also for S_2, when m = 7; j = 3, $\sigma = (a^3, a^4) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay (G, S) is not normal, the Case (52) of Theorem 1.1. When m = 14; j = 3,$\sigma = (a^2, a^{12})(a^5, a^9) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (52) of Theorem 1.1. Otherwise, Γ = Cay (G, S) is normal by Lemma 3.3(39, 40, 41). In the Case (2), when m = 5, 10 and 8 we have the Cases (21) and (19, m = 2) of Theorem 1.1 respectively. Otherwise, Γ = Cay(G, S) is normal by Lemma 3.3 (24). In the Cases (3) and (4), Γ = Cay (G, S) is normal by Lemma 3.3 (25, 26). Suppose $c = a^2b$. Then we have one of the following cases:
(1): $G = Z_m = \langle a\rangle$ (m≥ 7), $S = \{a, a^{-1}, a^j, a^{-j}, a^{2+j}, a^{-2-j}\}$, if m = 2k, 2 ≤j≤(m/2) − 3 and if m = 2k + 1, 2 ≤ j ≤ (m/2) − 1.

(2): $G = Z_m = \langle a\rangle$ (m≥ 7), $S_1=\{a^j, a^{-j}, a, a^{-1}, a^{2j+1}, a^{-2j-1}\}$,
2 ≤ j ≤ m − 2, j ≠ m/2 and 2j + 1 ≠ m/2, 0, 1, m −1, j, m − j (mod m)

(3): $G = Z_m \times Z_n = \langle a\rangle \times \langle b\rangle$ (m, n≥3),
$S = \{a, a^{-1}, b, b^{-1}, a^2b, a^{-2}b^{-1}\}$.

(4): $G = Z_{2m+1} \times Z_n = \langle a\rangle \times \langle b\rangle$ (m≥ 2, n≥3),
$S = \{ a^m, a^{m+1}, a, a^{-1}, b, b^{-1}\}$.

(5): $G = Z_{2m+1} \times Z_{2n+1} = \langle a\rangle \times \langle b\rangle$ (m, n≥1),
$S = \{ a^m b^{n+1}, a^m b^n, a, a^{-1}, b, b^{-1}\}$.

(6): $G = Z_2\times Z_{2m+1} \times Z_{2n+1} = \langle a\rangle\times\langle b\rangle\times\langle c\rangle$ (m, n≥1),
$S = \{ab^m c^{n+1}, ab^{m+1}c^n, b, b^{-1}, c, c^{-1}\}$.
In the Case (1), if m = 3k, k ≥ 3, j = k −1, $\sigma = (a^k, a^{2k}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (53) of Theorem 1.1. If m = 6k, k ≥3, j = 3k − 3, $\sigma = (a, a^{3k+1})(a^4, a^{3k+4})\ldots(a^{3k-2}, a^{6k-2}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ= Cay(G, S) is not normal, the Case (51 − S_2, m≥ 3) of Theorem 1.1. If m = 7; j = 2, $\sigma = (a^3, a^4) \in A_1$, but $\sigma \notin Aut(G, S)$, and if m = 14, j = 2, $\sigma = (a^2, a^{12})(a^5, a^9) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (52) of the main theorem.

Otherwise, Γ = Cay (G, S) is normal by Lemma 3.3(42, 43). In the Case (2), if m = 7, j = 4, $\sigma = (a^5, a^9) \in A_1$, but $\sigma \notin Aut(G, S)$, and if m = 14, j = 5,$\sigma = (a^2, a^{12})(a^5, a^9) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (52) of Theorem 1.1. If m = 3k, j = k − 1, k ≥ 3, $\sigma = (a^k, a^{2k}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (53) of Theorem 1.1. If m = 4j, j ≥ 2, $\sigma = (a^j, a^{3j}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (45) of Theorem 1.1. If m = 6k, j = 3k+1, k ≥ 3, $\sigma = (a, a^{3k+1})(a^4, a^{3k+4})\ldots(a^{3k-2}, a^{6k-2}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ= Cay(G, S) is not normal, the Case (51- S_1) of Theorem 1.1. If m = 8k + 4, k ≥ 1, for k = 2i − 1, j = 4i− 2, i ≥ 1,$\sigma = (a^2, a^{12i-1})(a^6, a^{12i+3})\ldots(a^{m-2}, a^{12i-5}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (54) of Theorem 1.1, and for k= 2i, j = 12i + 2, i ≥1, $\sigma = (a^2, a^{4i+3})(a^6, a^{4i+7})\ldots(a^{m-2}, a^{4i-1}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ=Cay(G, S) is not normal, the Case (55) of Theorem 1.1. In the Case (3), if m = n = 3, $\sigma = (ab, a^2b^2) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (56) of the main theorem. If m = 4, $\sigma = (ab^2, a^3b^2) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (50) of Theorem 1.1. Otherwise, Γ = Cay(G, S) is normal by Lemma 3.3(27) .
In the Case (4), if m = 2, we have the Case (21) of Theorem 1.1. if m≥ 3, Γ = Cay(G, S) is normal by Lemma 3.3(28). In the Case (5), if m = n = 1,$\sigma = (ab, a^2b^2) \in A_1$, but $\notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (56) of Theorem 1.1. Otherwise, Γ = Cay(G, S) is normal by Lemma 3.3(29). In the Case (6), Γ = Cay(G,S) is normal by Lemma 3.3(30).

4. Conclusion

Let Γ = Cay (G, S) be a connected Cayley graph of a abelian group G on S. In this paper we have shown all non-normal Cayley graph Γ with valency 6.

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