

# NORMAL 6-VALENT CAYLEY GRAPHS OF ABELIAN GROUPS

Mehdi Alaeiyan

**Abstract:** We call a Cayley graph  $\Gamma = \text{Cay}(G, S)$  normal for  $G$ , if the right regular representation  $R(G)$  of  $G$  is normal in the full automorphism group of  $\text{Aut}(\Gamma)$ . In this paper, a classification of all non-normal Cayley graphs of finite abelian group with valency 6 was presented.

**Keywords:** Cayley graph, normal Cayley graph, automorphism group.

## 1. Introduction

Let  $G$  be a finite group, and  $S$  be a subset of  $G$  not containing the identity element  $1_G$ . The Cayley digraph  $\Gamma = \text{Cay}(G, S)$  of  $G$  relative to  $S$  is defined as the graph with vertex set  $V(\Gamma) = G$  and edge set  $E(\Gamma)$  consisting of those ordered pairs  $(x, y)$  from  $G$  for which  $yx^{-1} \in S$ . Immediately from the definition we find that, there are three obvious facts: (1)  $\text{Aut}(\Gamma)$  contains the right regular representation  $R(G)$  of  $G$  and so  $\Gamma$  is vertex-transitive.

(2)  $\Gamma$  is connected if and only if  $G = \langle S \rangle$ . (3)  $\Gamma$  is an undirected if and only if  $S^{-1} = S$ .

A Cayley (di)graph  $\Gamma = \text{Cay}(G, S)$  is called normal if the right regular representation  $R(G)$  of  $G$  is a normal subgroup of the automorphism group of  $\Gamma$ .

The concept of normality of Cayley (di)graphs is known to be important for the study of arc-transitive graphs and half-transitive graphs (see[1,2]). Given a finite group  $G$ , a natural problem is to determine all normal or non-normal Cayley (di)graphs of  $G$ . This problem is very difficult and is solved only for the cyclic groups of prime order by Alspach [3] and the groups of order twice a prime by Du et al. [4], while some partial answers for other groups to this problem can be found in [5-8]. Wang et al. [8] characterized all normal disconnected Cayley's graphs of finite groups. Therefore the main work to determine the normality of Cayley graphs is to determine the normality of connected Cayley graphs. In [5, 6], all non-normal Cayley graphs of abelian groups with valency at most 5 were classified. The purpose of this paper is the following main theorem.

**Theorem 1.1** Let  $\Gamma = \text{Cay}(G, S)$  be a connected undirected Cayley graph of a finite abelian group  $G$  on  $S$  with valency 6. Then  $\Gamma$  is normal except when one of the following cases happens:

$$(1): G = Z_2^5 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle, \\ S = \{a, b, c, abc, d, e\}.$$

$$(2): G = Z_2^3 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 3), \\ S = \{a, b, c, abc, d, d^{-1}\}.$$

$$(3): G = Z_2^2 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, \\ S = \{a, b, ab, c^2, c, c^{-1}\}.$$

$$(4): G = Z_2^4 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle, \\ S = \{a, b, c, d, e, e^{-1}\}.$$

$$(5): G = Z_2^3 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \\ S_1 = \{a, b, c, d^2, d, d^{-1}\}, \\ S_2 = \{a, b, ab, c, d, d^{-1}\}, S_3 = \{a, b, c, ad^2, d, d^{-1}\}.$$

$$(6): G = Z_2^2 \times Z_6 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, \\ S = \{a, b, ab, c^3, c, c^{-1}\}.$$

$$(7): G = Z_2^3 \times Z_6 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle, \\ S = \{a, b, c, d^3, d, d^{-1}\}.$$

$$(8): G = Z_6 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2), \\ S = \{a^3, b^m, a, a^{-1}, b, b^{-1}\}.$$

$$(9): G = Z_2 \times Z_6 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 3), \\ S = \{a, b^3, b, b^{-1}, c, c^{-1}\}.$$

$$(10): G = Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2), \\ S = \{a, a^{-1}, a^2, b, b^{-1}, b^m\}.$$

$$(11): G = Z_2 \times Z_4 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 3), \\ S_1 = \{a, b, b^{-1}, b^2, c, c^{-1}\}, S_2 = \{a, b, b^{-1}, ab^2, c, c^{-1}\}.$$

$$(12): G = Z_2 \times Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 2), \\ S = \{a, b, b^{-1}, c, c^{-1}, c^m\}.$$

$$(13): G = Z_2^2 \times Z_4 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \\ (m \geq 3), S = \{a, b, c, c^{-1}, d, d^{-1}\}.$$

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**M Alaeiyan**, Department of Mathematics, Iran University of Science and Technology, alaeiyan@iust.ac.ir

$$(14): G = Z_2^3 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \ (m \geq 3),$$

$$S = \{a, b, cd, cd^{-1}, d, d^{-1}\}.$$

$$(15): G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \ (m = 5, 10),$$

$$S = \{a, b, c, c^{-1}, c^3, c^{-3}\}.$$

$$(16): G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \ (m \geq 2),$$

$$S = \{a, b, c, c^{-1}, c^{2m+1}, c^{2m-1}\}.$$

$$(17): G = Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \ (m \geq 3, m \text{ is odd}),$$

$$S = \{a, a^3, b, b^{-1}, b^{m+1}, b^{m-1}\}.$$

$$(18): G = Z_4^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \ (m \geq 3),$$

$$S = \{a, a^3, b, b^3, c, c^{-1}\}.$$

$$(19): G = Z_{4m} \times Z_n = \langle a \rangle \times \langle b \rangle \ (m \geq 2, n \geq 3),$$

$$S = \{a, a^{-1}, a^{2m+1}, a^{2m-1}, b, b^{-1}\}.$$

$$(20): G = Z_2 \times Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \ (m \geq 3, n \geq 3),$$

$$S = \{ab, a b^{-1}, b, b^{-1}, c, c^{-1}\}.$$

$$(21): G = Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \ (m = 5, 10, n \geq 3),$$

$$S = \{a, a^{-1}, a^3, a^{-3}, b, b^{-1}\}.$$

$$(22): G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle,$$

$$S = \{a, b, ab, c, abc, d\}.$$

$$(23): G = Z_2^2 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle,$$

$$S = \{a, b, ac^2, c, c^{-1}, c^2\}.$$

$$(24): G = Z_2^3 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle,$$

$$S = \{a, b, c, d, d^{-1}, abd^2\}.$$

$$(25): G = Z_2^2 \times Z_{3m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \ (m \geq 1),$$

$$S = \{a, b, ac^m, ac^{2m}, c, c^{-1}\}.$$

$$(26): G = Z_2 \times Z_{10} = \langle a \rangle \times \langle b \rangle, S = \{a, b, b^3, b^5, b^7, b^9\}.$$

$$(27): G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \ (m \geq 2),$$

$$S = \{ac, ac^{-1}, b, c^m, c, c^{-1}\}.$$

$$(28): G = Z_2 \times Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \ (m \geq 2),$$

$$S = \{a, b^2 c^m, b, b^{-1}, c, c^{-1}\}.$$

$$(29): G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \ (m \geq 3),$$

$$S = \{a, b^m, b, b^{-1}, b^{m+1}, b^{m-1}\}.$$

$$(30): G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \ (m \geq 2),$$

$$S = \{a, b, ac, ac^{-1}, c, c^{-1}\}.$$

$$(31): G = Z_2 \times Z_{6m} = \langle a \rangle \times \langle b \rangle \ (m \geq 3, m \text{ is odd}),$$

$$S = \{a, b^2, b^{-2}, b^m, b^{5m}, b^{3m}\}.$$

$$(32): G = Z_2^2 \times Z_{6m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \ (m \geq 2),$$

$$S = \{a, bc^m, bc^{3m}, bc^{5m}, c, c^{-1}\}.$$

$$(33): G = Z_2^3 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, S = \{a, b, c, ab, ac, abc\}.$$

$$(34): G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle,$$

$$S = \{a, b, c, d, abc, abd\}.$$

$$(35): G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \ (m \geq 2),$$

$$S = \{a, b, ac^m, bc^m, c, c^{-1}\}.$$

$$(36): G = Z_2^2 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle,$$

$$S_1 = \{a, b, ab, ac^2, c, c^{-1}\},$$

$$S_2 = \{a, b, ac^2, abc^2, c, c^{-1}\}.$$

$$(37): G = Z_2^3 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle,$$

$$S = \{a, b, c, abcd^2, d, d^{-1}\}.$$

$$(38): G = Z_2 \times Z_{6m} = \langle a \rangle \times \langle b \rangle \ (m \geq 2),$$

$$S = \{a, b^{3m}, ab^{2m}, ab^{4m}, b, b^{-1}\}.$$

$$(39): G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \ (m \geq 1),$$

$$S = \{a, ab^m, ab^{2m}, ab^{3m}, b, b^{-1}\}.$$

$$(40): G = Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \ (m \geq 2),$$

$$S = \{a, a^{-1}, b^m, a^2 b^m, b, b^{-1}\}.$$

$$(41): G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \ (m \geq 1),$$

$$S = \{a, ac^{2m}, bc^m, bc^{3m}, c, c^{-1}\}.$$

$$(42): G = Z_2 \times Z_{10} = \langle a \rangle \times \langle b \rangle,$$

$$S = \{a, ab^5, b, b^9, b^3, b^7\}.$$

$$(43): G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle,$$

$$S_1 = \{a, b, b^{-1}, b^m, ab, a b^{-1}\}, m \geq 2,$$

$$S_2 = \{a, ab^m, b, b^{-1}, ab, a b^{-1}\}, m \geq 2,$$

$$S_3 = \{ab^m, b^m, b, b^{-1}, ab, a b^{-1}\}, m \geq 2, S_4 = \{a, ab^m, b,$$

$$b^{-1}, b^{m+1}, b^{m-1}\}, m \geq 3, S_5 = \{a, b, b^{-1}, b^m, ab^{m+1}, ab^{m-1}\},$$

$$m \geq 3, S_6 = \{a, ab^m, b, b^{-1}, ab^{m+1}, ab^{m-1}\}, m \geq 3$$

$$S_7 = \{ab^m, b, b^{-1}, b^m, ab^{m+1}, ab^{m-1}\}, m \geq 3$$

$$(44): G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, S_1 = \{a, b, c, c^{-1},$$

$$abc, abc^{-1}\}, m \geq 3, S_2 = \{a, b, c, c^{-1}, ac^{k+1}, ac^{k-1}\}, m = 2k,$$

$$k \geq 3, S_3 = \{a, b, c, c^{-1}, abc^{k+1}, abc^{k-1}\}, m = 2k, k \geq 3,$$

$$S_4 = \{a, bc, b c^{-1}, ack, c, c^{-1}\}, m = 2k, k \geq 2,$$

$$S_5 = \{a, bc^{k+1}, bc^{k-1}, c^k, c, c^{-1}\}, m = 2k, k \geq 3,$$

$$S_6 = \{a, bc^{k+1}, bc^{k-1}, ac^k, c, c^{-1}\}, m = 2k, k \geq 3,$$

$$S_7 = \{a, b, c, c^{-1}, ac, ac^{-1}\}, m = 2k - 1, k \geq 2.$$

$$(45): G = Z_{4m} = \langle a \rangle \ (m \geq 2),$$

$$S = \{a, a^{-1}, a^m, a^{-m}, a^{2m+1}, a^{2m-1}\}.$$

$$(46): G = Z_{2m} = \langle a \rangle \ (m \geq 4),$$

$$S = \{a, a^{-1}, a^{m+1}, a^{m-1}, a^k, a^{-k}\} \ (2 \leq k \leq m - 2),$$

$$(m, k) = 1, \text{ if } l > 2 \text{ or } l = 2 \text{ for } m = 4i + 2; (k = 2i, \text{ with } i$$

$$\text{odd or } k = 2i + 2, \text{ with } i \text{ even}).$$

$$(47): G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle \ (m \geq 5),$$

$$S_1 = \{ab, ab^{-1}, b, b^{-1}, b^j, b^{-j}\} \ (2 \leq j < \frac{m}{2}), (m, j) = p >$$

$$2; m = (t + 1)p,$$

$S_2 = \{ab, ab^{-1}, b, b^{-1}, ab^j, ab^{-j}\}, (2 \leq j < \frac{m}{2}), (m, j) = p > 2; m = (t+1)p.$

(48):  $G = Z_2 \times Z_8 = \langle a \rangle \times \langle b \rangle,$   
 $S_1 = \{ab, ab^{-1}, b, b^{-1}, b^3, b^{-3}\},$   
 $S_2 = \{ab, ab^{-1}, b, b^{-1}, ab^3, ab^{-3}\}.$

(49):  $G = Z_{2m} \times Z_n = \langle a \rangle \times \langle b \rangle (m \geq 2, n \geq 3),$   
 $S = \{a, a^{-1}, a^m b, a^m b^{-1}, b, b^{-1}\}.$

(50):  $G = Z_{2m} \times Z_{2n} = \langle a \rangle \times \langle b \rangle (m \geq 3, n \geq 2),$   
 $S = \{a, a^{-1}, a^{m+1} b^n, a^{m-1} b^n, b, b^{-1}\}.$

(51):  $G = Z_{6m} = \langle a \rangle (m \geq 2), S_1 = \{a, a^{-1}, a^3, a^{-3}, a^{3m+1}, a^{3m-1}\},$   
 $S_2 = \{a, a^{-1}, a^{3m+1}, a^{3m-1}, a^{3m+3}, a^{3m-3}\}.$

(52):  $G = Z_m = \langle a \rangle (m = 7, 14), S = \{a, a^{-1}, a^3, a^{-3}, a^5, a^{-5}\}.$

(53):  $G = Z_{3m} = \langle a \rangle (m \geq 3),$   
 $S = \{a, a^{-1}, a^{m-1}, a^{m+1}, a^{2m-1}, a^{2m+1}\}.$

(54):  $G = Z_{16m-4} = \langle a \rangle (m \geq 1),$   
 $S = \{a, a^{-1}, a^{4m-2}, a^{12m-2}, a^{8m-3}, a^{8m-1}\}.$

(55):  $G = Z_{16m+4} = \langle a \rangle (m \geq 1),$   
 $S = \{a, a^{-1}, a^{4m+2}, a^{12m+2}, a^{8m+1}, a^{8m+3}\}.$

(56):  $G = Z_3 \times Z_3 = \langle a \rangle \times \langle b \rangle,$   
 $S = \{a, a^2, b, b^2, a^2 b, ab^2\}.$

(57):  $G = Z_2 \times Z_4 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle,$   
 $S = \{a, b, b^{-1}, c, c^{-1}, ab^2 c^2\}.$

## 2. Primary Analysis

**Proposition 2.1** [9, Proposition 1.5] Let  $\Gamma = \text{Cay}(G, S)$  be a Cayley graph of  $G$  over  $S$ , and  $A = \text{Aut}(\Gamma)$ . Let  $A_1$  be the stabilizer of the identity element 1 in  $A$ . Then  $\Gamma$  is normal if and only if every element of  $A_1$  is an automorphism of  $G$ .

**Proposition 2.2** [6, Theorem 1.1] Let  $G$  be a finite abelian group and  $S$  be a generating subset of  $G - 1_G$ . Assume  $S$  satisfies the condition that, if  $s, t, u, v \in S$  with  $1 \neq st = uv$ , implies  $\{s, t\} = \{u, v\}$ . Then the Cayley graph  $\text{Cay}(G, S)$  is normal.

Let  $X$  and  $Y$  be two graphs. The direct product  $X \times Y$  is defined as the graph with vertex set  $V(X \times Y) = V(X) \times V(Y)$  such that for any two vertices  $u = [x_1, y_1]$  and  $v = [x_2, y_2]$  in  $V(X \times Y)$ ,  $[u, v]$  is an edge in  $X \times Y$ , whenever  $x_1 = x_2$  and  $[y_1, y_2] \in E(Y)$  or  $y_1 = y_2$  and  $[x_1, x_2] \in E(X)$ . Two graphs are called relatively prime if they have no nontrivial common direct factor. The lexicographic product  $X[Y]$  is defined as the graph vertex set  $V(X[Y]) = V(X) \times V(Y)$  such that for any two vertices  $u = [x_1, y_1]$  and  $v = [x_2, y_2]$  in  $V(X[Y])$ ,  $[u, v]$  is an edge in  $X[Y]$  whenever  $[x_1, x_2] \in E(X)$  or  $x_1 = x_2$  and  $[y_1, y_2] \in E(Y)$ .

Let  $V(Y) = \{y_1, y_2, \dots, y_n\}$ . Then there is a natural embedding  $nX$  in  $X[Y]$ , where for  $1 \leq i \leq n$ , the  $i$ th copy of  $X$  is the subgraph induced on the vertex subset  $\{(x, y_i) | x \in V(X)\}$  in  $X[Y]$ . The deleted lexicographic product  $X[Y] - nX$  is the graph obtained by deleting all the edges of (this natural embedding of)  $nX$  from  $X[Y]$ . Let  $\Gamma$  be a graph and  $\alpha$  a permutation  $V(\Gamma)$  and  $C_n$  a circuit of length  $n$ . The twisted product  $\Gamma \times_\alpha C_n$  of  $\Gamma$  by  $C_n$  with respect to  $\alpha$  is defined by;

$V(\Gamma \times_\alpha C_n) = V(\Gamma) \times V(C_n) = \{(x, i) | x \in V(\Gamma), i = 0, 1, \dots, n-1\},$

$E(\Gamma \times_\alpha C_n) = \{[(x, i), (x, i+1)] | x \in V(\Gamma), i = 0, 1, \dots, n-2\} \cup \{[(x, n-1), (x^\alpha, 0)] | x \in V(\Gamma)\} \cup \{[(x, i), (y, i)] | [x, y] \in E(\Gamma), i = 0, 1, \dots, n-1\}.$

The graph  $Q_4^d$  denotes the graph obtained by connecting all long diagonals of 4-cube  $Q_4$ , that is, connecting all vertices  $u$  and  $v$  in  $Q_4$  such that  $d(u, v) = 4$ . The graph  $K_{m,m} \times_c C_n$  is the twisted product of  $K_{m,m}$  by  $C_n$  such that  $c$  is a cycle permutation on each part of the complete bipartite graph  $K_{m,m}$ . The graph  $Q_3 \times_d C_n$  is the twisted product of  $Q_3$  by  $C_n$  such that  $d$  transposes each pair of elements on long diagonals of  $Q_3$ . The graph  $C_{2m}^d[2K_1]$  is defined by:

$V(C_{2m}^d[2K_1]) = V(C_{2m}[2K_1]),$

$E(C_{2m}^d[2K_1]) = E(C_{2m}[2K_1]) \cup \{[(x_i, y_j), (x_{i+m}, y_j)] | i = 0, 1, \dots, m-1, j = 1, 2\},$  where  $V(C_{2m}) = \{x_0, x_1, \dots, x_{2m-1}\}$  and  $V(2K_1) = \{y_1, y_2\}.$

Let  $G = G_1 \times G_2$  be the direct product of two finite groups  $G_1$  and  $G_2$ , let  $S_1$  and  $S_2$  be subsets of  $G_1$  and  $G_2$ , respectively, and let  $S = S_1 \cup S_2$  be the disjoint union of two subsets  $S_1$  and  $S_2$ . Then we have,

### Lemma 2.3

(1)  $\text{Cay}(G, S) \cong \text{Cay}(G_1, S_1) \times \text{Cay}(G_2, S_2).$

(2) If  $\text{Cay}(G, S)$  is normal, then  $\text{Cay}(G_1, S_1)$  is also normal.

(3) If both of  $\text{Cay}(G_1, S_1)$  and  $\text{Cay}(G_2, S_2)$  are normal and relatively prime, then  $\text{Cay}(G, S)$  is normal.

## 3. Proof of the Main Theorem

In this section,  $\Gamma$  always denotes the Cayley graph  $\text{Cay}(G, S)$  of an abelian group  $G$  on  $S$  with valency 6. Let  $A = \text{Aut}(\Gamma)$ . Then  $A_1$  and  $A_1^*$  denote the stabilizer of 1 in  $A$  and the subgroup of  $A$  which fixes  $\{1\} \cup S$ , pointwise, respectively. In order to prove Theorem 1.1 we need several lemmas.

**Lemma 3.1** Let  $G = Z_{2m} = \langle a \rangle, (m \geq 5)$ , and  $S = \{a^i, a^{-i}, a^{m+i}, a^{m-i}, a, a^{-1}\}, 2 \leq i < \frac{m}{2}$ . Then  $\Gamma = \text{Cay}(G, S)$  is normal.

**Proof** Let  $\Gamma_2(1)$  be the subgraph of  $\Gamma$  with vertex set  $\{1\} \cup S \cup S^2$  and edge set  $\{[1, s], [s, st] \mid s, t \in S\}$ . By observing the subgraph  $\Gamma_2(1)$ , it is easy to prove that  $A_1^*$  fixes  $S^2$  pointwise, which implies that  $A_1^* = 1$ . Thus  $A_1$  acts faithfully on  $S$ . Observing the subgraph  $\Gamma_2(1)$  again,  $A_1$ , as a permutation group on  $S$ , is generated by  $(a, a^{-1})(a^{m+1}, a^{m-1})$ . So  $|A_1| = 2$  and  $\Gamma = \text{Cay}(G, S)$  is normal.

**Lemma 3.2:** Let  $G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ,  $m = 4k$ ,  $k \geq 2$  and  $S = \{a, b, c^k, c^{3k}, c, c^{-1}\}$ . Then  $\Gamma = \text{Cay}(G, S)$  is normal.

**Proof** Set  $G_1 = \langle a, b \rangle$ ,  $G_2 = \langle c \rangle$ ,  $S_1 = \{a, b\}$ ,  $S_2 = \{c^k, c^{3k}, c, c^{-1}\}$ . Then  $\Gamma_1 = \text{Cay}(G_1, S_1) \cong K_2 \times K_2$ . Note that  $\Gamma_1$  and  $\Gamma_2 = \text{Cay}(G_2, S_2)$  are relatively prime. By [5, Theorem 1.1] and [6, Theorem 1.2],  $\Gamma_1$  and  $\Gamma_2$  are normal and by Lemma 2.3,  $\Gamma = \text{Cay}(G, S)$  is normal.

With similar arguments as in Lemmas 3.1 and 3.2, we have the following lemma.

**Lemma 3.3** Let  $G$  and  $S$  be as the following. Then the Cayley graphs  $\Gamma = \text{Cay}(G, S)$  are normal.

$$(1): G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle, \\ S = \{a, b, c, d, ad, abc\}.$$

$$(2): G = Z_2^2 \times Z_6 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, \\ S = \{a, b, ac^3, c^3, c, c^{-1}\}.$$

$$(3): G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 2), \\ S = \{a, b, abc^m, c^m, c, c^{-1}\}.$$

$$(4): G = Z_2 \times Z_{6m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2), \\ S = \{a, ab^m, ab^{3m}, ab^{5m}, b, b^{-1}\}.$$

$$(5): G = Z_2 \times Z_{6m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2), \\ S = \{a, b^m, b^{3m}, b^{5m}, b, b^{-1}\}.$$

$$(6): G = Z_6 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 3), \\ S = \{a, a^{-1}, a^3, a^3b^m, b, b^{-1}\}.$$

$$(7): G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle, \\ S_1 = \{a, ab^2, ab^{-2}, b^m, b, b^{-1}\}, \quad (m \geq 4), \\ S_2 = \{a, ab^{m+2}, ab^{m-2}, ab^m, b, b^{-1}\}, \quad (m \geq 5), \\ S_3 = \{a, b^{m+2}, b^{m-2}, b^m, b, b^{-1}\}, \quad (m = 4, m \geq 6).$$

$$(8): G = Z_2 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle, \\ S_1 = \{a, ab^m, ab^{3m+2}, b^{2m+1}, b, b^{-1}\}, \\ (m \geq 2), S_2 = \{a, b, b^{-1}, b^m, b^{3m+2}, b^{2m+1}\}, \quad (m \geq 2), \\ S_3 = \{a, ab^{m+1}, ab^{3m+1}, b^{2m+1}, b, b^{-1}\}, \quad (m \geq 2), \\ S_4 = \{a, b, b^{-1}, b^{m+1}, b^{3m+1}, b^{2m+1}\}, \quad (m \geq 3).$$

$$(9): G = Z_4 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle \quad (m \geq 1), \\ S_1 = \{a^2b^{2m+1}, b^{2m+1}, ab^m, a^3b^{3m+2}, b, b^{-1}\}, \\ S_2 = \{a^2, a^2b^{2m+1}, ab^m, a^3b^{3m+2}, b, b^{-1}\}.$$

$$(10): G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, \quad (m \geq 4, m \neq 6), \\ S = \{a, b, c, c^{-1}, ac^2, ac^{-2}\}.$$

$$(11): G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2),$$

$$S_1 = \{a, ab^m, ab^{3m}, b^{2m}, b, b^{-1}\}, \\ S_2 = \{a, b, b^{-1}, b^m, b^{3m}, b^{2m}\}, \\ S_3 = \{a, ab^{2m}, b^m, b^{3m}, b, b^{-1}\}.$$

$$(12): G = Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 3), \\ S = \{a^2, a^2b^m, a, a^{-1}, b, b^{-1}\}.$$

$$(13): G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 2), \\ S_1 = \{a, b, abc^m, abc^{3m}, c, c^{-1}\}, S_2 = \{a, b, ac^m, ac^{3m}, c, c^{-1}\}, \\ S_3 = \{a, b, c^m, c^{3m}, c, c^{-1}\}, \\ S_4 = \{a, c^{2m}, bc^m, bc^{3m}, c, c^{-1}\}.$$

$$(14): G = Z_2^3 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 2), \\ S = \{a, b, cd^m, cd^{3m}, d, d^{-1}\}.$$

$$(15): G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m = 7, 9, m \geq 11), \\ S = \{a, b, c, c^{-1}, c^3, c^{-3}\}.$$

$$(16): G = Z_2 \times Z_4 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 1), \\ S = \{a, b^2c^{2m+1}, bc^m, b^3c^{3m+2}, c, c^{-1}\}.$$

$$(17): G = Z_2^2 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 2), \\ S = \{a, c^{2m+1}, bc^m, bc^{3m+2}, c, c^{-1}\}.$$

$$(18): G = Z_2 \times Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 3), \\ S = \{a, ac^m, b, b^{-1}, c, c^{-1}\}.$$

$$(19): G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 6), \\ S_1 = \{a, b^m, b, b^{-1}, b^3, b^{-3}\}, \\ S_2 = \{a, ab^m, b, b^{-1}, b^3, b^{-3}\}.$$

$$(20): G = Z_{4m} \times Z_n = \langle a \rangle \times \langle b \rangle \quad (m \geq 2, n \geq 3), \\ S = \{a, a^{-1}, a^m, a^{3m}, b, b^{-1}\}.$$

$$(21): G = Z_{4m} \times Z_{4n} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m, n \neq 4), \\ S = \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}.$$

$$(22): G = Z_4 \times Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m, n \neq 3), \\ S = \{a, a^3, b, b^{-1}, c, c^{-1}\}.$$

$$(23): G = Z_{2m} \quad (m \geq 5), \\ S = \{a, a^{-1}, a^j, a^{-j}, a^{m+j}, a^{m-j}\} \quad (2 \leq j < \frac{m}{2}).$$

$$(24): G = Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \quad (m = 7, 9, m \geq 11, n \geq 3), \\ S = \{a, a^{-1}, a^3, a^{-3}, b, b^{-1}\}.$$

$$(25): G = Z_{3m-1} \times Z_{3n} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2, n \geq 1), \\ S = \{a, a^{-1}, b, b^{-1}, a^mb^n, a^{2m-1}b^{2n}\}.$$

$$(26): G = Z_{3m+1} \times Z_{3n} = \langle a \rangle \times \langle b \rangle \quad (m, n \geq 1), \\ S = \{a, a^{-1}, b, b^{-1}, a^mb^{2n}, a^{2m+1}b^n\}.$$

$$(27): G = Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \quad (m \geq 5, n \geq 3), \\ S = \{a, a^{-1}, b, b^{-1}, a^2b, a^{-2}b^{-1}\}.$$

$$(28): G = Z_{2m+1} \times Z_n = \langle a \rangle \times \langle b \rangle \quad (m, n \geq 3), \\ S = \{a, a^{-1}, a^m, a^{m+1}, b, b^{-1}\}.$$

$$(29): G = Z_{2m+1} \times Z_{2n+1} = \langle a \rangle \times \langle b \rangle \quad (m, n \geq 2), \\ S = \{a, a^{-1}, b, b^{-1}, a^mb^{n+1}, a^{m+1}b^n\}.$$

$$(30): G = Z_2 \times Z_{2n+1} \times Z_{2m+1} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m, n \geq 1),$$

$$S = \{ab^m c^{n+1}, ab^{m+1} c^n, b, b^{-1}, c, c^{-1}\}.$$

$$(31): G = Z_{4m} = \langle a \rangle \quad (m \geq 2),$$

$$S = \{a, a^{-1}, a^k, a^{-k}, a^m, a^{-m}\}, \quad (1 < k < 2m, k \neq m, 2m-1).$$

$$(32): G = Z_4 \times Z_m = \langle a \rangle \times \langle b \rangle \quad (m \geq 3),$$

$$S = \{a, a^{-1}, b, b^{-1}, ab^j, a^{-1}b^{-j}\}, \quad 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor,$$

(When  $m \neq 2k$  for every  $j$  or  $m = 2k, j \neq k$ ).

$$(33): G = Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2),$$

$$S = \{a, a^{-1}, b, b^{-1}, a^2 b^j, a^2 b^{-j}\} \quad 1 \leq j < m$$

(for every  $j \neq 1, m-1$ ).

$$(34): G = Z_4 \times Z_{2m-1} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2),$$

$$S = \{a, a^{-1}, b, b^{-1}, a^2 b^j, a^2 b^{-j}\} \quad (1 < j < \frac{2m-1}{2}).$$

$$(35): G = Z_4 \times Z_m = \langle a \rangle \times \langle b \rangle \quad (m \geq 5),$$

$$S = \{a, a^{-1}, b, b^{-1}, b^j, b^{-j}\} \quad (1 < j < \frac{m}{2}),$$

when  $m \neq 2k, 5$  or  $m = 2k$  ( $k \geq 3, k \neq 5$ ),  $j \neq k-1$  or  $m = 10, j \neq 3$ .

$$(36): G = Z_{2m} = \langle a \rangle \quad (m \geq 4),$$

$$S = \{a, a^{-1}, a^j, a^{-j}, a^{m+1}, a^{m-1}\} \quad (2 \leq j \leq m-2),$$

when  $(m, j) = 1$  or  $(m, j) = 2, m \neq 4i+2$  ( $i \geq 1$ ).

$$(37): G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle \quad (m \geq 5, m \neq 8),$$

$$S_1 = \{ab, ab^{-1}, b, b^{-1}, b^j, b^{-j}\},$$

$$S_2 = \{ab, ab^{-1}, b, b^{-1}, ab^j, ab^{-j}\} \quad (2 \leq j < \frac{m}{2}), \text{ when}$$

$$(m, j) = p \leq 2.$$

$$(38): G = Z_2 \times Z_8 = \langle a \rangle \times \langle b \rangle,$$

$$S_1 = \{ab, ab^7, b, b^7, b^2, b^6\},$$

$$S_2 = \{ab, ab^7, b, b^7, ab^2, ab^6\}.$$

$$(39): G = Z_m = \langle a \rangle \quad (m \geq 9, m \neq 14),$$

$$S = \{a, a^{-1}, a^3, a^{-3}, a^j, a^{-j}\} \quad j \neq 3, 2 \leq j < \frac{m}{2} \text{ when}$$

$$m \neq 6k, \forall j \text{ or } m = 6k, j \neq 3k-1.$$

$$(40): G = Z_{14} = \langle a \rangle,$$

$$S = \{a, a^{-1}, a^3, a^{-3}, a^j, a^{-j}\} \text{ for } j = 2, 4, 6.$$

$$(41): G = Z_m = \langle a \rangle \quad (m \geq 7),$$

$$S = \{a, a^{-1}, a^{3j}, a^{-3j}, a^j, a^{-j}\}, \quad (2 \leq j < \frac{m}{2}, 3j \not\equiv 0, 1,$$

$$m-1, j, m-j, \frac{m}{2} \pmod{m}), \text{ when } m \neq 7, 14, 6k$$

$$(k \geq 2) \text{ and } m = 7; j = 2 \text{ or } m = 14; j = 2, 3, 4, 6 \text{ or } m = 6k; j \neq 3k-1.$$

$$(42): G = Z_m = \langle a \rangle \quad (m \geq 8, m \neq 14),$$

$$S = \{a, a^{-1}, a^{2+j}, a^{-2-j}, a^j, a^{-j}\} \text{ (if } m = 2k \text{ then } 2 \leq j \leq \frac{m}{2}-3 \text{ and if } m = 2k+1 \text{ then } 2 \leq j \leq \frac{m}{2}-1).$$

$$\text{When } m \neq 3k \text{ for every } j \text{ and when } m = 3k, \text{ for } k \text{ odd; } j \neq k-1$$

$$\text{and for } k \text{ even; } j \neq k-1, 3\frac{k}{2}-3.$$

$$(43): G = Z_{14} = \langle a \rangle,$$

$$S = \{a, a^{-1}, a^{2+j}, a^{-2-j}, a^j, a^{-j}\} \text{ for } j = 2, 4.$$

$$(44): G = Z_2 \times Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 3),$$

$$S = \{a, ab^2 c^m, b, b^{-1}, c, c^{-1}\}.$$

Now we are in a position to prove Theorem 1.1. Immediately from Lemma 2.3, [5, Theorem 1.1] and [6, Theorem 1.2], we have the Cases (1)-(32) of Theorem 1.1. Assume that  $\Gamma$  is not normal. In view of Proposition 2.2, we have the following assumption:  $\exists s, t, u, v \in S$  such that  $st = ub \neq 1$  but  $\{s, t\} \neq \{u, v\}$ . (\*).

We divide  $S$  into four cases:

**Case 1:**  $S = \{a, b, c, d, e, f\}$ , where  $a, b, c, d, e, f$  are involutions. In this case  $G$  is an elementary abelian 2-group and  $a, b, c, d, e, f$  are not independent by the assumption (\*). Consequently  $G = Z_2^3$  or  $G = Z_2^4$  or  $G = Z_2^5$ .

If  $G = Z_2^3 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$  by the assumption (\*) we can let  $S = \{a, b, c, ab, ac, abc\}$ . We have  $\sigma = (a, abc) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; and by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (33) of Theorem 1.1. If  $G = Z_4^2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$  by the assumption (\*) we see that  $S$  is one of the following cases:

(i)  $S_1 = \{a, b, c, d, abc, abd\}$ , (ii)  $S_2 = \{a, b, c, d, ab, abc\}$ ,

(iii)  $S_3 = \{a, b, c, d, abc, abc\}$ .

When  $S = S_1$ ,  $\sigma = (a, b) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (34) of Theorem 1.1. When  $S = S_2$ , we have the Case (22) of the main theorem. Also when  $S = S_3$ ,  $\Gamma$  is normal by Lemma 3.3. If  $G = Z_2^5 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$  we can let  $S = \{a, b, c, d, e, abc\}$  and hence  $\Gamma = \text{Cay}(G, S)$  is non-normal, the Case (1) of Theorem 1.1.

**Case 2:**  $S = \{a, b, c, d, e, e^{-1}\}$ , where  $a, b, c, d$  are involutions but  $e$  is not. In this case,  $S^2 - 1 = \{ab, ac, ad, ae, ae^{-1}, bc, bd, be, be^{-1}, cd, ce, ce^{-1}, de, de^{-1}, e^2, e^{-2}\}$ . By the assumption (\*)  $d = abc$ ,  $o(e) = 4$  or  $d = e^3$ .

Suppose  $d = abc$ . Then  $G = Z_2^2 \times Z_{2m}$ , ( $m \geq 2$ ) or

$$G = Z_2^3 \times Z_m, \quad (m \geq 3).$$

If  $G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ , ( $m \geq 2$ ), we can let

$$S = \{a, b, ac^m, bc^m, c, c^{-1}\} \text{ or}$$

$$S = \{a, b, c^m, abc^m, c, c^{-1}\}.$$

$$\text{When } S = \{a, b, ac^m, bc^m, c, c^{-1}\},$$

$\sigma = (ab, abc^m)(abc, abc^{m+1}) \dots (abc^{m-1}, abc^{2m-1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (35) of the main theorem.

When  $S = \{a, b, c^m, abc^m, c, c^{-1}\}$ ,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(3). If  $G = Z_2^3 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ , ( $m \geq 3$ ),  $S = \{a, b, c, abc, d, d^{-1}\}$ , the Case (2) of Theorem 1.1. Suppose  $o(e) = 4$ . Then  $G =$

$Z_2^2 \times Z_4$ ,  $Z_2^3 \times Z_4$  or  $Z_2^4 \times Z_4$ . If  $G = Z_2^2 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ , we have  $S$  is one of the following cases:  
 $S_1 = \{a, b, ab, ac^2, c, c^{-1}\}$ ,  $S_2 = \{a, b, ae^2, bc^2, c, c^{-1}\}$ ,  
 $S_3 = \{a, b, ac^2, abc^2, c, c^{-1}\}$ ,  
 $S_4 = \{a, b, ab, c^2, c, c^{-1}\}$ ,  
 $S_5 = \{a, b, ac^2, c^2, c, c^{-1}\}$ ,  
 $S_6 = \{a, b, abc^2, c^2, c, c^{-1}\}$ .

When  $S = S_1$ ,  $\sigma = (ac^2, c)(ac, c^2)(bc, abc^2)(abc, bc^2) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (36 –  $S_1$ ) of Theorem 1.1. When  $S = S_2$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (35,  $m = 2$ ) of Theorem 1.1. When  $S = S_3$ ,  $\sigma = (a, c)(ab, bc)(c^2, ac^3)(bc^3, abc^3) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.4,  $\Gamma = \text{Cay}(G, S)$  is not normal the Case (36 –  $S_2$ ) of Theorem 1.1. When  $S = S_4$ , we have the Case (3) of Theorem 1.1. When  $S = S_5$ , we have the Case (23) of Theorem 1.1. When  $S = S_6$ ,  $\Gamma$  is normal by Lemma 3.3 (3,  $m=2$ ). If  $G = Z_2^3 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ , we have  $S = \{a, b, c, d, d^{-1}, u\}$ , where  $u = d^2, ab, ad^2, abc, abd^2$  or  $abcd^2$ . When  $u = d^2$ , we have the Case (5 –  $S_1$ ) of Theorem 1.1. When  $u = ab$ , we have the Case (5 –  $S_2$ ) of Theorem 1.1. When  $u = ad^2$ , we have the Case (5 –  $S_3$ ) of Theorem 1.1. When  $u = abc$ , we have the Case (2) of Theorem 1.1. When  $u = abd^2$ , we have the Case (24) of Theorem 1.1. When  $u = abcd^2$ ,  $\sigma = (abcd^2, d)(bcd^2, ad)(acd^2, bd)(abd^2, cd)(abcd, d^2)(cd^2, abd)(bd^2, acd)$  and  $(bcd, ad^2) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (37) of Theorem 1.1. If  $G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$ ,  $S = \{a, b, c, d, e, e^{-1}\}$ , we have the Case (4) of Theorem 1.1. Now suppose  $d = e^3$ . Then  $G = Z_2^2 \times Z_6$  or  $G = Z_2^3 \times Z_6$ . If  $G = Z_2^2 \times Z_6 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ , we see that  $S$  is one of the following cases:  $S_1 = \{a, b, ab, c^3, c, c^{-1}\}$ ,  $S_2 = \{a, b, ac^3, c^3, c, c^{-1}\}$ ,  $S_3 = \{a, b, abc^3, c^3, c, c^{-1}\}$ . When  $S = S_1$ , we have the Case (6) of Theorem 1.1. For  $S_2$  and  $S_3$ , we have the Cases (2) and (3,  $m = 3$ ) of Lemma 3.3 respectively. If  $G = Z_2^3 \times Z_6 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ , then  $S = \{a, b, c, d^3, d, d^{-1}\}$ , the Case (7) of Theorem 1.1.

**Case 3:**  $S = \{a, b, c, c^{-1}, d, d^{-1}\}$ , where  $a, b$  are involutions but  $c, d$  are not. By the assumption (\*) and the symmetry of  $c, c^{-1}, d$  and  $d^{-1}$ , we have five sub cases (I)  $a = c^3$ , (II)  $a = c^2d$ , (III)  $o(c) = 4$ , (IV)  $c^3 = d$  and (V)  $c^2 = d^2$ . Suppose  $a = c^3$ , then  $G$  is isomorphic to one of the following:  $Z_2 \times Z_{6m}$  ( $m \geq 2$ ),  $Z_2 \times Z_6$ ,  $Z_6 \times Z_{2m}$  ( $m \geq 2$ ),  $Z_2^2 \times Z_{3m}$  ( $m \geq 1$ ),  $Z_2 \times Z_6 \times Z_m$  ( $m \geq 3$ ). If  $Z_2 \times Z_{6m} = \langle a \rangle \times \langle b \rangle$ , ( $m \geq 2$ ), we see that  $S$  is one of the following cases:  
 $S_1 = \{a, b^{3m}, ab^{2m}, ab^{4m}, b, b^{-1}\}$ ,  $S_2 = \{a, ab^{3m}, ab^m, ab^{5m}, b, b^{-1}\}$ ,  $S_3 = \{a, b^{3m}, b^m, b^{5m}, b, b^{-1}\}$ . When  $S = S_1$ ,  $\sigma = (a, ab^{2m}, ab^{4m})(ab, ab^{2m+1}, ab^{4m+1}) \dots (ab^{2m-1}, ab^{4m-1}, ab^{6m-1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by

Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (38) of the main theorem. For the Cases  $S = S_2$  and  $S = S_3$ , we have the Cases (4) and (5) of Lemma 3.3. If  $G = Z_2 \times Z_6 = \langle a \rangle \times \langle b \rangle$ , we see that  $S$  is one of the following cases:

$S_1 = \{a, b^3, ab^2, ab^4, b, b^{-1}\}$ ,  $S_2 = \{a, b^3, b, b^{-1}, b^2, b^4\}$ ,  
 $S_3 = \{a, b^3, b, b^{-1}, ab, ab^{-1}\}$ .  
When  $S = S_1$ ,  $\sigma = (a, ab^2, ab^4)(ab, ab^3, ab^5) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (43 –  $S_5$ ) of Theorem 1.1. When  $S = S_2$ , we have the Case (29,  $m=3$ ) of Theorem 1.1. When  $S = S_3$ ,  $\sigma = (b^5, ab^5)(b^2, ab^2) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (43 –  $S_1$ ) of Theorem 1.1. If  $G = Z_6 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$ , we see that  $S$  is one of the following cases:

$S_1 = \{a^3, b^m, a, a^{-1}, b, b^{-1}\}$ ,  $S_2 = \{a^3, a^3b^m, a, a^{-1}, b, b^{-1}\}$ .  
When  $S = S_1$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (8) of Theorem 1.1.  
For  $S = S_2$ , when  $m = 2$ ,  $\sigma = (b^2, a^3b)(ab^2, a^4b)(a^2b^2, a^5b)(a^3b^2, b)(a^4b^2, ab)(a^5b^2, a^2b) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ;  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (40,  $m=3$ ) of Theorem 1.1, and when  $m \geq 3$ ,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(6). If  $G = Z_2^2 \times Z_{3m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$  ( $m \geq 1$ ),  $S = \{a, b, ac^m, ac^{2m}, c, c^{-1}\}$ . Then we obtain the Case (25) of Theorem 1.1. If  $G = Z_2 \times Z_6 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$  ( $m \geq 3$ ),  $S = \{b^3, a, b, b^{-1}, c, c^{-1}\}$ . Then we obtain the Case (9) of Theorem 1.1. Suppose  $a = c^2d$ . Then we have one of the following cases:  
(1):  $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$  ( $m \geq 3$ ),  
 $S = \{a, b^m, b, b^{-1}, ab^{-2}, ab^2\}$ .

(2):  $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$ ,  
 $S_1 = \{ab^m, a, b, b^{-1}, ab^{m-2}, ab^{m+2}\}$  ( $m \geq 3$ ),  
 $S_2 = \{b^m, a, b, b^{-1}, b^{m-2}, b^{m+2}\}$ ,  $m \geq 4$ ,

(3):  $G = Z_2 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle$ ,  
 $S_1 = \{a, b, b^{-1}, b^{2m+1}, ab^m, ab^{3m+2}\}$  ( $m \geq 1$ ),  
 $S_2 = \{a, b, b^{-1}, b^{2m+1}, b^m, b^{3m+2}\}$ ,  $m \geq 2$   
 $S_3 = \{a, b, b^{-1}, b^{2m+1}, b^{3m+1}, b^{m+1}\}$  ( $m \geq 1$ ),  
 $S_4 = \{a, b^{2m+1}, ab^{3m+1}, ab^{m+1}, b, b^{-1}\}$ ,  $m \geq 1$ ,

(4):  $G = Z_4 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle$ ,  
 $S_1 = \{a^2b^{2m+1}, b^{2m+1}, ab^m, a^3b^{3m+2}, b, b^{-1}\}$ ,  $m \geq 1$   
 $S_2 = \{a^2b^{2m+1}, a^2, ab^m, a^3b^{3m+2}, b, b^{-1}\}$ ,  $m \geq 1$ .

(5):  $G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$  ( $m \geq 3$ ),  
 $S = \{a, b, c, c^{-1}, ac^{-2}, ac^2\}$ .

(6):  $G = Z_2 \times Z_4 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$  ( $m \geq 1$ ),  
 $S = \{a, b^2c^{2m+1}, bc^m, b^{-1}c^{-m}, c, c^{-1}\}$ .

(7):  $G = Z_2^2 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$  ( $m \geq 1$ ),  
 $S = \{a, c^{2m+1}, bc^m, bc^{-m}, c, c^{-1}\}$ .

In the Case (1), when  $m = 3$ ,  $\sigma = (b^2, b^4) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not

normal, the Case (43–  $S_5$ ,  $m = 3$ ) of Theorem 1.1. When  $m \geq 4$ ,  $\Gamma$  is normal by Lemma 3.3(7–  $S_1$ ). In the Case (2),  $S = S_1$  when  $m = 3$ ,  $\sigma = (b^2, ab^2)(b^5, ab^5) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (43–  $S_2$ ) of Theorem 1.1.

When  $m = 4$ ,  $\sigma = (b, b^7)(b^2, b^6)(b^3, b^7) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case 39 ( $m = 2$ ) of Theorem 1.1. When  $m \geq 5$ ,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3 (7–  $S_2$ ). In the Case (2),  $S = S_2$ , when  $m = 5$ , we have the Case (26) of Theorem 1.1. When  $m \geq 6$ ,  $\Gamma$  is normal by Lemma 3.3 (7–  $S_3$ ).

In the Case (3),  $S = S_1$ , when  $m = 1$ , we have the Case (43 –  $S_1$ ) of Theorem 1.1. When  $m \geq 2$ ,  $\Gamma$  is normal by Lemma 3.3 (8 –  $S_1$ ). In the Case (3),  $S = S_2$ ,  $\Gamma$  is normal by Lemma 3.3 (8 –  $S_2$ ). In the Case (3),  $S = S_3$ , when  $m = 1, 2$ , we have the Cases (29,  $m = 3, 5$ ) of Theorem 1.1 respectively. When  $m \geq 3$ ,  $\Gamma$  is normal by Lemma 3.3(8 –  $S_4$ ). In the Case (3),  $S = S_4$ , when  $m = 1$ ,  $\sigma = (ab, ab^5) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (29,  $m = 3$ ) of Theorem 1.1. When  $m \geq 2$ ,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(8 –  $S_3$ ). In the Case (4),  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(9). In the Case (5), when  $m = 3, 6$ , by Proposition 2.1,  $\Gamma$  is not normal, the Case (25,  $m = 1, 2$ ) of Theorem 1.1. Otherwise  $\Gamma$  is normal by Lemma 3.3(10). In the Case (6),  $\Gamma$  is normal by Lemma 3.3(16). In the Case (7), when  $m = 1$ , by Proposition 2.1,  $\Gamma$  is not normal, the Case 27 ( $m = 1$ ) of Theorem 1.1. When  $m \geq 2$ ,  $\Gamma$  is normal by Lemma 3.3 (17). Suppose  $o(c) = 4$ . Then we have one of the following cases:

(I)  $G = Z_2 \times Z_4 = \langle a \rangle \times \langle b \rangle$ ,  $S_1 = \{a, b^2, b, b^{-1}, ab, ab^{-1}\}$ ,

(II)  $G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle$ ,  $S_1 = \{a, b^{2m}, ab^m, ab^{3m}, b, b^{-1}\}$ , ( $m \geq 2$ ),  $S_2 = \{a, ab^{2m}, ab^m, ab^{3m}, b, b^{-1}\}$ , ( $m \geq 1$ ),  $S_3 = \{a, b^{2m}, b^m, b^{3m}, b, b^{-1}\}$ , ( $m \geq 2$ ),  $S_4 = \{a, ab^{2m}, b^m, b^{3m}, b, b^{-1}\}$ , ( $m \geq 2$ ).

(III)  $G = Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$  ( $m \geq 2$ ),  $S_1 = \{a^2, b^m, a, a^{-1}, b, b^{-1}\}$ ,  $S_2 = \{a^2, a^2b^m, a, a^{-1}, b, b^{-1}\}$ ,  $S_3 = \{a^2b^m, b^m, a, a^{-1}, b, b^{-1}\}$ .

(IV):  $G = Z_2^2 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ,  $S_1 = \{a, b, c, c^{-1}, ac, ac^{-1}\}$ ,  $S_2 = \{a, b, c, c^{-1}, abc, abc^{-1}\}$ .

(V):  $G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$  ( $m \geq 2$ ),  $S_1 = \{a, b, abc^m, abc^{3m}, c, c^{-1}\}$ ,  $S_2 = \{a, b, ac^m, ac^{3m}, c, c^{-1}\}$ ,  $S_3 = \{a, b, c^m, c^{3m}, c, c^{-1}\}$ .

(VI):  $G = Z_2 \times Z_4 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$  ( $m \geq 3$ ),  $S_1 = \{a, b^2, b, b^{-1}, c, c^{-1}\}$ ,  $S_2 = \{a, ab^2, b, b^{-1}, c, c^{-1}\}$ .

(VII):  $G = Z_2 \times Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$  ( $m \geq 2$ ),  $S_1 = \{a, c^m, b, b^{-1}, c, c^{-1}\}$ ,  $S_2 = \{a, ac^m, b, b^{-1}, c, c^{-1}\}$ ,  $S_3 = \{a, b^2c^m, b, b^{-1}, c, c^{-1}\}$ ,  $S_4 = \{a, ab^2c^m, b, b^{-1}, c, c^{-1}\}$ .

(VIII):  $G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$  ( $m \geq 1$ ),

$S_1 = \{a, c^{2m}, bc^m, bc^{3m}, c, c^{-1}\}$ ,  
 $S_2 = \{a, ac^{2m}, bc^m, bc^{3m}, c, c^{-1}\}$ .

(IX):  $G = Z_2^2 \times Z_4 \times \langle c \rangle \times \langle d \rangle$  ( $m \geq 3$ ),  
 $S = \{a, b, c, c^{-1}, d, d^{-1}\}$ .

(X):  $G = Z_2^3 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$  ( $m \geq 1$ ),  
 $S = \{a, b, cd^m, cd^{3m}, d, d^{-1}\}$ .

In the Case (I),  $\sigma = (ab, b^3) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (43 –  $S_1$ ) of Theorem 1.1. In the Case (II),  $S = S_1$ ,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(11 –  $S_1$ ). In the Case (II),  $S = S_2$ ,  $\sigma = (b, b^{-1})(b^2, b^{-2}) \dots (b^{2m-1}, b^{2m+1})(a, ab^m) \dots (ab^{2m+1}, ab^{-(m+1)}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (39) of Theorem 1.1. In the Case (II),  $S = S_3$ , and  $S = S_4$ ,  $\Gamma$  is normal by Lemma 3.3, the Case (11 –  $S_2, S_3$ ). In the Case (III), when  $S = S_1$ , we have the Case (10) of Theorem 1.1. When  $S = S_2$ ,  $m = 2$ ,  $\sigma = (a^2b^2, b)(a^3b^2, ab)(ab^2, a^3b)(b^2, a^2b) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (40,  $m = 2$ ) of Theorem 1.1. When  $S = S_2$ ,  $m \geq 3$ ,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(12). When  $S = S_3$ ,  $\sigma = (a^2, ab^m)(a^2b, ab^{m+1}) \dots (a^2b^{2m-1}, ab^{m+(2m-1)}) \in A_1$  but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (40) of Theorem 1.1.

In the Case (IV), when  $S = S_1$ ,  $\sigma = (c^2, ac^2)(bc^2, abc^2) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (44–  $S_2$ ) of Theorem 1.1. When  $S = S_2$ ,  $\sigma = (ac^2, bc^2) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (44–  $S_3$ ) of Theorem 1.1. In the Case (V),  $S = S_1$ , when  $m = 1$ , with an argument similar to the Case (IV –  $S_2$ ) we obtain the same result. When  $m \geq 2$ ,  $\Gamma$  is normal by Lemma 3.3 (13–  $S_1$ ). In the Case (V),  $S = S_2$ , when  $m = 1$ , with an argument similar to the Case (IV– $S_1$ ), we obtain the same result.

When  $m \geq 2$ ,  $\Gamma$  is normal by Lemma 3.3 (13 –  $S_2$ ). In the Case (V),  $S = S_3$ ,  $\Gamma$  is normal by Lemma 3.3(13–  $S_3$ ). In the Case (VI), we have the Case (11) of Theorem 1.1. In the Case (VII),  $S = S_1$ ,  $S = S_3$  and  $S = S_2$  ( $m = 2$ ), we have the Cases (12), (28) and (11 –  $S_2$ ,  $m = 4$ ) of Theorem 1.1 respectively. In the Case (VII),  $S = S_2$ ,  $m \geq 3$ ,  $\Gamma$  is normal by Lemma 3.3(18). In the Case (VII),  $S = S_4$ , for  $m = 2$ ,  $\sigma = (b^3, c)(ab^3, ac)(abc^2, ab^2c^3)(b^2, bc)(b^3c^3, c^2)(b^2c, b^2c^3)(ab^2, abc)(ab^3c^3, ac^2) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (57) of Theorem 1.1, and for  $m \geq 3$ ,  $\Gamma$  is normal by Lemma 3.3(44). In the Case (VIII),  $S = S_1$  when  $m = 1$ , we have the Case (21,  $m = 2$ ) of Theorem 1.1. If  $m \geq 2$ ,  $\Gamma$  is normal by Lemma 3.3 (13 –  $S_4$ ). In the Case (VIII),  $S = S_2$ ,  $\sigma = (ab, abc^{2m})(abc, abc^{2m+1}) \dots (abc^{2m-1}, abc^{4m-1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (41) of Theorem 1.1. In the Case

(IX), we have the Case (13) of Theorem 1.1. In the Case (X),  $m = 1$ , we have the Case (14) of Theorem 1.1, and for  $m \geq 2$ ,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(14). Suppose  $c^3 = d$ , then  $G = Z_2^2 \times Z_{2m}$ , ( $m \geq 4$ ) or  $G = Z_2^2 \times Z_m$  ( $m \geq 5$ ,  $m \neq 6$ ). If  $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$  ( $m \geq 4$ ), we can let  $S$  to be  $S_1 = \{a, b^m, b, b^{-1}, b^3, b^{-3}\}$  or  $S_2 = \{a, ab^m, b, b^{-1}, b^3, b^{-3}\}$ . Let  $S = S_1$ , for  $m = 4, 5$  we have the Cases (29), (26) of Theorem 1.1 respectively, and for  $m \geq 6$ ,  $\Gamma$  is normal by Lemma 3.3(19 –  $S_1$ ). Let  $S = S_2$ . When  $m = 4$ ,  $\sigma = (ab^2, ab^6) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (43 –  $S_4$ ),  $m = 4$  of Theorem 1.1. When  $m = 5$ ,  $\sigma = (b^3, b^7)(ab^3, ab^7)(b^2, b^8)(ab^2, ab^8) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (42) of Theorem 1.1. When  $m \geq 6$ ,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(19 –  $S_2$ ). If  $G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$  ( $m \geq 5$ ,  $m \neq 6$ ),  $S = \{a, b, c, c^{-1}, c^3, c^{-3}\}$ . When  $m = 5, 10$  and  $m = 8$  we have the Cases (15), and (16) of Theorem 1.1 respectively. When  $m = 7, 9, m \geq 11$ ,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(15). Suppose  $c^2 = d^2$ , then  $G = Z_2 \times Z_{2m}$ ,  $G = Z_2^2 \times Z_{2m}$  ( $m \geq 3$ )  $G = Z_2^2 \times Z_{2m-1}$  ( $m \geq 2$ ) or  $G = Z_2^2 \times Z_m$  ( $m \geq 3$ ). If  $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$  we see that  $S$  is one of the following cases:

- 1)  $S_1 = \{a, b^m, b, b^{-1}, ab, ab^{-1}\}$ ,  $m \geq 2$ ,
- 2)  $S_2 = \{a, ab^m, b, b^{-1}, ab, ab^{-1}\}$ ,  $m \geq 2$ ,
- 3)  $S_3 = \{a, b^m, b, b^{-1}, b^{m+1}, b^{m-1}\}$ ,  $m \geq 3$ ,
- 4)  $S_4 = \{a, ab^m, b, b^{-1}, b^{m+1}, b^{m-1}\}$ ,  $m \geq 3$ ,
- 5)  $S_5 = \{a, b^m, b, b^{-1}, ab^{m+1}, ab^{m-1}\}$ ,  $m \geq 3$ ,
- 6)  $S_6 = \{a, ab^m, b, b^{-1}, ab^{m+1}, ab^{m-1}\}$ ,  $m \geq 3$ ,
- 7)  $S_7 = \{ab^m, b^m, b, b^{-1}, ab, ab^{-1}\}$ ,  $m \geq 2$ ,
- 8)  $S_8 = \{ab^m, b^m, b, b^{-1}, ab^{m+1}, ab^{m-1}\}$ ,  $m \geq 2$ .

In the Case (1),  $m \geq 2$ , when  $m = 2i$ ,  $\sigma = (b^i, ab^i)(b^{3i}, ab^{3i}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$  and when  $m = 2i + 1$ ,  $\sigma = (b^{i+1}, ab^{i+1})(b^{3i+2}, ab^{3i+2}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (43 –  $S_1$ ) of Theorem 1.1. In the Case (2), similarly Case (1),  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (43 –  $S_2$ ) of Theorem 1.1. In the Case (3), we have the Case (29) of Theorem 1.1. In the Case (4), when  $m = 2i$ ,  $\sigma = (ab^i, ab^{3i}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$  and when  $m = 2i + 1$ ,  $\sigma = (ab^{i+1}, ab^{3i+2}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (43 –  $S_4$ ) of Theorem 1.1. In the Case (5), when  $m = 2i$ ,  $\sigma = (b^{3i}, ab^i) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$  and when  $m = 2i + 1$ ,  $\sigma = (b^{i+1}, ab^{3i+2}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (43 –  $S_5$ ) of Theorem 1.1. In the Case (6), when  $m = 2i$ ,  $\sigma = (b^i, ab^{3i})(b^{3i}, ab^i) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$  and when  $m = 2i + 1$ ,  $\sigma = (b^{i+1}, ab^{3i+2})(b^{3i+2}, ab^{i+1}) \in A_1$ ,

but  $\sigma \notin \text{Aut}(G, S)$ . Hence by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (43 –  $S_6$ ) of Theorem 1.1.

In the Case (7), for  $m = 2i$  and  $m = 2i + 1$ ,  $\sigma = (b^{i+1}, ab^{i+1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (43 –  $S_3$ ) of Theorem 1.1. In the Case (8), for  $m = 2i$  and  $m = 2i - 1$ ,  $\sigma = (b^i, ab^{i+m})(b^{m+i}, ab^i) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (43 –  $S_1$ ) of Theorem 1.1. If  $G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ , we can let  $S$  to be one of the following cases:

- (1):  $S_1 = \{a, b, c, c^{-1}, ac, ac^{-1}\}$ ,  $m \geq 2$ ,
- (2):  $S_2 = \{a, b, c, c^{-1}, abc, abc^{-1}\}$ ,  $m \geq 2$ ,
- (3):  $S_3 = \{a, b, c, c^{-1}, c^{m+1}, c^{m-1}\}$ ,  $m \geq 3$ ,
- (4):  $S_4 = \{a, b, c, c^{-1}, ac^{m+1}, ac^{m-1}\}$ ,  $m \geq 2$ ,
- (5):  $S_5 = \{a, b, c, c^{-1}, abc^{m+1}, abc^{m-1}\}$ ,  $m \geq 2$ ,
- (6):  $S_6 = \{a, cm, c, c^{-1}, bc, bc^{-1}\}$ ,  $m \geq 2$ ,
- (7):  $S_7 = \{a, ac^m, c, c^{-1}, bc, bc^{-1}\}$ ,  $m \geq 2$ ,
- (8):  $S_8 = \{a, c^m, c, c^{-1}, bc^{m+1}, bc^{m-1}\}$ ,  $m \geq 2$ ,
- (9):  $S_9 = \{a, ac^m, c, c^{-1}, bc^{m+1}, bc^{m-1}\}$ ,  $m \geq 2$ .

In the Case (1),  $\Gamma$  is not normal, the Case (30) of Theorem 1.1. In the Case (2),  $\sigma = (ac^{m-1}, bc^{m-1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (44 –  $S_1$ ) of Theorem 1.1. In the Case (3), when  $m = 2i$ ,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (16) of Theorem 1.1.

When  $m = 2i + 1$ ,  $\Gamma = \text{Cay}(G, S)$  is not normal, we have the Case 14 (with  $m$  odd) of Theorem 1.1. In the Case (4), when  $m = 2i$ ,  $i \geq 2$ ,  $\sigma = (c^i, ac^{3i})(ac^i, c^{3i})(bc^i, abc^{3i})(abc^i, bc^{3i}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , and when  $m = 2i + 1$ ,  $\sigma = (c^{i+1}, ac^{3i+2})(ac^{i+1}, c^{3i+2})(bc^{i+1}, abc^{3i+2})(abc^{i+1}, bc^{3i+2}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (44 –  $S_2$ ) of Theorem 1.1. In the Case (5), when  $m = 2i$ ,  $i \geq 2$ ,  $\sigma = (c^{3i}, abc^i)(ac^{3i}, bc^i)(bc^{3i}, ac^i)(abc^{3i}, c^i) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$  and when  $m = 2i + 1$ ,  $\sigma = (c^{3i+2}, abc^{i+1})(ac^{3i+2}, bc^{i+1})(bc^{3i+2}, ac^{i+1})(abc^{3i+2}, c^{i+1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ ; by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (44 –  $S_3$ ) of Theorem 1.1. In the Case (6),  $m \geq 2$ ,  $\Gamma$  is not normal, we have the Case (27) of Theorem 1.1. In the Case (7), if  $m \geq 3$ , for  $m = 2i$  and  $m = 2i - 1$ ,  $\sigma = (ci, bci)(aci, abci)(ci+m, bci+m)(aci+m, abci+m) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , and if  $m = 2$ ,  $\sigma = (b, bc^2)(ab, abc^2) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ . Then by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (44 –  $S_4$ ) of Theorem 1.1. In the Case (8), for  $m = 2i$  and  $m = 2i - 1$ ,  $\sigma = (c^i, bc^{i+m})(ac^i, abc^{i+m})(c^{i+m}, bc^i)(ac^{i+m}, abc^i) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (44 –  $S_5$ ) of Theorem 1.1. In the Case (9), similarly Case (8),  $\Gamma = \text{Cay}(G, S)$  is not normal. We have the Case (44 –  $S_6$ ) of Theorem 1.1.

If  $G = Z_2^2 \times Z_{2m-1} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ , ( $m \geq 2$ ), then  $S$  is  $S_1 = \{a, b, c, c^{-1}, ac, ac^{-1}\}$  or  $S_2 = \{a, b, c, c^{-1}, abc, abc^{-1}\}$ . When  $S = S_1$ ,  $\sigma = (cm, acm)(bcm, abcm) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (44 –  $S_7$ ) of the main theorem.



When  $S = S_2$ ,  $\sigma = (ac^{m-1}, bc^{m-1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (44–  $S_1$ ) of Theorem 1.1. If  $G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ , we can consider  $m \geq 3$ ,  $S = \{a, b, d, d^{-1}, cd, cd^{-1}\}$ . In this case for  $m = 2i$  and  $m = 2i-1$ , ( $i \geq 2$ )  $\sigma = (d^i, cd^i)(ad^i, acd^i)(bd^i bcd^i)(abd^i, abcd^i) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$  and by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal the Case (14) of Theorem 1.1.

**Case 4:**  $S = \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$ , where the elements of the set  $S$  are not involution. By the assumption (\*),  $o(a) = 4$ ,  $a^2 = b^2$ ,  $a^3 = b$  or  $c = a^2b$ . Suppose  $o(a) = 4$ , then  $G$  is isomorphic to one of the following:  $Z_{4m}$  ( $m \geq 2$ ),  $Z_4 \times Z_m$ ,  $Z_{4m} \times Z_n$  ( $m \geq 2, n \geq 3$ ),  $Z_{4m} \times Z_{4n}$  ( $m \geq 1, n \geq 1$ ),  $Z_4 \times Z_m \times Z_n$  ( $m, n \geq 3$ ). If  $G = Z_{4m} = \langle a \rangle$  ( $m \geq 2$ ), we can let  $S = \{a^m, a^{-m}, a, a^{-1}, a^j, a^{-j}\}$ , where  $1 < j < 2m$ ,  $j \neq m$ . When  $j = 2m - 1$ ,  $\sigma = (a^m, a^{-m}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (45) of Theorem 1.1. When  $j \neq 2m - 1$ ,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(31). If  $G = Z_4 \times Z_m = \langle a \rangle \times \langle b \rangle$ , we can let  $S$  to be one of the following cases:

(1):  $S_1 = \{a, a^3, b, b^{-1}, ab^j, a^3b^{-j}\}$ ,  $m \geq 3$ ,  $1 \leq j \leq [m/2]$ ,

(2):  $S_2 = \{a, a^3, b, b^{-1}, a^2b^j, a^2b^{-j}\}$ ,  $m \geq 2$ ,  $1 \leq j \leq (m/2)$ ,

(3):  $S_3 = \{a, a^3, b, b^{-1}, b^j, b^{-j}\}$ ,  $m \geq 5$ ,  $1 < j < (m/2)$ .

When  $S = S_1$ , for  $m = 2j$ ,  $\sigma = (a^2, a^{2j})(a^{2j}b, a^{2j}b^{-1}) \dots (a^{2j}b^{j-1}, a^{2j}b^{2j-1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (49) of the main theorem. Otherwise,  $\Gamma$  is normal by Lemma 3.3(32). When  $S = S_2$ ,  $j = 1$  for  $m = 2k$  and  $m = 2k - 1$ ,  $k \geq 2$ ,  $\sigma = (ab^k, a^3b^k) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , and when  $j = k - 1$ ,  $m = 2k$  ( $k \geq 3$ ),  $\sigma = (b^{k-1}, a^{2j}b^{-1})(ab^{k-1}, a^3b^{-1})(a^{2j}b^{k-1}, b^{-1})(a^3b^{k-1}, ab^{-1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , then these graphs are non-normal and we have the Cases (49, 50) of Theorem 1.1. Otherwise,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3 (33, 34). When  $S = S_3$ , for  $j = k - 1$ ,  $m = 2k$ , if  $k$  is odd we have the Case (17) of Theorem 1.1 and if  $k$  is even we have the Case 19 ( $m = 4$ ) of the main theorem. For  $m = 5$ ;  $j = 2$  and  $m = 10$ ;  $j = 3$  we have the Case 21 ( $m = 4$ ) of the main theorem.

Otherwise,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3 (35). If  $G = Z_{4m} \times Z_n = \langle a \rangle \times \langle b \rangle$  ( $m \geq 2, n \geq 3$ ),  $S = \{a^m, a^{-m}, a, a^{-1}, b, b^{-1}\}$ , then  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(20). If  $G = Z_{4m} \times Z_{4n} = \langle a \rangle \times \langle b \rangle$  ( $m \geq 1, n \geq 1$ ),  $S = \{a^m b^n, a^{-m} b^{-n}, a, a^{-1}, b, b^{-1}\}$ , then  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(21). If  $G = Z_4 \times Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$  ( $m, n \geq 3$ ), we can consider  $S = \{a, a^3, b, b^{-1}, c, c^{-1}\}$ . In this case, for  $m = 4$ ,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (18) of Theorem 1.1, and for  $m, n \neq 4$ ,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(22). Suppose  $a^2 = b^2$ . Then  $G$  is isomorphic to one of the following:  $Z_{2m}$ ,  $Z_2 \times Z_m$  ( $m \geq 5$ ),  $Z_{2m} \times Z_{2n+1}$ ,  $Z_{2m} \times Z_{2n}$  ( $m \geq 3, n \geq 2$ ),  $Z_2 \times Z_n$  ( $m \geq 3, n \geq 3$ ). If  $G = Z_{2m} = \langle a \rangle$ , we can let  $S$  to be  $S_1 = \{a^j, a^{-j}, a^{m+j}, a^{m-j}, a, a^{-1}\}$ ,

$2 \leq j \leq m/2$ ,  $m \geq 5$ , or  $S_2 = \{a, a^{-1}, a^{m+1}, a^{m-1}, a^j, a^{-j}\}$ ,  $2 \leq j \leq m - 2$ ,  $m \geq 4$ . When  $S = S_1$ ,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(23). When  $S = S_2$ , ( $m, j$ ) = 2, for  $m = 4i + 2$ ,  $j = 2i$  (with  $i$  odd) and  $j = 2i + 2$  (with  $i$  even),  $\sigma = (a^2, a^{2+m/2})(a^6, a^{6+m/2}) \dots (a^{2m-2}, a^{m/2-2}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , and when ( $m, j$ ) = 1 > 2, then  $\sigma = (a^2, a^{m+2})(a^{2+1}, a^{m+2+1}) \dots (a^{m+2-1}, a^{2-1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , then by Proposition 2.1 these graphs are non-normal, and we have the Case (46) of the main theorem. Otherwise,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3 (36). If  $G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle$   $m \geq 5$ , we can let  $S$  to be  $S_1 = \{b, b^{-1}, ab, ab^{-1}, b^j, b^{-j}\}$ ,  $2 \leq j > m/2$  or  $S_2 = \{b, b^{-1}, ab, ab^{-1}, ab^j, ab^{-j}\}$ ,  $2 \leq j > m/2$ . Let  $S = S_1$ . When ( $m, j$ ) =  $p > 2$ ;  $m = (t + 1)p$ ,  $\sigma = (b, ab)(b^{p+1}, ab^{p+1}) \dots (b^{tp+1}, ab^{tp+1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (47–  $S_1$ ) of the main theorem.

When  $m = 8$ ,  $j = 3$ ,  $\sigma = (b^2, b^6)(ab, ab^7)(ab^3, ab^5) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (48–  $S_1$ ) of Theorem 1.1. Otherwise,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(37, 38–  $S_1$ ). Let  $S = S_2$ . When ( $m, j$ ) =  $p > 2$ ;  $m = (t + 1)p$ ,  $\sigma = (b, ab)(b^{p+1}, ab^{p+1}) \dots (b^{tp+1}, ab^{tp+1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (47–  $S_2$ ) of Theorem 1.1. When  $m = 8$ ,  $j = 3$ ,  $\sigma = (b^2, b^6)(b^3, b^5)(b, b^7) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (48–  $S_2$ ) of main theorem. Otherwise,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(37, 38–  $S_2$ ). If  $G = Z_{2m} \times Z_n = \langle a \rangle \times \langle b \rangle$ , we can let  $S$  to be one of the following cases:

(1):  $S_1 = \{a, a^{-1}, a^{m-1}, a^{m-1}, b, b^{-1}\}$ ,  $m \geq 3$ ,

(2):  $S_2 = \{b, b^{-1}, a^m b, a^m b^{-1}, a, a^{-1}\}$ ,  $m \geq 2$ ,

(3):  $S_3 = \{b, b^{-1}, a^{m+1} b^l, a^{m-1} b^l, a, a^{-1}\}$ ,  $n = 2l$ ,  $l \geq 2$ .

Let  $S = S_1$ . When  $m = 2i$ ,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (19) of Theorem 1.1. When  $m = 2i + 1$ ,  $\sigma = (a^{m-1}, a^{2m-1})(a^{m-1}b, a^{2m-1}b) \dots (a^{m-1}b^{n-1}, a^{2m-1}b^{n-1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.4,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case 20 (with  $m$  odd) of Theorem 1.1. Let  $S = S_2$ . When  $n = 2j$ ,  $2j - 1$  ( $j \geq 2$ ),  $\sigma = (b^j, a^m b^j)(ab^j, a^{m+1} b^j) \dots (a^{m-1} b^j, a^{2m-1} b^j) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (49) of Theorem 1.1. When  $S = S_3$ ,  $\sigma = (a^{m-1}, a^{-1} b^l)(a^{m-1} b, a^{-1} b^{l+1}) \dots (a^{m-1} b^{2l-1}, a^{-1} b^{l-1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (50) of Theorem 1.1. If  $G = Z_2 \times Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ,  $m \geq 3, n \geq 3$ ,  $S = \{b, b^{-1}, ab, ab^{-1}, c, c^{-1}\}$ , we have the Case (20) of the main theorem. Suppose  $a^3 = b$ , then we have one of the following cases:

(1):  $G = Z_m = \langle a \rangle$ ,  $m \geq 7$ ,  $S_1 = \{a, a^{-1}, a^3, a^{-3}, a^j, a^{-j}\}$ , ( $j \neq 3, 2 \leq j \leq m/2$ ),

$S_2 = \{a^j, a^{-j}, a^{3j}, a^{-3j}, a, a^{-1}\}$ , ( $2 \leq j \leq m/2, 3j \neq 0, 1, m - 1, j, m - j, m/2 \pmod{m}$ ).

(2):  $G = Z_m \times Z_n = \langle a \rangle \times \langle b \rangle$ , ( $n \geq 3, m \geq 5, m \neq 6$ ),  $S = \{a, a^{-1}, a^3, a^{-3}, b, b^{-1}\}$ .

(3):  $G = Z_{3m-1} \times Z_{3n} = \langle a \rangle \times \langle b \rangle$ , ( $m \geq 2, n \geq 1$ ),

$$S = \{a^m b^n, a^{2m-1} b^{2n}, a^3, a, a^{-1}, b, b^{-1}\}.$$

$$(4): G = Z_{3m+1} \times Z_{3n} = \langle a \rangle \times \langle b \rangle, (m, n \geq 1), S = \{a^{2m+1} b^n, a^m b^{2n}, a, a^{-1}, b, b^{-1}\}.$$

In the Case (1), when  $m = 6k, j = 3k-1, k \geq 2, \sigma = (a, a^{3k+1})(a^4, a^{3k+4}) \dots (a^{3k-2}, a^{6k-2}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (51) of Theorem 1.1. In this case for  $S_1$ , when  $m = 7, j = 2, \sigma = (a^2, a^5) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (52) of Theorem 1.1. When  $m = 8, j = 2, \sigma = (a^2, a^6) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (45) of the main theorem.

When  $m = 14; j = 5, \sigma = (a^2, a^{12})(a^5, a^9) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (52) of Theorem 1.1. Also for  $S_2$ , when  $m = 7; j = 3, \sigma = (a^3, a^4) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (52) of Theorem 1.1. When  $m = 14; j = 3, \sigma = (a^2, a^{12})(a^5, a^9) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (52) of Theorem 1.1. Otherwise,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(39, 40, 41). In the Case (2), when  $m = 5, 10$  and  $8$  we have the Cases (21) and (19,  $m = 2$ ) of Theorem 1.1 respectively. Otherwise,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3 (24). In the Cases (3) and (4),  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3 (25, 26). Suppose  $c = a^2 b$ . Then we have one of the following cases:

$$(1): G = Z_m = \langle a \rangle (m \geq 7), S = \{a, a^{-1}, a^j, a^{-j}, a^{2+j}, a^{-2-j}\}, \text{ if } m = 2k, 2 \leq j \leq (m/2) - 3 \text{ and if } m = 2k + 1, 2 \leq j \leq (m/2) - 1.$$

$$(2): G = Z_m = \langle a \rangle (m \geq 7), S_1 = \{a^j, a^{-j}, a, a^{-1}, a^{2j+1}, a^{-2j-1}\}, 2 \leq j \leq m-2, j \neq m/2 \text{ and } 2j+1 \neq m/2, 0, 1, m-1, j, m-j \pmod{m}$$

$$(3): G = Z_m \times Z_n = \langle a \rangle \times \langle b \rangle (m, n \geq 3), S = \{a, a^{-1}, b, b^{-1}, a^2 b, a^{-2} b^{-1}\}.$$

$$(4): G = Z_{2m+1} \times Z_n = \langle a \rangle \times \langle b \rangle (m \geq 2, n \geq 3), S = \{a^m, a^{m+1}, a, a^{-1}, b, b^{-1}\}.$$

$$(5): G = Z_{2m+1} \times Z_{2n+1} = \langle a \rangle \times \langle b \rangle (m, n \geq 1), S = \{a^m b^{n+1}, a^m b^n, a, a^{-1}, b, b^{-1}\}.$$

$$(6): G = Z_2 \times Z_{2m+1} \times Z_{2n+1} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle (m, n \geq 1), S = \{ab^m c^{n+1}, ab^{m+1} c^n, b, b^{-1}, c, c^{-1}\}.$$

In the Case (1), if  $m = 3k, k \geq 3, j = k-1, \sigma = (a^k, a^{2k}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (53) of Theorem 1.1. If  $m = 6k, k \geq 3, j = 3k-3, \sigma = (a, a^{3k+1})(a^4, a^{3k+4}) \dots (a^{3k-2}, a^{6k-2}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (51 -  $S_2, m \geq 3$ ) of Theorem 1.1. If  $m = 7; j = 2, \sigma = (a^3, a^4) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , and if  $m = 14, j = 2, \sigma = (a^2, a^{12})(a^5, a^9) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (52) of the main theorem.

Otherwise,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(42, 43). In the Case (2), if  $m = 7, j = 4, \sigma = (a^5, a^9) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , and if  $m = 14, j = 5, \sigma = (a^2, a^{12})(a^5, a^9) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (52) of Theorem 1.1. If  $m = 3k, j = k-1, k \geq 3, \sigma = (a^k, a^{2k}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (53) of Theorem 1.1. If  $m = 4j, j \geq 2, \sigma = (a^j, a^{3j}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (45) of Theorem 1.1. If  $m = 6k, j = 3k+1, k \geq 3, \sigma = (a, a^{3k+1})(a^4, a^{3k+4}) \dots (a^{3k-2}, a^{6k-2}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (51 -  $S_1$ ) of Theorem 1.1. If  $m = 8k+4, k \geq 1$ , for  $k = 2i-1, j = 4i-2, i \geq 1, \sigma = (a^2, a^{12i-1})(a^6, a^{12i+3}) \dots (a^{m-2}, a^{12i-5}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (54) of Theorem 1.1, and for  $k = 2i, j = 12i+2, i \geq 1, \sigma = (a^2, a^{4i+3})(a^6, a^{4i+7}) \dots (a^{m-2}, a^{4i-1}) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (55) of Theorem 1.1. In the Case (3), if  $m = n = 3, \sigma = (ab, a^2 b^2) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (56) of the main theorem. If  $m = 4, \sigma = (ab^2, a^3 b^2) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (50) of Theorem 1.1. Otherwise,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(27).

In the Case (4), if  $m = 2$ , we have the Case (21) of Theorem 1.1. if  $m \geq 3, \Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(28). In the Case (5), if  $m = n = 1, \sigma = (ab, a^2 b^2) \in A_1$ , but  $\sigma \notin \text{Aut}(G, S)$ , by Proposition 2.1,  $\Gamma = \text{Cay}(G, S)$  is not normal, the Case (56) of Theorem 1.1. Otherwise,  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(29). In the Case (6),  $\Gamma = \text{Cay}(G, S)$  is normal by Lemma 3.3(30).

#### 4. Conclusion

Let  $\Gamma = \text{Cay}(G, S)$  be a connected Cayley graph of an abelian group  $G$  on  $S$ . In this paper we have shown all non-normal Cayley graph  $\Gamma$  with valency 6.

#### References

- [1] Alspach, B., Conder, M., Marusic, D., Xu, M.Y., "Two-Arc Transitive Circulants", Journal of Algebraic Combinatorics, 5, 1996, PP. 83-86.
- [2] Xu, M.Y., "Half-Transitive Graphs of Prime-Cube Order", Journal of Algebraic Combinatorics, 1, 1992, PP. 275-292.
- [3] Alspach, B., "Point-Symmetric Graphs and Digraphs of Prime Order and Transitive Permutation Groups of Prime Order", J. Combin. Theory, 15, 1973, PP. 12-17.
- [4] Du, S.F., Wang, R.J., Xu, M.Y. "On Normality of Cayley Digraphs of Order Twice a Prime", Australian Journal of Combinatorics, 18, 1998, PP. 227-234.

- [5] Baik, Y.G., Feng, Y.Q., Sim, H.S., “*The Normality of Cayley Graphs of Finite Abelian Group with Valency 5*”, Sys. Sci. and Math. Sci., 13(4), 2000, PP. 420-431.
- [6] Baik, Y.G., Feng, Y.Q., Sim, H.S., Xu, M.Y., “*On The Normality of Cayley Graphs of Abelian Group*”, Algebra Colloq. 5, 1998, PP. 297-304.
- [7] Feng, Y.Q., Gao, T.P., “*Automorphism Groups and Isomorphism of Cayley Digraphs of Abelian Groups*”, Australian Journal of Combinatorics, 16 1997, PP. 183-187.
- [8] Wang, C.Q., Wang, D.J., Xu, M.Y., “*On Normal Cayley Graphs of Finite Groups*”, Science in China (A), 28(2), 1998, PP. 131-139.
- [9] Xu, M.Y., “*Automorphism Groups and Isomorphism of Cayley Graphs*”, Discrete Math, 182, 1998, PP. 309-319.