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RESEARCH PAPER

Pricing, Advertising, and Service Policies in a Manufacturer-Retailer Supply Chain with Nash, Stackelberg-Retailer, and Cooperative Games

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ABSTRACT
This paper examines the simultaneous decisions regarding advertising, pricing, and service to supply chain coordination involving one manufacturer and one retailer. Demand is impacted by these decisions, with service playing a crucial role in enhancing customer loyalty and boosting sales. The study employs three well-known game theory approaches—Nash, Stackelberg-Retailer, and Cooperative games—to analyze their effects on the supply chain. Optimal strategies for both the manufacturer and the retailer are identified within each approach, and the strategies' results are compared. Results show that the retailer manufacturer, and the entire system achieves higher profits through the Stackelberg-Retailer game compared to the Nash game, while the Cooperative game results in the highest overall profits. Finally, the Nash bargaining model is outlined and analyzed to assess opportunities for sharing profits.

KEYWORDS: Supply chain; Services; Advertising; Pricing; Game theory.

1. Introduction

Various strategies exist for coordinating supply chains, including pricing, advertising, and after-sales services. Numerous studies have explored the impact of two strategies together on the coordination and profit enhancement of supply chain members. In this section, we review recent studies that simultaneously examine the effects of advertising and after-sales services, advertising and pricing, as well as pricing and after-sales services.

1.1. Pricing and service decisions

Shu et al. investigated how pricing and logistics service decisions can be coordinated in a supply chain comprising one manufacturer and one online retailer, suggesting revenue-sharing and cost-sharing contracts to address coordination issues and improve overall chain profitability [1]. Hosseini-Motlagh et al. explored pricing, economic incentives, and customer service levels in a competitive closed-loop supply chain (CLSC), proposing a new expense-sharing contract for coordination [2]. Chen et al. analyzed the best pricing and green strategies in a dynamic green supply chain, comparing scenarios of competitive and supportive retail services [3]. Jiang et al. examined the best pricing decisions within an omni-channel supply chain, specifically where a dominant physical retailer offers buy-online-and-pickup-in-store (BOPS) options [4]. Ren et al. evaluated pricing and service cooperation strategies in a dual-channel supply chain involving two manufacturers and one retailer selling complementary products [5]. Additionally, Kang et al. and Zhang and Zhu explored pricing and service level strategies in a reverse supply chain with a dual-channel recycling structure [6-7].

The research conducted by Mohammadzadeh et al. focused on optimal pricing and free periodic maintenance service strategies in an automotive supply chain involving electric and fuel vehicles. Their focus was on the total ownership cost and customer usage patterns to determine vehicle market shares and environmental impacts [8]. Liu et al. explored pricing and logistics in a closed-loop supply chain of electronic products [9]. Tian and Wu examined the impact of pricing power and service strategies on dual-channel supply chains [10].

Zhang et al. explored pricing decision models in a dual-channel supply chain by regarding service

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levels and returns, emphasizing the impact of service levels on consumer return behavior and overall supply chain profitability [11]. Sarkar and Pal modeled pricing and service strategies in a dual-channel supply chain featuring a return-refund policy and retail services [12]. He et al. examined a dual-channel supply chain emphasizing on underservice issues, where retailers struggle to meet consumer expectations for personalized services [13]. Qu et al. investigated the interaction between misreporting behavior and fairness concerns within a supply chain context [14]. Ma and Hong examined the dynamic game involving pricing and service strategies within a dual-channel supply chain, considering risk attitudes and the free-ride effect [15].

Ullah et al. explored pricing decisions within a risk-averse supply chain, focusing on after-sales maintenance services for repairable products. They examined how demand uncertainty and risk preferences influence the pricing strategies of both manufacturers and agents, shedding light on optimal pricing for maximizing utility [16]. Zhai et al. modeled the role of different power structures in a supply chain on service and pricing policies under demand disruptions. Their study focused on the robustness of service levels and production quantities under specific conditions, with implications for both centralized and decentralized systems [17].

Xi and Zhang investigated pricing strategies, service levels, and emission reduction choices in an e-commerce supply chain consisting of an e-commerce platform and a low-carbon manufacturer, applying various game scenarios [18]. Wen and He explored the interplay between pricing and service levels in a supply chain for fresh agricultural products, factoring in partial integration [19].

Lin and Januardi delved into two-period pricing and service utilization decisions in dual-channel service-only supply chains, highlighting the effect of dynamic pricing and capacity utilization on demand functions [20]. Yao et al. investigated pricing decisions in the cloud service supply chain, focusing on the influence of market structures, competition levels, and service quality on pricing equilibrium [21]. Gu et al. focused on pricing coordination in a multichannel supply chain, specifically considering the influence of offline in-sale services [22].

The research conducted by Gong et al. focused on how free-riding behavior and backward revenue-sharing ratios impact pricing and service decisions of retailers in a closed-loop supply chain, employing Stackelberg game theory to explore their effects on both online and offline prices and service investments [23]. Xu et al. modeled dual-channel supply chain strategies under information sharing, showing that while information sharing can increase prices and service levels, its benefits vary [24]. De and Singh developed a model for resilient pricing and service quality decisions in the fresh agricultural product supply chain during the post-COVID-19 era [25].

Liu et al. examined the optimal pricing strategy within a dual-channel supply chain, focusing on how online reviews and in-sale services affect consumer behavior. They compared decentralized and centralized pricing decisions and proposed a two-part tariff coordination mechanism to enhance profits for both manufacturers and retailers [26]. Lu et al. examined pricing strategies in the dual-channel pharmaceutical supply chain, emphasizing the effect of service levels on pricing decisions [27]. Li et al. analyzed optimal pricing and promotional approaches in a dual-channel green supply chain, addressing concerns regarding after-sales service cooperation and free-riding among online and offline retailers [28].

1.2. Pricing and advertising decisions

Sadjadi and Alirezaee used a game-theoretic model to analyze the impact of pricing structures and cooperative advertising on supply chain coordination in a two-echelon system consisting of one manufacturer and two retailers [29]. Alirezaee and Sadjadi explored optimal cooperative advertising and pricing strategies within a supply chain comprising one manufacturer and several retailers, utilizing a game-theoretic model [30]. Taleizadeh et al. analyzed pricing, advertising, and the selection of reverse channels in a probabilistic closed-loop supply chain (CLSC) under the influence of fuzzy demand parameters [31].

Farshbaf-Geranmayeh and Zaccour modeled pricing and advertising strategies in a supply chain with strategic and myopic consumers, focusing on how to manage price markdowns to optimize profits [32]. Khorshidvand et al. presented a two-stage model for a sustainable closed-loop supply chain (CLSC) that incorporates pricing, green quality, and advertising decisions. The first stage focused on optimal strategies, while the second stage utilized a fuzzy multi-objective Mixed Integer Linear Programming (MILP) method [33]. In addition, Khorshidvand et al. analyzed revenue management strategies in a multi-level, multi-channel supply chain, addressing the integration of pricing, greening, and advertising decisions for both online and traditional platforms [34].

Pan et al. studied a sustainable production-inventory model where pricing and advertising are influenced by demand and carbon emission policies [35]. Mozafari et al. investigated how pricing and cooperative advertising can be coordinated in a two-echelon supply chain with uncertain demand and manufacturing costs, analyzing three different scenarios related to market power dynamics [36]. Li et al. modeled cooperative advertising in an O2O supply chain with a BOPS strategy, finding that BOPS can partially substitute the incentive effects of cooperative advertising [37]. Zarei et al. examined the coordination of pricing, lot-sizing, and advertising in a dual-channel supply chain using game theory [38].

Asghari et al. explored the pricing and advertising choices in a direct-sales closed-loop supply chain, taking into account the remanufacturing and recycling processes. They introduced a new optimization model enhanced by a particle swarm optimization algorithm based on crowd-learning theory, demonstrating superior performance in computational experiments [39]. Karray and Martin-Herran studied the impact of introducing a private brand in a supply chain with competeting generic brand manufacturers, focusing on the strategic timing of pricing and advertising decisions. Using a game-theoretic model, their research highlighted how strategic timing adjustments can mitigate or exacerbate the effects of store brand arrival on national brands and retailers [40].

He et al. modeled a platform service supply chain in the hospitality and tourism industries, focusing on three decision modes [41]. Xie et al. analyzed cooperative advertising strategies in a two-period online supply chain using a Stackelberg game model [42]. Wang et al. explored coordination mechanisms to combat online advertising fraud in supply chains [43]. Yan et al. studied how dual-channel retailers adjust pricing and advertising strategies under discounted online advertising effectiveness [44]. Huo et al. examined pricing decisions in a low-carbon dual-channel supply chain by utilizing behavioral economics and optimization models. They evaluated both fair and neutral decentralized and centralized decision-making models, analyzing how low-carbon advertising levels influence pricing strategies for online and offline retailers [45]. Chan et al. applied a multi-methodological approach to analyze optimal pricing, green advertising efforts, and advanced technology investments in sustainable fashion supply chains [46].

Chaab and Demirag explored the influence of consumer loyalty and product compatibility with online shopping on cooperative advertising and pricing strategies in a dual-channel supply chain. Their study also analyzed how direct sales channels impact profits [47]. Chen and Xu analyzed the integration of optimal pricing and advertising strategies in a fashion supply chain using the ODM strategy, with an emphasis on fashion level and brand goodwill. They introduced dynamic wholesale price contracts to enhance coordination, ultimately improving design investment, promotion efforts, and product demand [48]. Mohamadi Zanjirani et al. studied the impact of pricing and advertising on competition between manufacturers and retailers in the coffee supply chain, taking into account direct online sales [49].

Apornak and Keramati investigated pricing and collaborative advertising efforts in a dual-channel supply chain [50]. Yan and He examined pricing and advertising coordination in a two-period supply chain, focusing on the retailer's second-period price discount strategy [51]. Wu et al. applied optimal control theory to analyze pricing and advertising coordination in a consignment-based supply chain. They compared strategies in centralized and decentralized settings and proposed a contract scheme to enhance channel member profits [52].

Shi examined supply chain pricing decisions considering advertising effects and market encroachment using a game framework. She modeled scenarios in which an incumbent manufacturer and retailer invest in advertising, particularly when facing a new entrant manufacturer [53]. Gu et al. explored how consumer regret influences pricing, advertising decisions, and profits in an O2O supply chain involving e-retailers and brick-and-mortar stores offering buy online, pick up in-store (BOPS) services [54]. Yue et al. investigated, through a game theory model, how consumer information investment impacts supply chain advertising and pricing [55]. Mahdi utilized a bi-level programming approach to analyze optimal resource allocation and pricing in a dual-level supply chain with collaborative advertising efforts. He focused on a manufacturer's sales through two retailers and online, considering the impacts of global and local advertising as well as budget constraints [56]. Jafari utilized game theory to examine pricing and cooperative advertising strategies for two substitutable products in a supply chain comprising two manufacturers and two retailers [57]. Wang et al., coordinated supply chain with advertising and pricing in an online advertising fraud [58].

1.3. Advertising and service decisions

Yousefi et al. developed a machine learning-optimization hybrid model to enhance supply chain performance by integrating pricing and social network advertising, specifically through influencer marketing. They emphasized the importance of accurately selecting influencers based on effectiveness rather than network size, validating their approach using Instagram data [59]. Wang et al. utilized a continuous-time Stackelberg game model to analyze the optimal timing for introducing a distributor in a dynamic cooperative advertising supply chain [60]. Shi et al. studied a two-stage tourism supply chain during regular COVID-19 prevention, focusing on a tourist attraction and a travel agency. Using theoretical game models, they examined how service and advertising levels impact retail prices, service quality and profits [61].

1.4. Problem expression

In this paper, we aim to align the supply chain and maximize members' profits by investigating three key strategies—pricing, advertising, and service level— together. We examine these strategies within two non-cooperative games: 1) the Nash game and 2) the Stackelberg-Retailer (SR) game, as well as within a cooperative game. The demand function is influenced by all these policies, allowing us to analyze their simultaneous effects on market demand.

The layout of the paper is as follows: Section 2 explains the model. Section 3 considers both non-cooperative and cooperative games. Visual results of the models are presented in Section 4. Section 5 explores the application of the Nash bargaining problem for profit sharing. Ultimately, Section 6 delivers the conclusions, points out directions for future research, and recaps the findings. The Appendix includes detailed proofs of the propositions.

Tab. 1. The relevant studies and the proposed model

	Equality of margins	Price demand	Advertising demand	Services	Game structures
[62]	Assumed	$\alpha - \beta p$	$A - Ba^{-\gamma}A^{-\delta}$	----	N, SR, SM and Co
[63]	Assumed	$(\alpha - \beta p)^{\frac{1}{v}}$	$k_r\sqrt{a} + k_m\sqrt{A}$	----	N, SR, SM and Co
[64]	Relaxed	$(\alpha - \beta p)^{\frac{1}{v}}$	$k_r\sqrt{a} + k_m\sqrt{A}$	----	N, SR, SM and Co
Proposed Model	Assumed	$(\alpha - \beta p)^{\frac{1}{v}}$	$k_r\sqrt{a} + k_m\sqrt{A}$	$b_r s_r + b_m s_m$	N, SR and Co

2. The Model and the Notations

In this supply chain framework, a single manufacturer supplies products through a single retailer channel. The manufacturer sets the wholesale price, national advertising budget, participation rate, and the budget for manufacturer services. The retail price, local advertising budget, and retailer services are determined by the retailer. National ads involve promoting the business across the country while local ads are focused on a small and specific geographic area. The framework of the supply chain under consideration is illustrated in Fig. 1. Table 2 presents the decision variables and parameters used in this paper.

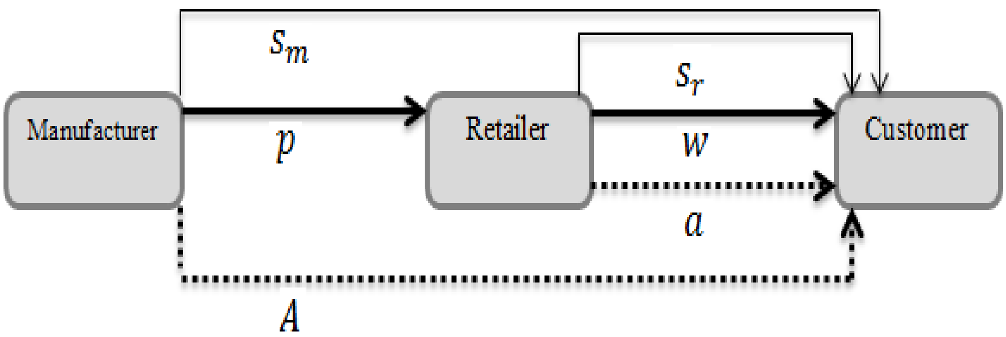


Fig. 1. The structure of the considered supply chain

Tab. 2. Symbols used

Variable		Parameters	
w	Wholesale price	α	Price demand potential
A	National advertising	β	Price sensitivity
s_m	Manufacturer's services	ν	Demand graph shape parameter
p	Retail price	k_m	Effectiveness of national advertising
a	Local advertising	k_r	Effectiveness of local advertising
s_r	Retailer's services	b_m	Effectiveness of manufacturer's services
Π_m	Manufacturer's profit	b_r	Effectiveness of retailer's services
Π_r	Retailer's profit	η_m	Manufacturer's services cost factor
Π_{m+r}	System's profit	η_r	Retailer's services cost factor
		c	Manufacturer's unit production cost
		d	Retailer's unit handling cost

This paper employs the multiplicative form of the relationship between advertising and pricing, which is well-established in the published works. The additive format between services and pricing, based on the model proposed by Tsay and Agrawal, is assumed here [65]. Future studies could investigate the multiplicative form further. Finally, we present the demand functions as bellow; however, alternative forms could be explored in future research:

$$D = D_0(g(p)h(a, A) + I(s_r, s_m)) \tag{1}$$

D_0 is defined as the base level of demand. The effects of the retail price, advertising, and services cost on the demand are expressed by $g(p), h(a, A)$, and $I(s_r, s_m)$, in the mentioned order. The demand changes when the price changes within an reverse relationship. Based on SeyedEsfahani et al. $g(p)$ and $h(a, A)$ are as bellow [63]

$$g(p) = (\alpha - \beta p)^{\frac{1}{\nu}} \tag{2}$$

This basic form can be convex, linear, or concave according to whether $\nu < 1$, $\nu = 1$, or $\nu > 1$, respectively.

$$h(a, A) = k_r\sqrt{a} + k_m\sqrt{A} \tag{3}$$

In a significant number of publications, the services function form is relies on Tsay and Agrawal's [65] as follows:

$$I(s_r, s_m) = b_r s_r + b_m s_m \tag{4}$$

In the formulated demand equation it is presumed that with the price enhancement, the market demand decreases and Stabilizes at a fixed level of $D_0 I(s_r, s_m)$ rather zero. Due to the essential nature of certain goods, customers are compelled to purchase them even when prices increase. According to Dan et al. (2012), the cost providing sales effort level (s) is represented by $\frac{\eta s^2}{2}$. The quadratic form is utilized to represent the diminishing returns associated with expenditures on sales efforts. Diminishing returns are expected, especially when considering the substantial impact of store-level inventory on service levels. According to standard inventory models, increasing the fill rate from 97% to 99% generally necessitates a significantly greater incremental investment compared to enhancing it from 95% to 97%. For other service concepts, we assume that a rational manager will prioritize the "lowest-hanging fruit," making subsequent improvements increasingly challenging [65].

Based on Eqs. (1)-(4), the demand function is defined as follows:

$$D(p, a, A, s_r, s_m) = D_0[(\alpha - \beta p)^{\frac{1}{\nu}}(k_r\sqrt{a} + k_m\sqrt{A}) + (b_r s_r + b_m s_m)] \tag{5}$$

To mitigate the adverse impact of pricing and advertising on the demand when they are jointly committed, the following condition needs to be confirmed:

$$p < \frac{\alpha}{\beta} \tag{6}$$

The profit functions for the channel members and the overall system are defined as follows:

$$\Pi_m(w, A, t, s_m) = D_0(w - c)\left[(\alpha - \beta p)^{\frac{1}{\nu}}\left(k_r\sqrt{a} + k_m\sqrt{A}\right) + (b_r s_r + b_m s_m)\right] - A - a - \frac{\eta_m s_m^2}{2}; \tag{7}$$

$$\Pi_r(p,a,s_r) = D_0(p-w-d)\left[(\alpha-\beta p)^{\frac{1}{v}}\left(k_r\sqrt{a}+k_m\sqrt{A}\right)+(b_r s_r+b_m s_m)\right]-a-\frac{\eta_r s_r^2}{2}; \quad (8)$$

$$\Pi_{m+r}(p,A,a,s_m,s_r) = D_0(p-c-d)\left[(\alpha-\beta p)^{\frac{1}{v}}\left(k_r\sqrt{a}+k_m\sqrt{A}\right)+(b_r s_r+b_m s_m)\right]-A-a-\left(\frac{\eta_r s_r^2}{2}+\frac{\eta_m s_m^2}{2}\right); \quad (9)$$

In this paper, m, r, and m+r denote the manufacturer, the retailer, and the system, respectively. Inspired by SeyedEsfahani et al., Eqs. must validate the subsequent condition to avoid backwash effects [63]:

$\Pi_m > 0 \rightarrow w > c$;

$\Pi_r > 0 \rightarrow p > w + d > w$

And based on Eq. it can be rewritten as $\alpha - \beta(c + d) > 0$

$\Pi_{m+r} > 0 \rightarrow p > c + d$;

The variables are changed similar to the model of SeyedEsfahani et al. to simplify the analysis [63]. Assume:

$$\alpha' = \alpha - \beta(c+d)$$

$$w' = \frac{\beta}{\alpha'}(w-c)$$

$$p' = \frac{\beta}{\alpha'}(p-(c+d))$$

$$k_r' = D_0\frac{\alpha'^{1+\frac{1}{v}}}{\beta}k_r$$

$$b_r' = D_0\frac{\alpha'}{\beta}b_r$$

$$b_m' = D_0\frac{\alpha'}{\beta}b_m$$

$$k_m' = D_0\frac{\alpha'^{1+\frac{1}{v}}}{\beta}k_m$$

Reflecting the changes mentioned, Eqs. (7)-(9) can be restated as follows:

$$\Pi_m'(w',A,t,s_m) = w'\left[(1-p')^{\frac{1}{v}}\left(k_r'\sqrt{a}+k_m'\sqrt{A}\right)+(b_r's_r+b_m's_m)\right]-A-\frac{\eta_m s_m^2}{2}; \quad (10)$$

$$\Pi_r'(p',a,s_r) = (p'-w')\left[(1-p')^{\frac{1}{v}}\left(k_r'\sqrt{a}+k_m'\sqrt{A}\right)+(b_r's_r+b_m's_m)\right]-a-\frac{\eta_r s_r^2}{2}; \quad (11)$$

$$\Pi_{m+r}'(p',A,a,s_m,s_r) = p'\left[(1-p')^{\frac{1}{v}}\left(k_r'\sqrt{a}+k_m'\sqrt{A}\right)+(b_r's_r+b_m's_m)\right]-A-a-\left(\frac{\eta_r s_r^2}{2}+\frac{\eta_m s_m^2}{2}\right); \quad (12)$$

To simplify the sequence of equations, the superscript (') has been omitted.

3. Three Game Models

In this section, three games, consisting of two non-cooperative games (i.e., the Nash and Stackelberg-retailer (SR)) and one cooperative are described. Future studies could examine the Stackelberg-manufacturer game, a widely recognized non-cooperative model in which the manufacturer serves as the leader over the retailer.

3. 1. The nash game

The Nash game is particularly suited for scenarios where participants hold equal power and make decisions simultaneously and independently. This game's solution, called the 'Nash equilibrium,' is determined by solving these two models:

$$\max \Pi_m(w,A,t,s_m) = w\left[(1-p)^{\frac{1}{v}}\left(k_r\sqrt{a}+k_m\sqrt{A}\right)+(b_r s_r+b_m s_m)\right]-A-\frac{\eta_m s_m^2}{2};$$

$$st: \quad 0\le w\le 1 \quad 0\le A \quad 0\le t\le 1 \quad 0\le s_m$$

$$\max \Pi_r(p,a,s_r) = (p-w)\left[(1-p)^{\frac{1}{v}}\left(k_r\sqrt{a}+k_m\sqrt{A}\right)+(b_r s_r+b_m s_m)\right]-a-\frac{\eta_r s_r^2}{2};$$

$$s.t: \quad w\le p\le 1 \quad 0\le s_r$$

Since w has a positive coefficient, its optimal value is *p*, but $p > w$; so, To secure profit for both parties, we assume the retailer will not proceed with sales unless they receive a minimum unit margin. This method resembles the models proposed by Xie and Neyret and SeyedEsfahani et al. [62-63]. They regarded the manufacturer's minimum unit margin as the minimum standard and replaced the wholesale price constraint with $\mu_r > \mu_m \rightarrow p - w \ge w \rightarrow w \le \frac{p}{2}$.

The retailer's and manufacturer's unit margins are expressed as $\mu_r = p - w$ and $\mu_m = w$ so the highest possible value for w is $\frac{p}{2}$. As the model is rather complicated, three examples of ν values, namely $\nu = 1, \nu = \frac{1}{2}$ and $\nu = 2$ representing examples of linear, convex, and concave price-demand curves, respectively, are engaged here to ascertain the equilibrium.

Proposition 1. The Nash equilibrium is obtained as follows (The method for achieving equilibrium is explained in the attachment.):

Case 1. $\nu = 1$

$$p^N = \frac{5-\sqrt{1-24y}}{6} \qquad w^N = \frac{5-\sqrt{1-24y}}{12}$$

$$s_m^N = \frac{b_m}{\eta_m}\left(\frac{5-\sqrt{1-24y}}{12}\right) \qquad A^N = \left(\frac{k_m}{144}(5-\sqrt{1-24y})(1+\sqrt{1-24y})\right)^2$$

$$s_r^N = \frac{b_r}{\eta_r}\left(\frac{5-\sqrt{1-24y}}{12}\right) \qquad a^N = \left(\frac{k_r}{144}(5-\sqrt{1-24y})(1+\sqrt{1-24y})\right)^2$$

Case 2. $\nu = 2$

$$p^N = 0.8(1+y) \qquad w^N = 0.4(1+y)$$

$$s_m^N = \frac{b_m}{\eta_m}(0.4+0.4y) \qquad A^N = \frac{k_m^2}{16}(0.8+0.8y)^2(0.2-0.8y)$$

$$s_r^N = \frac{b_r}{\eta_r}(0.4+0.4y) \qquad a^N = \frac{k_r^2}{16}(0.8+0.8y)^2(0.2-0.8y)$$

Case 3. $\nu = \frac{1}{2}$

$$w^N = \frac{28-\sqrt{3}x-\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}}{48}$$

$$p^N = \frac{1}{24}\left(21-\sqrt{3}x-\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}\right.$$

$$A^N = \frac{k_m^2}{16}\left(\frac{21-\sqrt{3}x-\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}}{24}\right)^2\left(\frac{3+\sqrt{3}x+\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}}{24}\right.$$

$$a^N = \frac{k_r^2}{16}\left(\frac{21-\sqrt{3}x-\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}}{24}\right)^2\left(\frac{3+\sqrt{3}x+\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}}{24}\right.$$

$$s_m^N = \frac{b_m}{\eta_m}\left(\frac{28-\sqrt{3}x-\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}}{48}\right.$$

$$s_r^N = \frac{b_r}{\eta_r}\left(\frac{7}{16}-\frac{\sqrt{3}}{48}x-\frac{1}{48}\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}\right.$$

To simplify matters in all the aforementioned cases, we assume that:

$$y = 2\frac{\frac{b_r^2}{\eta_r}+\frac{b_m^2}{\eta_m}}{k_r^2+k_m^2} \qquad z = (108y+12\sqrt{81y^2-6144y^3})^{\frac{1}{3}} \qquad x = \sqrt{3+4z+384\frac{y}{z}}$$

The Appendix contains the proofs for all the propositions.

3. 2. The stackelberg-retailer game

In this game, the participants are categorized as a leader and a follower. In the Stackelberg game, the leader first determines their output, after which the follower, equipped with knowledge of the leader's decision, selects their own quantity [66]. In the Stackelberg-retailer game, the retailer holds greater power than the manufacturer. The outcome of this game is referred to as the 'Stackelberg-retailer equilibrium.' In the SR game, the manufacturer's optimal response mirrors that of the Nash game, as it arises from the first-order condition of the manufacturer's profit function:

$$A^* = \left(\frac{1}{2}\mathrm{k_m} \mathrm{w}(1-\mathrm{p})^2\right)^2 \tag{13}$$

$$\mathrm{w}^* = \frac{\mathrm{p}}{2} \tag{14}$$

To determine the Stackelberg-retailer (SR) equilibrium, the retailer's decision problem is addressed by utilizing the optimal values of t, w, and A.

Proposition 2. The equilibrium of the Stackelberg-retailer game is derived as follows (details on the method for achieving equilibrium can be found in the attachment):***Case 1.*** $\nu = 1$

$$p^{SR} = \frac{3-\sqrt{1-8\mathrm{y}}}{4} \qquad w^{SR} = \frac{3-\sqrt{1-8\mathrm{y}}}{8}$$

$$s_m^{SR} = \frac{\mathrm{b_m}}{\eta_\mathrm{m}}\left(\frac{3-\sqrt{1-8\mathrm{y}}}{8}\right) \qquad A^{SR} = \left(\frac{\mathrm{k_m}}{64}(3-\sqrt{1-8\mathrm{y}})(1+\sqrt{1-8\mathrm{y}})\right)^2$$

$$s_r^{SR} = \frac{\mathrm{b_r}}{\eta_\mathrm{r}}\left(\frac{3-\sqrt{1-8\mathrm{y}}}{8}\right) \qquad a^{SR} = \left(\frac{\mathrm{k_r}}{64}(3-\sqrt{1-8\mathrm{y}})(1+\sqrt{1-8\mathrm{y}})\right)^2$$

Case 2. $\nu = 2$

$$p^{SR} = \frac{2(1+\mathrm{y})}{3} \qquad w^{SR} = \frac{1+\mathrm{y}}{3}$$

$$s_m^{SR} = \frac{\mathrm{b_m}}{3\eta_\mathrm{m}}(1+\mathrm{y}) \qquad A^{SR} = \frac{\mathrm{k_m^2}}{16}\left(\frac{2(1+\mathrm{y})}{3}\right)^2\left(\frac{1-2\mathrm{y}}{3}\right)$$

$$s_r^{SR} = \frac{\mathrm{b_r}}{3\eta_\mathrm{r}}(1+\mathrm{y}) \qquad a^{SR} = \frac{\mathrm{k_r^2}}{16}\left(\frac{2(1+\mathrm{y})}{3}\right)^2\left(\frac{1-2\mathrm{y}}{3}\right)$$

Case 3. $\nu = \frac{1}{2}$

$$w^{SR} = \frac{5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{12}$$

$$p^{SR} = \frac{1}{6}(5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}$$

$$A^{SR} = \frac{k_m^2}{16}\left(\frac{5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6}\right)^2 \left(\frac{1+x+\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6}\right.$$

$$a^{SR} = \frac{k_r^2}{16}\left(\frac{5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6}\right)^2 \left(\frac{1+x+\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6}\right.$$

$$s_m^{SR} = \frac{b_m}{\eta_m}\left(\frac{5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{12}\right.$$

$$s_r^{SR} = \frac{b_r}{\eta_r}\left(\frac{5}{12}-\frac{x}{12}-\frac{1}{12}\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}\right.$$

For the sake of simplification in all previously mentioned cases, we make the following assumptions:

$$y = 2\frac{\frac{b_r^2}{\eta_r}+2\frac{b_m^2}{\eta_m}}{k_r^2+2k_m^2} \qquad z = (4y+4\sqrt{y^2-32y^3})^{\frac{1}{3}} \qquad x = \sqrt{1+3z+24\frac{y}{z}}$$

The proofs for all the propositions are presented in the Appendix.

3. 3. The cooperative game

In this game, the manufacturer and retailer first collaborate to maximize the total system profit, after which they negotiate to allocate the shared profit.

Proposition 3. The equilibrium of the cooperative game is obtained as follows (The method for achieving equilibrium is explained in the attachment.):

Case 1. $\nu = 1$

$$p^{Co} = \frac{3-\sqrt{1-8y}}{4}$$

$$s_m^{Co} = \frac{b_m}{\eta_m}\left(\frac{3-\sqrt{1-8y}}{4}\right) \qquad A^{Co} = \left(\frac{\mathrm{k_m}}{32}(3-\sqrt{1-8y})(1+\sqrt{1-8y})\right)^2$$

$$s_r^{Co} = \frac{b_r}{\eta_r}\left(\frac{3-\sqrt{1-8y}}{4}\right) \qquad a^{Co} = \left(\frac{\mathrm{k_r}}{32}(3-\sqrt{1-8y})(1+\sqrt{1-8y})\right)^2$$

Case 2. $\boldsymbol{\nu = 2}$

$$p^{Co} = \frac{2(1+y)}{3}$$

$$s_m^{Co} = \frac{2b_m}{3\eta_m}(1+y) \qquad A^{Co} = \frac{k_m^2}{4}\left(\frac{2(1+y)}{3}\right)^2\left(\frac{1-2y}{3}\right)$$

$$s_r^{Co} = \frac{2b_r}{3\eta_r}(1+y) \qquad a^{Co} = \frac{k_r^2}{4}\left(\frac{2(1+y)}{3}\right)^2\left(\frac{1-2y}{3}\right)$$

Case 3. $\nu = \frac{1}{2}$

$$p^{Co} = \frac{1}{6}\left(5 - x - \sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}\right)$$

$$A^{Co} = \frac{k_m^2}{4}\left(\frac{5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6}\right)^2\left(\frac{1+x+\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6}\right)^4$$

$$a^{Co} = \frac{k_r^2}{4}\left(\frac{5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6}\right)^2\left(\frac{1+x+\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6}\right)^4$$

$$s_m^{Co} = \frac{b_m}{\eta_m}\left(\frac{5}{6}-\frac{x}{6}-\frac{1}{6}\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}\right)$$

$$s_r^{Co} = \frac{b_r}{\eta_r}\left(\frac{5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6}\right)$$

To simplify all cases discussed, we assume the following conditions:

$$y = 2\frac{\frac{b_r^2}{\eta_r}+\frac{b_m^2}{\eta_m}}{k_r^2+k_m^2} \qquad z = (4y+4\sqrt{-32y^3+y^2})^{\frac{1}{3}} \qquad x = \sqrt{1+3z+\frac{24y}{z}}$$

In the table 3 the optimal solutions in three game models are summarized.

Tab. 3. Summary of the optimal solutions in three game models

	Nash	SR	Co
y	$2\frac{\frac{b_r^2}{\eta_r}+\frac{b_m^2}{\eta_m}}{k_r^2+k_m^2}$	$2\frac{\frac{b_r^2}{\eta_r}+2\frac{b_m^2}{\eta_m}}{k_r^2+2k_m^2}$	$2\frac{\frac{b_r^2}{\eta_r}+\frac{b_m^2}{\eta_m}}{k_r^2+k_m^2}$
z	$(108y+12\sqrt{81y^2-6144y^3})^{\frac{1}{3}}$	$(4y+4\sqrt{y^2-32y^3})^{\frac{1}{3}}$	$(4y+4\sqrt{-32y^3+y^2})^{\frac{1}{3}}$

x	$\sqrt{3+4z+384\frac{y}{z}}$	$\sqrt{1+3z+24\frac{y}{z}}$	$\sqrt{1+3z+\frac{24y}{z}}$
		ν = 1	
	Nash game	Stackelberg game	Cooperative game
Wholesale price w	$\frac{5-\sqrt{1-24y}}{12}$	$\frac{3-\sqrt{1-8y}}{8}$	------
Retail price p	$\frac{5-\sqrt{1-24y}}{6}$	$\frac{3-\sqrt{1-8y}}{4}$	$\frac{3-\sqrt{1-8y}}{4}$
National advertising A	$(\frac{k_m}{144}(5-\sqrt{1-24y})(1+\sqrt{1-24y}))^2$	$(\frac{k_m}{64}(3-\sqrt{1-8y})(1+\sqrt{1-8y}))^2$	$(\frac{k_m}{32}(3-\sqrt{1-8y})(1+\sqrt{1-8y}))^2$
Local advertising a	$(\frac{k_r}{144}(5-\sqrt{1-24y})(1+\sqrt{1-24y}))^2$	$(\frac{k_r}{64}(3-\sqrt{1-8y})(1+\sqrt{1-8y}))^2$	$(\frac{k_r}{32}(3-\sqrt{1-8y})(1+\sqrt{1-8y}))^2$
Manufacturer's service $\boldsymbol{s_m}$	$\frac{b_m}{\eta_m}(\frac{5-\sqrt{1-24y}}{12})$	$\frac{b_m}{\eta_m}(\frac{3-\sqrt{1-8y}}{8})$	$\frac{b_m}{\eta_m}(\frac{3-\sqrt{1-8y}}{4})$
Retailer's service $\boldsymbol{s_r}$	$\frac{b_r}{\eta_r}(\frac{5-\sqrt{1-24y}}{12})$	$\frac{b_r}{\eta_r}(\frac{3-\sqrt{1-8y}}{8})$	$\frac{b_r}{\eta_r}(\frac{3-\sqrt{1-8y}}{4})$
		ν = 2	
Wholesale price w	$0.4(1+y)$	$\frac{1+y}{3}$	------
Retail price p	$0.8(1+y)$	$\frac{2(1+y)}{3}$	$\frac{2(1+y)}{3}$
National advertising A	$\frac{k_m^2}{16}(0.8+0.8y)^2(0.2-0.8y)$	$\frac{k_m^2}{16}(\frac{2(1+y)}{3})^2(\frac{1-2y}{3})$	$\frac{k_m^2}{4}(\frac{2(1+y)}{3})^2(\frac{1-2y}{3})$
Local advertising a	$\frac{k_r^2}{16}(0.8+0.8y)^2(0.2-0.8y)$	$\frac{k_r^2}{16}(\frac{2(1+y)}{3})^2(\frac{1-2y}{3})$	$\frac{k_r^2}{4}(\frac{2(1+y)}{3})^2(\frac{1-2y}{3})$
Manufacturer's service $\boldsymbol{s_m}$	$\frac{b_m}{\eta_m}(0.4+0.4y)$	$\frac{b_m}{3\eta_m}(1+y)$	$\frac{2b_m}{3\eta_m}(1+y)$
Retailer's service $\boldsymbol{s_r}$	$\frac{b_r}{\eta_r}(0.4+0.4y)$	$\frac{b_r}{3\eta_r}(1+y)$	$\frac{2b_r}{3\eta_r}(1+y)$

ν = 0.5

Wholesale price w	Nash	$\frac{28-\sqrt{3}x-\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}}{48}$
	SR	$\frac{5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{12}$
	Co	------
Retail price p	Nash	$\frac{1}{24}(21-\sqrt{3}x-\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}})$
	SR	$\frac{1}{6}(5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}})$
	Co	$\frac{1}{6}(5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}})$
National advertising A	Nash	$\frac{k_m^2}{16}\left(\frac{21-\sqrt{3}x-\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}}{24}\right)^2(\frac{3+\sqrt{3}x+\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}}}{24}$
	SR	$\frac{k_m^2}{16}\left(\frac{5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6}\right)^2(\frac{1+x+\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6})$
	Co	$\frac{k_m^2}{4}\left(\frac{5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6}\right)^2(\frac{1+x+\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6})^4$

Local advertising a	Nash	$\frac{k_r^2}{16}\left(\frac{21-\sqrt{3}x-\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}}{24}\right)^2(\frac{3+\sqrt{3}x+\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}}}{24}$
	SR	$\frac{k_r^2}{16}\left(\frac{5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6}\right)^2(\frac{1+x+\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6})$
	Co	$\frac{k_r^2}{4}\left(\frac{5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6}\right)^2(\frac{1+x+\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6})^4$
Manufacturer's service s_m	Nash	$\frac{b_m}{\eta_m}(\frac{28-\sqrt{3}x-\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}}{48})$
	SR	$\frac{b_m}{\eta_m}(\frac{5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{12})$
	Co	$\frac{b_m}{\eta_m}(\frac{5}{6}-\frac{x}{6}-\frac{1}{6}\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}})$
Retailer's service s_r		$\frac{b_r}{\eta_r}(\frac{7}{16}-\frac{\sqrt{3}}{48}x-\frac{1}{48}\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}})$
	Nash	$\frac{b_r}{\eta_r}(\frac{5}{12}-\frac{x}{12}-\frac{1}{12}\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}})$
		$\frac{b_r}{\eta_r}(\frac{5-x-\sqrt{2-3z-24\frac{y}{z}+\frac{2}{z}}}{6})$

4. Discussion of The results

This section provides a comparison of the optimal solutions between a cooperative game and two non-cooperative games. Due to the significant complexity involved in the computations, all comparisons are presented using the three values of $\nu = 1, 2$, and $\frac{1}{2}$ for the linear, convex, and concave price-demand curves, respectively. In order to be able to plot the comparison results, as an example the three values of k_r^2 ($k_r^2 = k_m^2$, $k_r^2 = 2k_m^2$ and $k_r^2 = 0.5k_m^2$) are considered for illustration. Other parameters are given as functions of k_m^2. Comparisons are conducted among price, national advertising spending, and profits in the aforementioned games. The results indicate that the retail price comparisons in the Nash, SR, and cooperative games align with those of the wholesale price, as well as the service levels of both the manufacturer and retailer, due to the direct correlation these variables share with retail price.

Similarly, the results of the comparison for national and local advertising expenditures are consistent. In all aspects of the aforementioned games, the price derived from the equilibrium solution yields higher retailer profits compared to the prices at both the beginning and end of the feasible price interval. Thus, it is demonstrated that the price obtained from solving the equations represents the equilibrium price of the games.

4. 1. Comparisons of the profits

In Figure 2, it is evident that the retailer's profit in the SR game surpasses that of the Nash game. Moreover, both the retailer's profit and the disparity between profits in the SR and Nash games rise as the values of ν increases. Also, the retailer's profit increases with increasing $k_r^2(k_m^2)$, or when the proportion of $\frac{\frac{b_r^2}{\eta_r}+\frac{b_m^2}{\eta_m}}{k_r^2+k_m^2}$ decreases. Because of the high complexity of the computations u and q are presented numerically for $\nu = 0.5$ to draw the plot.

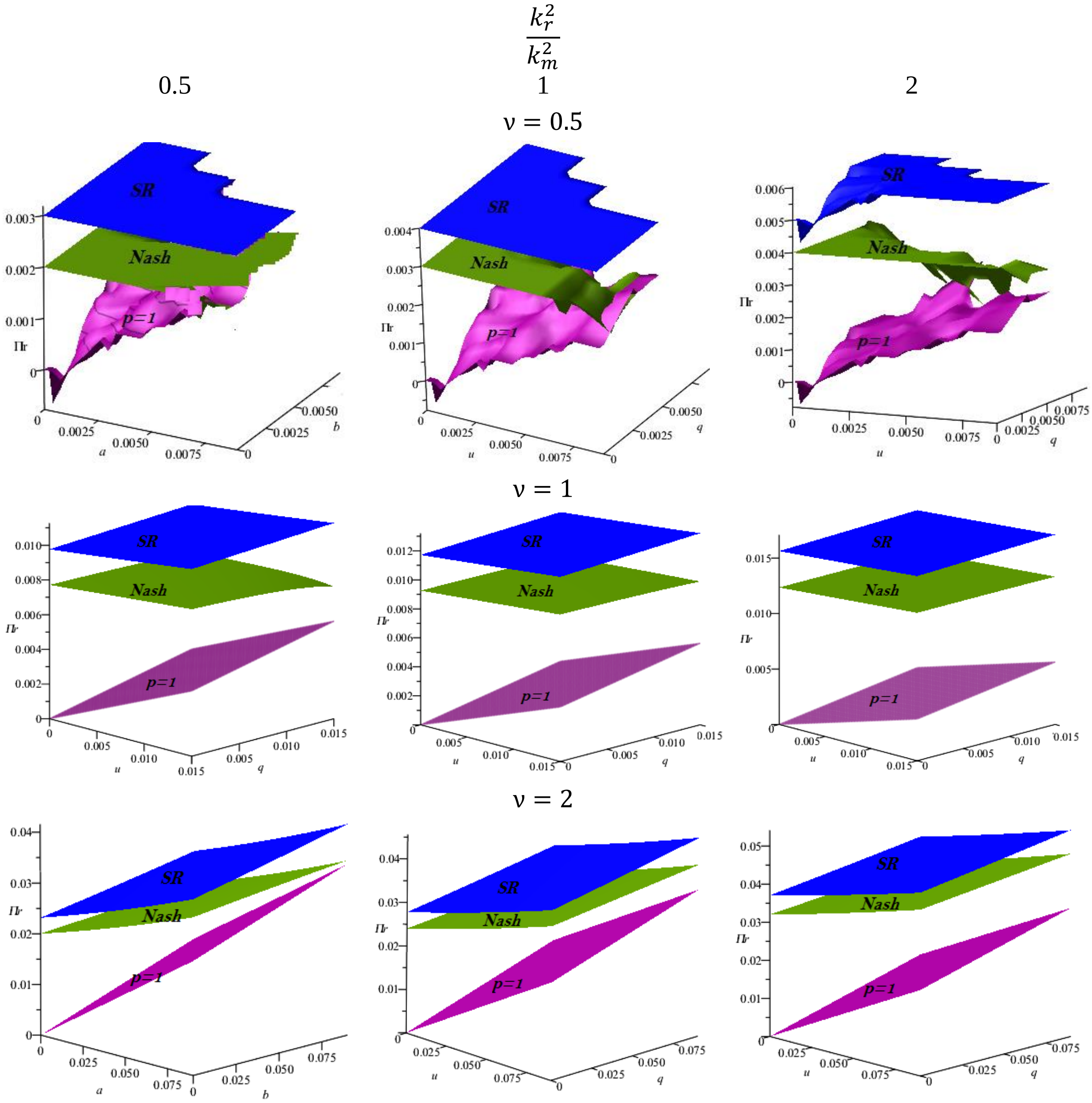
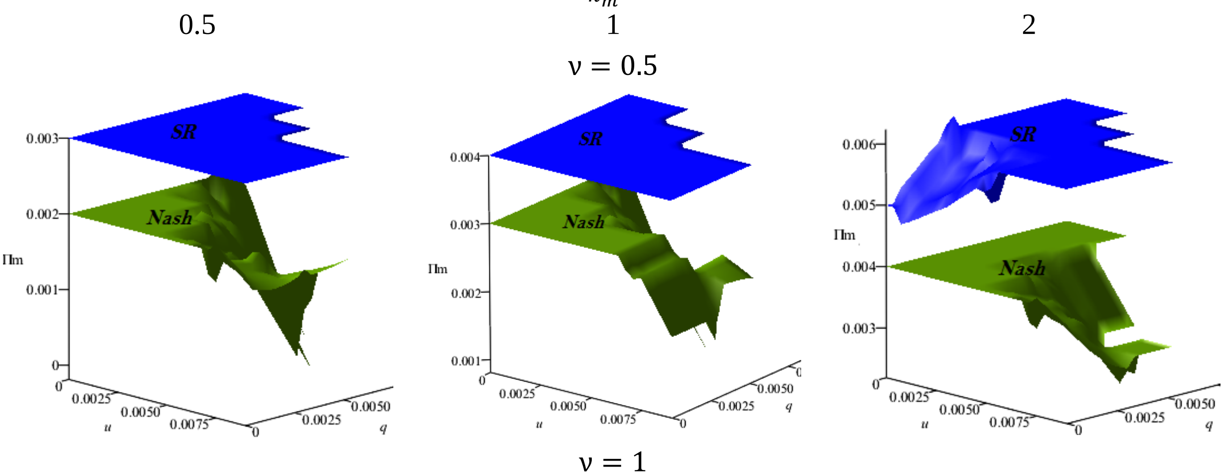


Fig. 2. Comparison of retailer profit.

The results comparing the manufacturer's profit function across the three games mirror those of the retailer's profit function, as illustrated in Figure 3.$\frac{k_r^2}{k_m^2}$



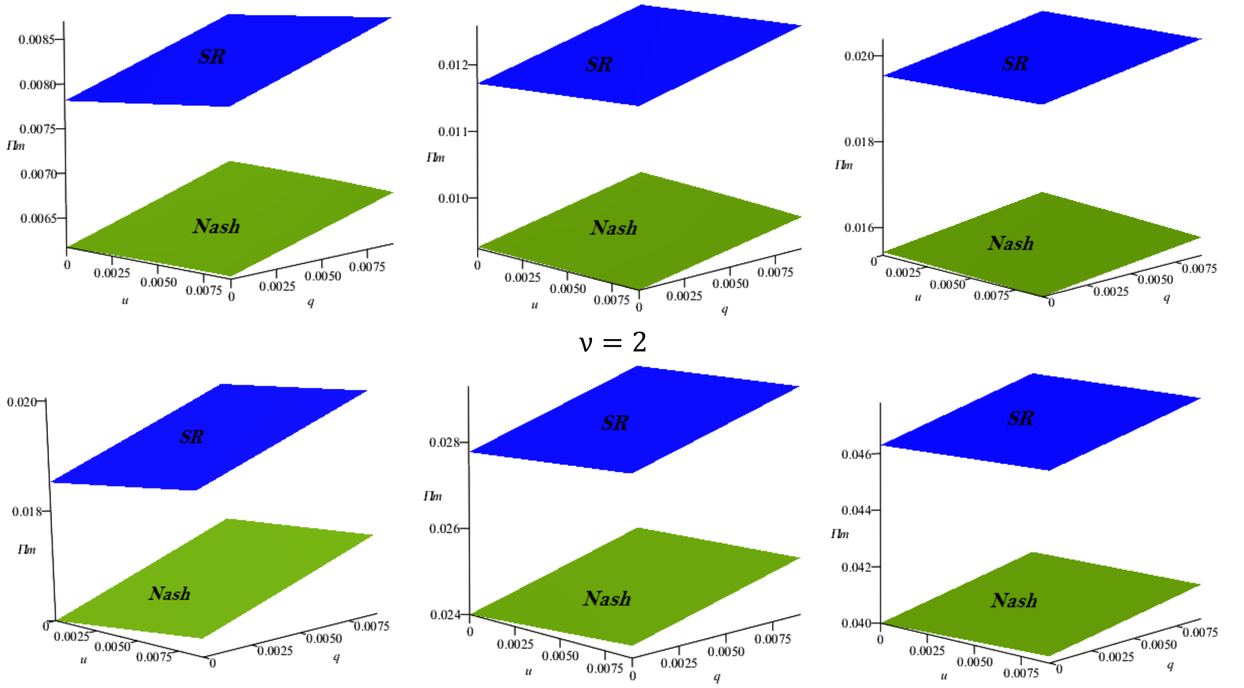


Fig. 3. Comparison of manufacturer profit.

$$\Pi^{SR}_{m+r} = \frac{3}{4}\Pi^{co}_{m+r} \rightarrow \Pi^{co}_{m+r} > \Pi^{SR}_{m+r} > \Pi^{N}_{m+r}$$

4. 2. Comparison of prices

Figure 4 shows that the price in the Nash game is higher than those in the cooperative and SR games. The prices in the SR and cooperative games are nearly identical and can be regarded as virtually equal. When ν increases, the price also increases. If k_r^2 (k_m^2) increases, i.e. if the proportion of $\frac{\frac{b_r^2}{\eta_r}+\frac{b_m^2}{\eta_m}}{k_r^2+k_m^2}$ decreases, the prices also decrease. Below is the outcome for the prices:

$p^N > p^{SR} \simeq p^{co} \rightarrow s_r^N > s_r^{SR} \simeq s_r^{co}$

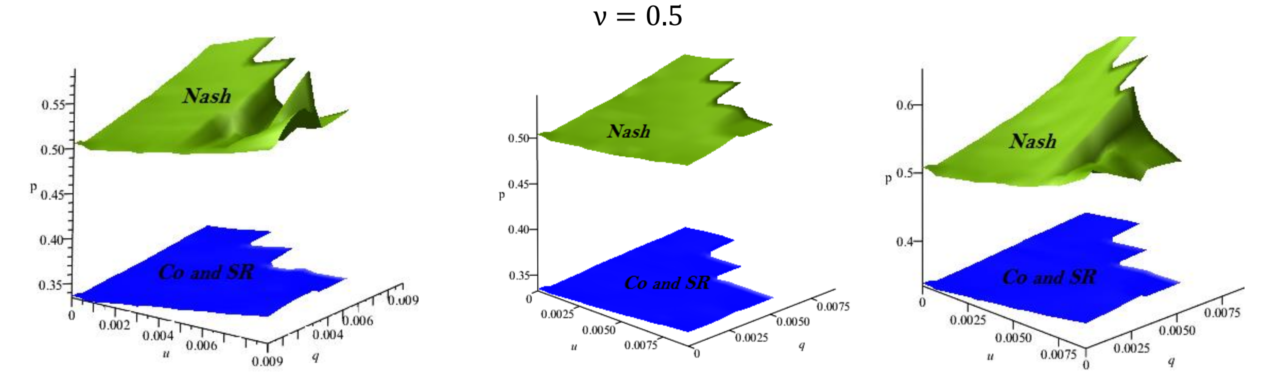
$s_m^N > s_m^{SR} \simeq s_m^{co}$

$w^N > w^{SR} \simeq w^{co}$

$\frac{k_r^2}{k_m^2}$

0.5 | 1 | 2

ν = 0.5



ν = 1

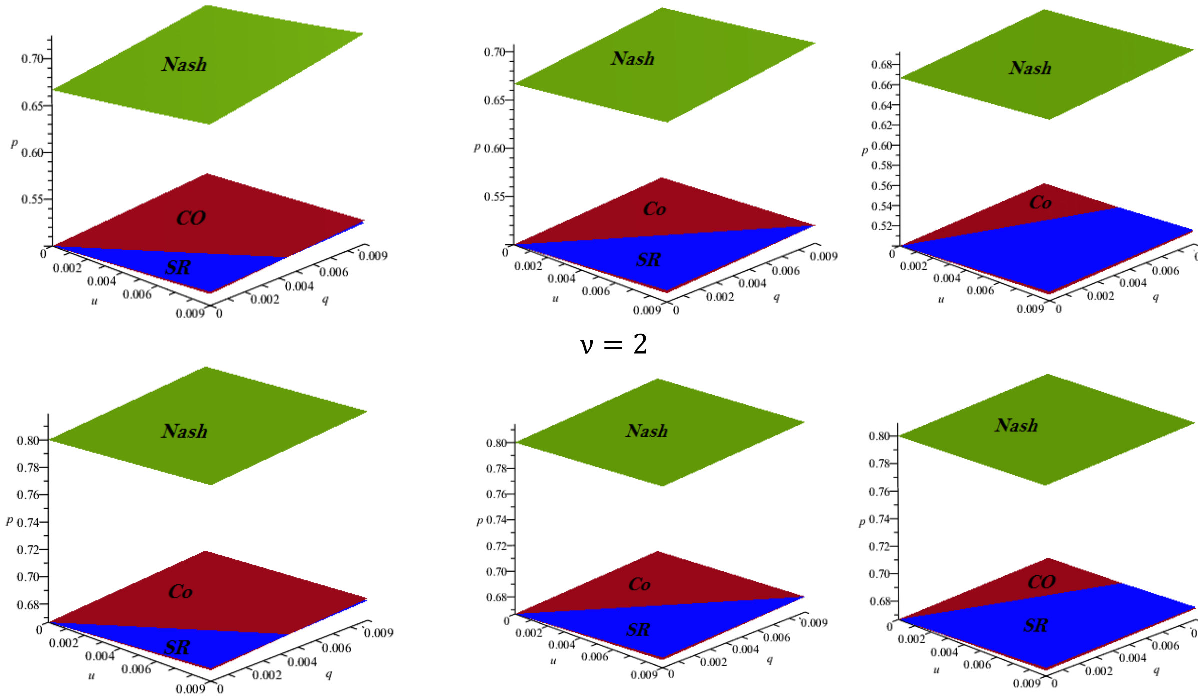


Fig. 4. Comparison of the prices

4. 3. Comparisons of advertising expenditures

Figure 5 illustrates a comparison of advertising expenditures, which are not influenced by the proportion of $\frac{k_r^2}{k_m^2}$.

$A^{co} > A^{SR} > A^N \rightarrow a^{co} > a^{SR} > a^N$

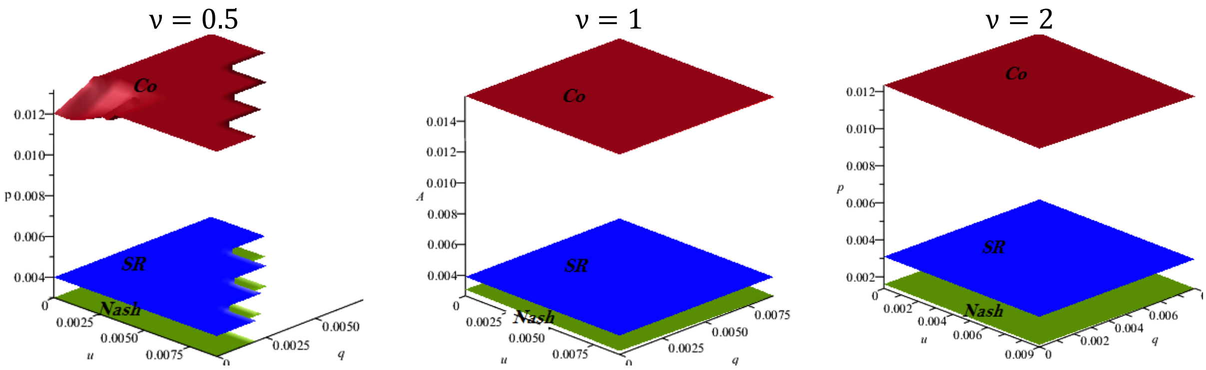


Fig. 5. Comparison of advertising expenditure

4.4. Feasibility of the cooperative game

The SR game generates higher profits for the participants compared to the Nash game. By comparing the profit results, we can confirm the viability of the cooperative game. For the game to be considered feasible, the following conditions need to be satisfied:

$$\Pi_m^{co} = \Pi_m(p^{co}, w^{co}, a^{co}, A^{co}) \geq \max(\Pi_m^{SR}, \Pi_m^N) = \Pi_m^{max} = \Pi_m^{SR} \tag{15}$$

$$\Pi_r^{co} = \Pi_r(p^{co}, w^{co}, a^{co}, A^{co}) \geq \max(\Pi_r^{SR}, \Pi_r^N) = \Pi_r^{max} = \Pi_r^{SR} \tag{16}$$

We integrate Eqs.:

$$\Pi_{m+r}^{co} = \Pi_m^{co} + \Pi_r^{co} \geq \Pi_m^{max} + \Pi_r^{max} = \Pi_m^{SR} + \Pi_r^{SR} \tag{17}$$

In the equation below, Δ represents the relative difference in profits between the cooperative and non−cooperative games. Given that this parameter has a positive value, the condition stated in Eq. (17) is satisfied, ensuring the existence of a feasible solution.

$$\Delta = \frac{\Pi_{m+r}^{co} - (\Pi_m^{SR} + \Pi_r^{SR})}{\Pi_{m+r}^{co}} \times 100 = \frac{\Pi_{m+r}^{co} - \Pi_{m+r}^{SR}}{\Pi_{m+r}^{co}} \times 100 = \frac{0.25\Pi_{m+r}^{co}}{\Pi_{m+r}^{co}} \times 100 = 25 > 0 \quad (18)$$

The viability of the cooperative game indicates that both the manufacturer and the retailer are inclined to collaborate. In the subsequent section, we will explore the Nash bargaining model for distributing the additional profits generated.

5. Bargaining Problem

In this section, we apply the Nash bargaining model similarly to the approach used by SeyedEsfahani et al. to assess how profits can be distributed among the members [63]. To begin, we will establish the feasible range for the variable *w*. The additional profits of the members are outlined below:

$$\Delta\Pi_m = \Pi_m^{co} - \Pi_m^{max} = w\left[(1-p^{co})^{\frac{1}{v}}\left(k_r\sqrt{a^{co}} + k_m\sqrt{A^{co}}\right) + (b_r s_r^{co} + b_m s_m^{co})\right] - A^{co} - \frac{\eta_m {s_m^{co}}^2}{2} - \Pi_m^{max} = wB - C > 0, \quad (19)$$

$$\begin{aligned}\Delta\Pi_m = \Pi_r^{co} - \Pi_r^{max} \\ &= (p^{co} - w)\left[(1-p^{co})^{\frac{1}{v}}\left(k_r\sqrt{a^{co}} + k_m\sqrt{A^{co}}\right) + (b_r s_r^{co} + b_m s_m^{co})\right] - a^{co} - \frac{\eta_r {s_r^{co}}^2}{2} - \Pi_r^{max} \quad (20) \\ &= -wB + D > 0,\end{aligned}$$

$$B = (1-p^{co})^{\frac{1}{v}}\left(k_r\sqrt{a^{co}} + k_m\sqrt{A^{co}}\right) + (b_r s_r^{co} + b_m s_m^{co}) > 0,$$

$$C = A^{co} + \frac{\eta_m {s_m^{co}}^2}{2} + \Pi_m^{max} > 0,$$

$$D = p^{co}B - a^{co} - \frac{\eta_r {s_r^{co}}^2}{2} - \Pi_r^{max} > 0,$$

The feasible interval for w is between the inequalities (**Error! Reference source not found.**) and (**Error! Reference source not found.**) and is $\frac{C}{B} < w < \frac{D}{B}$. The manufacturer gains more from the extra profit if the solution gets nearer to $\Pi_m = \Pi_m^{max}$ $(\frac{D}{B})$ and the retailer's share will be less, or vice versa.

According to Nash, the optimal values of *w* are found by maximizing the product of the members' utility function [67]. In our case, the utility function is assumed to be the same as the one used in SeyedEsfahani et al. [63]:

$U_m(w) = \Delta\Pi_m(w)^{\lambda_m}$

$U_r(w) = \Delta\Pi_r(w)^{\lambda_r}$

The parameter λ shows the member's attitude to the risk. If $\lambda = 1$, the player is indifferent to the risk; if $\lambda > 1$, the player is a risk seeker; and if $\lambda < 1$, he will be a risk averser. The members gain the profit according to their risk attitude. A higher risk seeking attitude leads to a higher profit. The Nash bargaining model is solved as follows:

$\text{Max } U_m(w)U_m(w) = \Delta\Pi_m(w)^{\lambda_m}\Delta\Pi_r(w)^{\lambda_r}$

The profit is divided with respect to λ.

$$\Delta\Pi_m(w^*) = \frac{\lambda_m}{\lambda_m + \lambda_r}\Delta\Pi = \frac{\lambda_m}{\lambda_m + \lambda_r}(D - c)$$

$$\Delta\Pi_r(w^*) = \frac{\lambda_r}{\lambda_m + \lambda_r}\Delta\Pi = \frac{\lambda_r}{\lambda_m + \lambda_r}(D - c)$$

$$\Rightarrow w^*B = \frac{C\lambda_m + D\lambda_r}{\lambda_m + \lambda_r}. \quad (21)$$

The optimal value for w can be obtained only if the other variable can be determined.

6. Conclusion

This study investigates a supply chain involving a single retailer and a single manufacturer, where pricing, advertising, and service strategies are implemented concurrently to affect customer demand. The optimal outcomes obtained from the Nash, Stackelberg (SR), and Cooperative game frameworks reveal that the retail price, wholesale price, and service levels from both the retailer and manufacturer are consistently elevated when the retailer assumes a leadership role or when the members collaborate. Furthermore, the decision variables in the SR and Cooperative games exhibit a strong correlation.

National and local advertising expenditures reach their peak in the Cooperative game, while they are at their lowest in the Nash game. In the Cooperative game, the retailer and manufacturer, as well as the entire system, attain greater profits. In contrast, the Nash game results in the lowest profits for all participants, indicating that cooperation is more advantageous. If cooperation is not an option, the manufacturer would rather adopt a following role behind the retailer than engage in the Nash game, where power is shared equally.

As ν increases, there is a corresponding rise in price levels, advertising expenditures, and service costs. This results in greater profits for both members and the overall system, while also amplifying the profit disparities across the three game scenarios.

This issue in multi-member or multi-channel supply chains presents a promising area for future research. Additionally, exploring other games or bargaining methods to address the same problem could be valuable. Finally, employing different demand functions may also provide new insights into the problem.

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Appendix

Proof of Proposition 1

Initially, the first partial derivatives of Π_m and Π_r with respect to their respective variables need to be calculated. After this step, all equations should be solved together. The outcome is negative, indicating that the maximum value occurs at the start of the interval.

$$\frac{\partial \Pi_m}{\partial w} = (1-p)^{\frac{1}{\nu}}\left(k_r\sqrt{a} + k_m\sqrt{A}\right) + (b_r s_r + b_m s_m) > 0 \rightarrow = w^* = \frac{p}{2} \tag{A. 1}$$

The outcome is positive, indicating that the maximum value occurs at the conclusion of the interval.

$$\frac{\partial \Pi_m}{\partial A} = \frac{1}{2} w(1-p)^{\frac{1}{\nu}} k_m A^{-\frac{1}{2}} - 1 \rightarrow \frac{\partial \Pi_m}{\partial A} = 0 \longrightarrow A^* = (\frac{1}{2} k_m w (1-p)^{\frac{1}{\nu}})^2 \tag{A. 2}$$

$$\frac{\partial \Pi_m}{\partial s_m} = w b_m - \eta_m s_m \rightarrow \frac{\partial \Pi_m}{\partial s_m} = 0 \longrightarrow s_m^* = \frac{w b_m}{\eta_m} \tag{A. 3}$$

$$\frac{\partial \Pi_r}{\partial a} = \frac{1}{2}(p-w)(1-p)^{\frac{1}{\nu}} k_r a^{-\frac{1}{2}} \rightarrow \frac{\partial \Pi_r}{\partial a} = 0 \longrightarrow a^* = (k_r (p-w)(1-p)^{\frac{1}{\nu}})^2 \tag{A. 4}$$

$$\frac{\partial \Pi_r}{\partial s_r} = (p-w) b_r - \eta_r s_r \rightarrow \frac{\partial \Pi_r}{\partial s_r} = 0 \longrightarrow s_r^* = \frac{(p-w) b_r}{\eta_r} \tag{A. 5}$$

$$\frac{\partial \Pi_r}{\partial p} = \frac{1}{\nu}(1-p)^{\frac{1}{\nu}-1}\left(k_r\sqrt{a} + k_m\sqrt{A}\right)(\nu - p(\nu+1) + w) + (b_r s_r + b_m s_m) \tag{A. 6}$$

To determine the optimal value of p, the equation mentioned above must be equated to zero. Given the complexity of solving this equation, three values of ν (i.e., 1, 2, and 0.5) representing the linear, convex, and concave shapes of the price-demand function, respectively, are survived.

$1.\nu = 1$

$$w^* = \frac{p}{2}$$

$$A^* = \left(\frac{1}{2}k_m w(1-p)\right)^2$$

$$s_m^* = \frac{wb_m}{\eta_m}$$

$$a^* = \left(\frac{k_r(p-w)(1-p)}{2}\right)^2$$

$$s_r^* = \frac{(p-w)b_r}{\eta_r}$$

$$p^* = \frac{1}{2}\left(1 + w + \frac{b_r s_r + b_m s_m}{k_r\sqrt{a} + k_m\sqrt{A}}\right)$$

All calculations are performed using Maple software, where the equations mentioned are solved simultaneously, and yielding three distinct values for p.

$$y = 2\frac{\frac{b_r^2}{\eta_r} + \frac{b_m^2}{\eta_m}}{k_r^2 + k_m^2} \left(y \le \frac{1}{48}\right)$$

$$p_1 = 0$$

$$p_2 = \frac{5}{6} - \frac{1}{6}\sqrt{1-48y}$$

$$p_3 = \frac{5}{6} + \frac{1}{6}\sqrt{1-48y}$$

One method to identify whether an extreme point is a maximum, minimum, or saddle point is by utilizing the Hessian matrix. If the odd-order minors are negative while the even-order minors are positive, this indicates that the Hessian matrix corresponds to a concave curve, leading to a maximum point. The Hessian matrix for the manufacturer demonstrates that their function is concave; confirming that the extreme point is indeed a maximum.

$$H(\Pi_m) = \begin{bmatrix} \frac{\partial^2 \Pi_m}{\partial A^2} & \frac{\partial^2 \Pi_m}{\partial A\,\partial s_m} \\ \frac{\partial^2 \Pi_m}{\partial s_m\,\partial A} & \frac{\partial^2 \Pi_m}{\partial {s_m}^2} \end{bmatrix} = \begin{bmatrix} -\frac{k_m}{4}A^{\frac{-3}{2}}w(1-p)^{\frac{1}{\nu}} & 0 \\ 0 & -\eta_m \end{bmatrix} \tag{A. 7}$$

The odd minor is negative:

$$-\frac{k_m}{4}A^{\frac{-3}{2}}w(1-p)^{\frac{1}{2}} < 0 \tag{A. 8}$$

The even minor is positive:

$$\frac{k_m}{4}A^{\frac{-3}{2}}w\,\eta_m(1-p)^{\frac{1}{2}} > 0 \tag{A. 9}$$

Determining whether the retailer's Hessian matrix is positive or negative poses a challenge.

$$H(\Pi_r) = \begin{bmatrix} \frac{\partial^2 \Pi_r}{\partial a^2} & \frac{\partial^2 \Pi_r}{\partial a\,\partial s_r} & \frac{\partial^2 \Pi_r}{\partial a\,\partial p} \\ \frac{\partial^2 \Pi_r}{\partial s_r\,\partial a} & \frac{\partial^2 \Pi_r}{\partial {s_r}^2} & \frac{\partial^2 \Pi_r}{\partial s_r\,\partial p} \\ \frac{\partial^2 \Pi_r}{\partial p\,\partial a} & \frac{\partial^2 \Pi_r}{\partial p\,\partial s_r} & \frac{\partial^2 \Pi_r}{\partial p^2} \end{bmatrix} \tag{A10}$$

$$
=\begin{bmatrix}
-\frac{k_r}{4}a^{\frac{-3}{2}}(p-w)(1-p)^{\frac{1}{\nu}} & 0 & \frac{k_r}{2}a^{\frac{-1}{2}}\left((1-p)^{\frac{1}{\nu}}-\frac{1}{\nu}(1-p)^{\frac{1-\nu}{\nu}}(p-w)\right) \\
0 & -\eta_r & b_r \\
\frac{k_r}{2\nu}a^{\frac{-1}{2}}(\nu-p(\nu+1)+w)(1-p)^{\frac{1-\nu}{\nu}} & b_r & \frac{(1-p)^{\frac{1}{\nu}-1}\left(k_r\sqrt{a}+k_m\sqrt{A}\right)}{\nu}\left((1-\frac{1}{\nu})(1-p)^{-1}(\nu-p(\nu+1)+w)-(\nu+1)\right)
\end{bmatrix}
$$

In certain areas of the feasible interval, Π_r is concave, and in another, it is convex. The second partial derivative of Π_m w.r.t. A and s_m is negative. The second partial derivative of Π_r w.r.t. a and s_r is negative, too; so, the extreme points show the maximum point. But $\frac{\partial^2\Pi_r}{\partial p^2}$ can vary between negative and positive. To establish which extreme point of p maximizes Π_r, we replace the optimal values of the variables obtained from the first partial derivative of Π_m and Π_r w.r.t its variables which maximize the profit of the members in the function of Π_r. The one p value should now be found which maximizes Π_r among the three values of p obtained by $\frac{\partial\Pi_r}{\partial p}=0$ should be found. After replacing the variables, Π_r is a quartic function of p and the coefficient of the biggest p is positive.

$$
\begin{aligned}
\Pi_r(p,a,s_r) &= (p-w)\left[(1-p)\left(k_r\sqrt{a}+k_m\sqrt{A}\right)+(b_r s_r+b_m s_m)\right]-a-\frac{\eta_r s_r^2}{2} \\
&= p^2(1-p)^2\left(\frac{2k_m^2+k_r^2}{16}\right)+p^2\left(\frac{b_m^2}{4\eta_m}+\frac{b_r^2}{8\eta_r}\right) \\
&= p^4\left(\frac{2k_m^2+k_r^2}{16}\right)-2p^3\left(\frac{2k_m^2+k_r^2}{16}\right)+p^2\left(\frac{b_m^2}{4\eta_m}+\frac{b_r^2}{8\eta_r}+\frac{2km^2+k_r^2}{16}\right)
\end{aligned}
\tag{A. 11}
$$

The shape of the quartic functions with positive coefficient of the biggest power will be as follows. Since we have three extreme points, the function shape is the right one in Fig. 6. After sorting the extreme points of p, the second root will be the relative maximum and the equilibrium can be found by comparing the relative maximum with the first and last points of the feasible interval of p (p=0 which leads to the minimum profit of the members and p=1).

Fig. 6. The shapes of the quartic functions with positive coefficient of the biggest power.

The following set defines the Nash equilibrium:

$$
p^N=\frac{5}{6}-\frac{1}{6}\sqrt{1-48y} \qquad w^N=\frac{5}{12}-\frac{1}{12}\sqrt{1-48y}
$$

$$
s_m^N=\frac{b_m}{\eta_m}\left(\frac{5}{12}-\frac{1}{12}\sqrt{1-48y}\right) \qquad A^N=\frac{k_m^2}{16}\left(\frac{5}{6}-\frac{1}{6}\sqrt{1-48y}\right)^2\left(\frac{1}{6}+\frac{1}{6}\sqrt{1-48y}\right)^2
$$

$$
s_r^N=\frac{b_r}{\eta_r}\left(\frac{5}{12}-\frac{1}{12}\sqrt{1-48y}\right) \qquad a^N=\frac{k_r^2}{16}\left(\frac{5}{6}-\frac{1}{6}\sqrt{1-48y}\right)^2\left(\frac{1}{6}+\frac{1}{6}\sqrt{1-48y}\right)^2
$$

The subsequent proofs adhere to the same reasoning outlined previously.

2. $\nu = 2$

In this case, two roots are identified as the extreme points for p are obtained.

$$
y=2\frac{\frac{b_r^2}{\eta_r}+\frac{b_m^2}{\eta_m}}{k_r^2+k_m^2}\ (y\leq 0.25) \qquad p_1=0 \qquad p_2=0.8+0.8y
$$

$$
\begin{aligned}
\Pi_r(p,a,s_r) &= (p-w)\left[(1-p)^{\frac{1}{2}}\left(k_r\sqrt{a}+k_m\sqrt{A}\right)+(b_r s_r+b_m s_m)\right]-a-\frac{\eta_r {s_r}^2}{2} \\
&= p^2(1-p)\left(\frac{2k_m^2+k_r^2}{16}\right)+p^2\left(\frac{b_m^2}{4\eta_m}+\frac{b_r^2}{8\eta_r}\right) \\
&= -p^3\left(\frac{2k_m^2+k_r^2}{16}\right)+p^2\left(\frac{b_m^2}{4\eta_m}+\frac{b_r^2}{8\eta_r}+\frac{2k_m^2+k_r^2}{16}\right)
\end{aligned}
\tag{A. 12}
$$

The cubic polynomial equation, featuring a negative coefficient for the third degree and two extreme points, will have the characteristics illustrated in Fig. 7. Therefore, after identifying the roots, the second one will represent the relative maximum, and the equilibrium can subsequently be determined by comparing it to the endpoint of the feasible interval. The Nash equilibrium set is presented below:

$$s_m^N = \frac{b_m}{\eta_m}(0.4 + 0.4y) \qquad w^N = 0.4 + 0.4y \qquad A^N = \frac{{k_m}^2}{16}(0.2 - 0.8y)(0.8 + 0.8y)^2$$

$$s_r^N = \frac{b_r}{\eta_r}(0.4 + 0.4y) \qquad p^N = 0.8 + 0.8y \qquad a^N = \frac{{k_r}^2}{16}(0.2 - 0.8y)(0.8 + 0.8y)^2$$

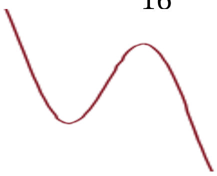

Fig. 7. The cubic polynomials function with the negative coefficient of the third degree and two extreme points.

3. $\boldsymbol{\nu = 0.5}$

Upon simultaneously solving the equations, we identify five extreme points for p as follows:

$$y = 2\frac{\frac{b_r^2}{\eta_r}+\frac{b_m^2}{\eta_m}}{k_r^2+k_m^2}\,(y \le 0.013) \qquad z = (108y + 12\sqrt{81y^2 - 6144y^3})^{\frac{1}{3}} \qquad X = \sqrt{3 + 4z + 384\frac{y}{z}}$$

$$p_1 = 0$$

$$p_2 = \frac{7}{8} - \frac{\sqrt{3}}{24}x - \frac{1}{24}\sqrt{18 - 12z - 1152\frac{y}{z} + \frac{18}{x}\sqrt{3}}$$

$$p_3 = \frac{7}{8} + \frac{\sqrt{3}}{24}x - \frac{1}{24}\sqrt{18 - 12z - 1152\frac{y}{z} - \frac{18}{x}\sqrt{3}}$$

$$p_4 = \frac{7}{8} - \frac{\sqrt{3}}{24}x + \frac{1}{24}\sqrt{18 - 12z - 1152\frac{y}{z} + \frac{18}{x}\sqrt{3}}$$

$$p_5 = \frac{7}{8} + \frac{\sqrt{3}}{24}x + \frac{1}{24}\sqrt{18 - 12z - 1152\frac{y}{z} - \frac{18}{x}\sqrt{3}}$$

These roots are quite complex; therefore, we will classify them by numbering them as y. Among them, two roots are irrational, while the remaining three are real roots.

Tab. A.1. The price value in the nash game obtained by numerating y for ν=0.5

y	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.010	0.011	0.012	0.013
p_1	0	0	0	0	0	0	0	0	0	0	0	0	0
p_2	0.504	0. 508	0.512	0.518	0.523	0.529	0.534	0.541	0.549	0.558	0.568	0.582	0.607
p_3	-	-	-	-	-	-	-	-	-	-	-	-	-
p_4	0.891	0.859	0.835	0.815	0.796	0.779	0.763	0.747	0.731	0.714	0.696	0.675	0.644
p_5	-	-	-	-	-	-	-	-	-	-	-	-	-

It can be inferred thatp_2 represents the relative maximum within the feasible interval.

$$\Pi_r(p,a,s_r)=\left(p-\frac{p}{2}\right)\left[(1-p)^2(km^2+kr^2)\left(\frac{p}{4}(1-p)^2\right)+\frac{p}{2}\left(\frac{b_m^2}{\eta_m}+\frac{b_r^2}{\eta_r}\right)\right] -\frac{p(1-p)^2}{4}kr^2-\frac{p^2}{8}\frac{b_r^2}{\eta_r}=0\rightarrow p^6\left(\frac{k_m^2+k_r^2}{8}\right)-p^5\left(\frac{k_m^2+k_r^2}{2}\right) +\frac{3}{4}p^4(k_m^2+k_r^2)-p^3\left(\frac{k_m^2+2k_r^2}{4}\right)+p^2\left(\frac{b_m^2}{4\eta_m}+\frac{b_r^2}{8\eta_r}+\frac{k_m^2+5k_r^2}{8}\right) -p\frac{k_r^2}{4} \tag{A. 13}$$

The sixth-degree polynomial with a positive leading coefficient can take one of the forms illustrated below: The curve of the polynomial mentioned above corresponds to the one depicted on the right side of Fig. 6, as it features three extreme points for p. So, p_2 is the relative maximum and the equilibrium should be determined by comparing it with the first point of the feasible interval of p (p=1). The optimal values are:

$$w^N=\frac{7}{16}-\frac{\sqrt{3}}{48}x-\frac{1}{48}\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}$$

$$A^N=\frac{k_m^2}{16}.\left(\frac{7}{8}-\frac{\sqrt{3}}{24}x-\frac{1}{24}\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}\right)^2(\frac{1}{8}+\frac{\sqrt{3}}{24}x +\frac{1}{24}\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}})$$

$$s_m^N=\frac{b_m}{\eta_m}.\left(\frac{7}{16}-\frac{\sqrt{3}}{48}x-\frac{1}{48}\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}\right)$$

$$p^N=\frac{7}{8}-\frac{\sqrt{3}}{24}x-\frac{1}{24}\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}$$

$$a^N=\frac{k_r^2}{16}.\left(\frac{7}{8}-\frac{\sqrt{3}}{24}x-\frac{1}{24}\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}\right)^2.(\frac{1}{8}+\frac{\sqrt{3}}{24}x +\frac{1}{24}\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}})$$

$$s_r^N=\frac{b_r}{\eta_r}.\left(\frac{7}{16}-\frac{\sqrt{3}}{48}x-\frac{1}{48}\sqrt{18-12z-1152\frac{y}{z}+\frac{18}{x}\sqrt{3}}\right).$$

□

The proofs for propositions 2 and 3 follow the same approach as that of proposition 1.

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Parinaz Esmaeili & Morteza Rasti Barzoki. Pricing, Advertising, and Service Decisions in a Manufacturer-Retailer Supply Chain with Nash, Stackelberg-Retailer, and Cooperative Games. IJIEPR 2024; 35 (4) :1-24
URL: http://ijiepr.iust.ac.ir/article-1-2108-en.html