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EXPECTED DURATION OF DYNAMIC MARKOV PERT NETWORKS

Amir Azaron and S.M.T. Fatemi Ghomi

Abstract*: In this paper, we apply the stochastic dynamic programming to approximate the mean project completion time in dynamic Markov PERT networks. It is assumed that the activity durations are independent random variables with exponential distributions, but some social and economical problems influence the mean of activity durations. It is also assumed that the social problems evolve in accordance with the independent semi-Markov processes over the planning horizon. By using the stochastic dynamic programming, we find a dynamic path with maximum expected length from the source node to the sink node of the stochastic dynamic network. The expected value of such path can be considered as an approximation for the mean project completion time in the original dynamic PERT network.*

Keywords*: Dynamic Programming, Stochastic Processes, Longest Path, Graph Theory*

1. Introduction

Project Scheduling has been a major objective of most models proposed to aid planning and management of projects. The most important method to schedule a project assuming deterministic durations is the well-known CPM – Critical Path Method. However, most durations have the random natures and therefore, PERT was proposed to determine the distribution of the total duration, *T*. This method is based on the substitution of the network by the CPAD – critical path assuming that each activity has a fixed duration equal to its mean (critical path using average durations). The mean and the variance of the CPAD are given by the sum of the means and of the variances of its activities, respectively, and therefore these results considered the mean and the variance of the total duration of the network.

This paper presents a new methodology to approximate the mean project completion time in dynamic Markov PERT networks. It is assumed that the activity durations are independent random variables with exponential distributions**,** but upon starting to do each activity, some social and economical problems like strike, war or inflation influence the mean of activity duration, and consequently its exponential distribution's parameter.

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Amir Azaron is with the Department of Industrial Engineering, University of Bu-Ali-Sina, Hamadan, a.azaron@dal.ca
S.M.T. Fatemi Ghomi is with the Department of Industrial Engineering, Amirkabir University of Technology, fatemi@aut.ac.ir

It is also assumed that the social problems evolve in accordance with the independent continuous-time Markov processes over the planning horizon.

By using the stochastic dynamic programming, we obtain the expected value of the dynamic longest path in the stochastic dynamic network, which would be approximately equal to the mean of project completion time in the original dynamic PERT network.

Although we could not find any paper about the longest path analysis in dynamic PERT networks, there are several papers about analysis of project completion time in PERT networks. Charnes, Cooper and Thompson [1] developed a chance-constrained programming.

They assumed exponential activity durations. For polynomial activity durations, Martin [2] provided a systematic way of analyzing the problem through series-parallel reductions. Kulkarni and Adlakha [3] developed an analytical procedure for PERT networks with independent and exponentially distributed activity durations. They modeled such networks as finite-state, absorbing continuous-time Markov chains with upper triangular generator matrices. Then, they proved the time until absorption into this absorbing state is equal to the length of the longest path in the original network provided it starts from the initial state. Elmaghraby [4] provided lower bounds for the true expected project completion time. Fulkerson [5], Clingen [6], Robillard [7] and Perry and Creig [8] have done the similar works. The analytical and approximation methods above are all static, because they consider the activity durations as the independent random variables and ignore their dependence on the dynamic behavior of social problems.There are also a few papers about the shortest path analysis in stochastic dynamic networks, see Hall [9], Bertsimas and Van Ryzin [10], Psaraftis and Tsitsiklis [11] and Azaron and Kianfar [12]. In this paper, we apply the model worked by Azaron and Kianfar [12] in order to find a dynamic path with maximum expected length in the dynamic Markov PERT network. The expected length of such path would be approximately equal to the mean project completion time in the dynamic PERT network. The true mean project completion time would be equal or greater than such estimate. Therefore, our approximation can be considered as a lower bound for the true expected project completion time. The remainder of this paper is organized in the following way. The analysis of Dynamic Markov PERT networks is illustrated in Section 2. In Section 3, the method is illustrated through solving a numerical example. Finally, we draw the conclusion of the paper in Section 4.

2. Analysis of Dynamic Markov PERT Networks

In this section, we present an approximation method to obtain the mean project completion time in Dynamic Markov PERT networks. Let $G=(V,A)$ be a PERT network, in which V and A represent the set of nodes and activities of G, respectively. The number of effective social and economical problems, which influence the mean of activity durations, is equal to N. Duration of activity $(l,j)\in A$ is an exponential random variable with parameter λ_{lj} . This parameter is a function of $(s^1,s^2,...,s^N)$ or the state vector of system in node l, which means the states of the social problems upon starting to do the activities originating from node l. The social problems evolve in accordance with the independent continuous-time Markov processes over the planning horizon. Clearly, the state vector of system is known only at the source node of the network, because, at the beginning of project, we know the initial states of the social problems. The other assumptions are as follows:

1. The number of states of ith social problem is equal to N_i (these states are in this order: $s_1^i, s_2^i, \ldots, s_{N_i}^i$), and $P_{m_ik_i}^i$ represents the probability of transition of this social problem from state $s_{m_i}^i$ to state $s_{k_i}^i$.

2. $t_{m_ik_i}^i$ represents the transition time of the ith social problem from state $s_{m_i}^i$ to state $s_{k_i}^i$. Each $t_{m_ik_i}^i$ is a random variable with exponential density function $f_{m_ik_i}^i(t)$, because it was assumed that each social problem evolves in accordance with a continuous-time Markov process. Then, if we consider $t_{m_i}^i$ as the staying time of the ith social problem in state $s_{m_i}^i$, its density function or $w_{m_i}^i(t)$ would be

$$w_{m_i}^i(t)=\sum_{k_i=1}^{N_i} P_{m_ik_i}^i f_{m_ik_i}^i(t) \tag{1}$$

3. All imbedded Markov processes corresponding to the indicated continuous-time Markov processes have the ergodic property.

4. $\phi_{m_ik_i}^i(t)$ represents the conditional probability that ith social problem moves to state $s_{k_i}^i$, given that at time zero, it was in state $s_{m_i}^i$.

How can a process that started by entering state $s_{m_i}^i$ at time zero be in state $s_{k_i}^i$ at time t. One way this can is for $s_{m_i}^i$ and $s_{k_i}^i$ to be the same state and for the process never to have left state $s_{m_i}^i$ throughout the period $(0,t)$. This requires that the process make its first transition after time t. Every other way to get from state $s_{m_i}^i$ to state $s_{k_i}^i$ in the interval $(0,t)$ requires that the process make at least one transition during that interval. For example, the process could have made its first transition from state $s_{m_i}^i$ to some state $s_{l_i}^i$ at a time τ , $0<\tau<t$, and then by some succession of transitions have made its way to state $s_{k_i}^i$ at time t. These considerations lead us to Eq. (2) for computing $\phi_{m_ik_i}^i(t)$:

$$\phi_{m_ik_i}^i(t)=\delta_{m_ik_i}\int_t^\infty w_{m_i}^i(\tau)d\tau+\sum_{l_i=1}^{N_i} P_{m_il_i}^i\int_0^t f_{m_il_i}^i(\tau)\phi_{l_ik_i}^i(t-\tau)d\tau ,$$

$$\delta_{m_ik_i}=\begin{cases}1 & if \quad m_i=k_i,\\ 0 & otherwise.\end{cases} \tag{2}$$

Certainly, we cannot directly compute $\phi_{m_ik_i}^i(t)$ from Eq. (2), but since the second integral of Eq. (2) is a convolution of two functions, we can compute $\phi_{m_ik_i}^i(t)$ by the Laplace transform. Let $f_{m_il_i}^{ie}(s)$ represent the Laplace transform of $f_{m_il_i}^i(t)$ and $\phi_{m_ik_i}^{ie}(s)$ represent the Laplace transform of $\phi_{m_ik_i}^i(t)$, which is computed from Eq. (3).

$$\phi_{m_ik_i}^{ie}(s)=\delta_{m_ik_i}\int_0^\infty\int_t^\infty e^{-st}w_{m_i}^i(\tau)d\tau dt+\sum_{l_i=1}^{N_i} P_{m_il_i}^i f_{m_il_i}^{ie}(s)\phi_{l_ik_i}^{ie}(s) \tag{3}$$

Now, we can compute $\phi_{m_ik_i}^i(t)$ by getting the inverse Laplace of $\phi_{m_ik_i}^{ie}(s)$, see Howard [13] for more details.

Let $\overline{P}_{m_1m_2\ldots m_N}^{ik_ilj}$ represent the conditional probability that after doing the activity (l,j), the state of ith social problem changes to $s_{k_i}^i$, given that upon starting to do this activity, the state of ith social problem and the state vector of system have been $s^i_{m_i}$ and $(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N})$, respectively.

Lemma 1. $\overline{P}^{ik_ilj}_{m_1m_2...m_N}$ is given by:

$$\overline{P}^{ik_ilj}_{m_1m_2...m_N} = \int_0^\infty \phi^i_{m_ik_i}(t)\lambda_{lj}(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N})e^{-\lambda_{lj}(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N})t}dt \qquad (4)$$

Proof. $\overline{P}^{ik_ilj}_{m_1m_2...m_N}$ is computed by conditioning on the duration of activity *(l,j)*. The probability of transition the *i*th social problem from state $s^i_{m_i}$ to state $s^i_{k_i}$ after a time *t*, upon finishing the activity *(l,j)*, given that the duration of activity *(l,j)* is equal to *t*, would be $\phi^i_{m_ik_i}(t)$, because the continuous-time Markov process corresponding to the transitions of *i*th social problem is memoryless. Since, the activity duration *(l,j)* has exponential distribution with parameter $\lambda_{lj}(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N})$, then, Lemma 1 is proved.

Let *A(l)* be the set of forward adjacent nodes of node *l*, and $V_l(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N})$ represent the maximum expected length form node *l* to the sink node of the dynamic PERT network, if the state vector of system in node *l*, which includes the states of all social and economical problems upon starting to do the activities *(l,j)*, $j \in A(l)$, is $(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N})$.

Theorem 1. $V_l(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N})$ for $i = 1,2,...,N$ and $m_i = 1,2,...,N_i$ is given by

$$V_l(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N}) = \max_{j \in A(l)} \left\{ \frac{1}{\lambda_{lj}(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N})} + \sum_{k_1=1}^{N_1}\sum_{k_2=1}^{N_2}...\sum_{k_N=1}^{N_N} \overline{P}^{1k_1lj}_{m_1m_2...m_N}\overline{P}^{2k_2lj}_{m_1m_2...m_N}...\overline{P}^{Nk_Nlj}_{m_1m_2...m_N} V_j(s^1_{k_1}, s^2_{k_2}, ..., s^N_{k_N}) \right\} \qquad (5)$$

Proof. The expected length of arc *(l,j)* is $\frac{1}{\lambda_{lj}(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N})}$, because the duration of activity *(l,j)* has exponential distribution with parameter $\lambda_{lj}(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N})$.

The probability that the state vector of system changes from $(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N})$ in node *l* to $(s^1_{k_1}, s^2_{k_2}, ..., s^N_{k_N})$, after doing the activity *(l,j)*, would be $\overline{P}^{1k_1lj}_{m_1m_2...m_N}\overline{P}^{2k_2lj}_{m_1m_2...m_N}...\overline{P}^{Nk_Nlj}_{m_1m_2...m_N}$, because the continuous-time Markov processes corresponding to the transitions of the social problems are independent. If the arc *(l,j)* belongs to the path with maximum expected length, which approximately considered as the longest path, then, the state vector of system in node *j* actually changes to $(s^1_{k_1}, s^2_{k_2}, ..., s^N_{k_N})$ with the indicated probability. Finally, by conditioning on the state vector of system in node *j*, Theorem 1 is proved.

Without losing the generality, we assume that the nodes of the network are numbered from *1* to *n*, in which there should be no directed path from node *j* to node *l* for *j>l*. The following algorithm can be used to approximate the mean project completion time in dynamic Markov PERT networks.

Algorithm

Step 1. Begin from node *l=n*. It is clear that $V_n(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N}) = 0$ for $m_i = 1,2,...,N_i$, $i = 1,2,...,N$.

Step 2. Set *l=n-1*. Then, compute $V_{n-1}(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N})$ for $m_i = 1,2,...,N_i$, $i = 1,2,...,N$, from the recursive function (5). In this case, the second term of (5) would be equal to *0*, because $A(n-1) = n$.

Step 3. Compute $\phi^i_{m_ik_i}(t)$ for $m_i = 1,2,...,N_i$, $k_i = 1,2,...,N_i$, $i = 1,2,...,N$, by getting the inverse Laplace of $\phi^{ie}_{m_ik_i}(s)$ in Eq. (3).

Step 4. Set *l=l-1*. Then, compute $\overline{P}^{ik_ilj}_{m_1m_2...m_N}$ for $m_i = 1,2,...,N_i$, $k_i = 1,2,...,N_i$, $i = 1,2,...,N$ and $j \in A(l)$, except for *j=n*, from Eq. (4).

Step 5. Compute $V_l(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N})$ for $m_i = 1,2,...,N_i$, $i = 1,2,...,N$, from the recursive function (5).

Step 6. If *l>1*, then, go to 4. Otherwise, go to 7.

Step 7. Stop. The approximation of mean project completion time would be equal to $V_1(s^1_{m_1}, s^2_{m_2}, ..., s^N_{m_N})$.

In the worst case, assume that we have a complete stochastic network, in which for each *l*, *A(l)={l+1,l+2,…,n}*. In this case, the time complexity of the algorithm in steps 2 and 3 would be clearly $O\left(\prod_{i=1}^N N_i\right)$ and $O\left(\sum_{i=1}^N {N_i}^2\right)$, respectively. The time complexity of the algorithm in step 4 is $O\left(\frac{(n-2)(n-1)}{2}\prod_{i=1}^N N_i \sum_{i=1}^N N_i\right)$, because in each node *l=1,2,…,n-2,* there are $(n-l-1)\prod_{i=1}^N N_i \sum_{i=1}^N N_i$ combinations for all values of $\overline{P}^{ik_ilj}_{m_1m_2...m_N}$, and $\sum_{l=1}^{n-2}\left((n-l-1)\prod_{i=1}^N N_i \sum_{i=1}^N N_i\right) = \frac{(n-2)(n-1)}{2}\prod_{i=1}^N N_i \sum_{i=1}^N N_i$.

With the same reason, the time complexity of the algorithm in step 5 would be $O\left(\frac{(n-2)(n+1)}{2}\prod_{i=1}^N N_i\right)$.

Therefore, in the worst case, the time complexity of the algorithm in step 3 is polynomial, but the time complexity of the algorithm in steps 2, 4 and 5 would be exponential. In practice, the stoshatic network is not complete, and there is also a limited number of effective social and economical problems in real world problems. Therefore, this algorithm would be efficient for approximating the mean project completion time in dynamic Markov PERT networks.

3. Numerical Example

Consider the dynamic PERT network depicted in Fig. 1. Strike and inflation rate are two social and economical problems, which influence the activity durations. These problems evolve in accordance with two independent continuous-time Markov processes over the planning horizon. Strike has two states, in which s_1^1 refers to existing and s_2^1 refers to non-existing the strike, in the society. Inflation rate has also two states, in which s_1^2 refers to high inflation rate and s_2^2 refers to low inflation rate. The activity durations are independent random variable with exponential distributions, but the social and econimical problems infulence their parameters, upon starting to do these activities. These parameters can be estimated from the previous data related to the similar activities, which have done before in similar conditions. Table 1 shows the values of these parameters (time unit is in year). The objective is to approximate the mean project completion time in this dynamic Markov PERT network.

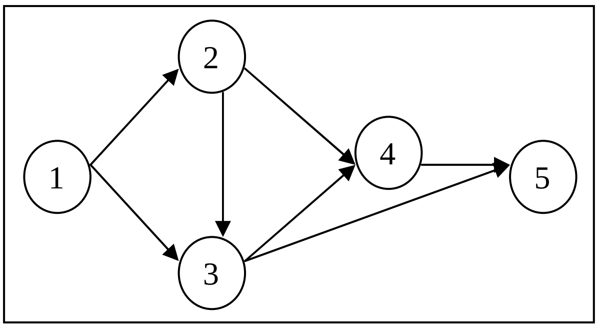

Fig 1. The Dynamic Markov PERT Network

The transition matrices are

$$P^1 = \begin{bmatrix} 0 & 1 \\ 0.4 & 0.6 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}.$$

It is also assumed that $f^i_{m_ik_i}(t)$ for m_i=1,2, k_i=1,2, *i=1,2*, are as follows:

$$f^1_{11}(t) = f^1_{12}(t) = f^1_{21}(t) = f^1_{22}(t) = e^{-t} \quad t>0$$

$$f^2_{11}(t) = 4e^{-4t} \quad t>0$$

$$f^2_{12}(t) = 2e^{-2t} \quad t>0$$

$$f^2_{21}(t) = 3e^{-3t} \quad t>0$$

$$f^2_{22}(t) = e^{-t} \quad t>0$$

Table 1. Parameters of activity durations

	λ_{12}	λ_{13}	λ_{23}	λ_{24}	λ_{34}	λ_{35}	λ_{45}
(s_1^1, s_1^2)	1	2	3	5	1	3	4
(s_1^1, s_2^2)	2	3	4	6	4	4	6
(s_2^1, s_1^2)	4	4	7	7	3	5	8
(s_2^1, s_2^2)	6	5	9	10	7	8	10

According to step 1 of the proposed algorithm, $V_5(s^1_{m_1}, s^2_{m_2}) = 0$ for $m_i = 1,2$, $i = 1,2$. Then, we go to step 2, set *l=4*, and compute $V_4(s^1_{m_1}, s^2_{m_2})$ for $m_i = 1,2$, $i = 1,2$, from (5). Then, we go to step 3 and compute $\phi^i_{m_ik_i}(t)$ for $m_i = 1,2$, $k_i = 1,2$, $i = 1,2$, by getting the inverse Laplace of $\phi^{ie}_{m_ik_i}(s)$ in (3). The results are as follows:

$\phi^1_{11}(t) = 0.29+0.71e^{-1.4t}$, $\phi^1_{12}(t) = 1-\phi^1_{11}(t)$

$\phi^1_{21}(t) = 0.29-0.29e^{-1.4t}$, $\phi^1_{22}(t) = 1-\phi^1_{21}(t)$

$\phi^2_{11}(t) = 0.36+0.73e^{-0.89t}-0.09e^{-2.81t}$, $\phi^2_{12}(t) = 1-\phi^2_{11}(t)$

$\phi^2_{21}(t) = 0.36-0.06e^{-0.89t}-0.3e^{-2.81t}$, $\phi^2_{22}(t) = 1-\phi^2_{21}(t)$

Then, we go to step 4, set *l=3*, and compute $\overline{P}^{ik_ilj}_{m_1m_2\ldots m_N}$ for $m_i = 1,2$, $k_i = 1,2$, $i = 1,2$ and $j = 4$, from (4). Taking into account the results, we go to step 5 and compute $V_3(s^1_{m_1}, s^2_{m_2})$ for $m_i = 1,2$, $i = 1,2$, from (5).

This process continuous until *l=1*. In this stage, the state vector of system is actually known and assumed to be (s_2^1, s_2^2), and $V_1(s_2^1, s_2^2) = 0.597$. So, the approximation of mean project completion time in the dynamic Markov PERT network of Fig. 1 is equal to *0.597* year.

4. Conclusion

In this paper, we developed an algorithm based on semi-Markovian decision processes and network flows theory to find a dynamic path with maximum expected length from the source node to the sink node of dynamic Markov PERT networks. The expected length of such path is approximately equal to the mean project completion time in the dynamic PERT network.

Unfortunately, our method has the same disadvantages of the classical PERT method, and the true mean project completion time would be equal or greater than such estimate. Therefore, our approximation can be considered as a good lower bound for the true mean project completion time. Another limitation of this model is that the time complexity of some steps of the proposed algorithm, in the worst case examples, would be exponential. In practice, there is a limited number of effective social and economical problems. Therefore, our algorithm would be an efficient algorithm for approximating the mean project completion time in real situation PERT networks.

Our methodology can be extended to general PERT networks. We can also consider the choice of temporary stopping the activities originating from a node, when the situation of social problems in that node is not good and therefore these activities would have the long durations. In this case, it might be better to stop doing these activities until the next transitions of the social problems, for encountering more favorable conditions. This new model can be easily developed, by extending the model of Azaron and Kianfar [12].

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