



Designing a Single Stage Acceptance Sampling Plan Based on the Control Threshold Policy

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ABSTRACT

In this research, a new control policy for the acceptance sampling problem is introduced. Decision is made based on the number of defectives items in an inspected batch. The objective of the model is to find a constant control level that minimizes the total costs, including the cost of rejecting the batch, the cost of inspection and the cost of defective items. The optimization is performed by approximating the negative binomial distribution with Poisson distribution and using the properties of binomial distribution. A solution method along with numerical demonstration on the application of the proposed methodology is presented. Furthermore, the results of sensitivity analysis show that the proposed method needs a large sample size.

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1. Introduction

Acceptance sampling is a procedure used for sentencing incoming batches. Sampling plan consist of a sample size and a decision making rule. The sample size is the number of items to sample or the number of measurements to take. The decision making rule involves the acceptance threshold and a description of how to use the sample result to accept or reject the lot. Acceptance sampling plans are also practical tools for quality control applications, which involve quality contracting on product orders between the vendor and the buyer.

Those sampling plans provide the vendor and the buyer rules for lot sentencing while meeting their preset requirements on product quality. Various methods of inspection to improve the quality of items are presented by many researchers. Klassen [1] proposed a credit-based acceptance sampling system in which the credit of the producer was defined as the total number of items accepted since last rejection. Tagaras [2] studied the process control and machine maintenance problem of a Markovian deteriorating machine. Kuo [3] developed an adaptive control policy for machine

maintenance and product quality control. Ferrell and Chhoker[4] proposed an economically acceptance sampling plan based on Taguchi loss function to quantify deviations between a quality characteristic and its target level.

Pearn and Wu [5] introduced a variable sampling plan for unilateral processes based on the one-sided process capability indices. Niaki and Fallahnezhad [6] used the Bayesian inferences concept to design an optimal sampling plan.

They formulated the problem into a stochastic dynamic programming model to minimize the ratio of total discounted system cost to a discounted system correct choice probability.

Moskowitz and Tang [7] proposed acceptance-sampling plans based on Taguchi loss function and Bayesian approach. Aminzadeh [8] proposed acceptance-sampling plans based on the assumption that consecutive observations on a quality characteristic are auto correlated. He obtained the sampling plans based on the ARMA model and suggested two types of acceptance sampling plans: (1) non-sequential acceptance sampling and (2) sequential acceptance sampling based on the concept of sequential probability ratio test (SPRT). McWilliams et al. [9] provided a method of finding exact designs for single sample acceptance sampling plans. William et al. [10] developed mathematical models that can be

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used to design both 100% inspection and single sampling plans.

In their research, inspection error is explicitly included in the model as is the ability to mitigate the consequences by the expending resources.

In this research, a new policy for the acceptance sampling problem is introduced. The objective of the model is to find a constant control level that minimizes total costs, including the cost of rejecting the batch, the cost of inspection and the cost of defectives items. With attribute sampling plans, these accept/reject decision performs based on counting of the number of defective items in a batch.

The assumptions and derivations of the proposed method are presented in section 2, the solution algorithm along with numerical demonstration on the application is presented in section 3, and discussion and conclusion of the results are presented in sections 4.

2. Proposed Model

We suppose a batch of size n is received which its proportion of the defectives items is equal to p . For a batch of size n , random variable Y is defined as the number of inspected items and z is defined as the number of items classified as 'defective' after inspection. The number of inspected items has an upper threshold equal to m .

For $Y = 1, 2, \dots, m$ inspected items ($m \leq n$) the batch will be rejected if $x \leq z$ where x is the upper control level for batch acceptance. In the other words, when the number of defective items in the inspected items gets more than the control threshold x then decision making process stops and the batch is rejected.

The probability distribution function of Y is determined by the following equations,

$$\Pr\{Y\} = \begin{cases} \sum_{z=0}^x \Pr\{z\} = \\ \Pr\{z \leq x-1\} + \Pr\{z = x\} \\ = \sum_{z=0}^{x-1} \binom{m}{z} p^z (1-p)^{m-z} + \binom{m-1}{x-1} p^x (1-p)^{m-x} & Y = m \\ \binom{Y-1}{x-1} p^x (1-p)^{Y-x} & x \leq Y < m \end{cases} \quad (1)$$

In equations 1, $Y = m$ indicates that all items are inspected therefore, the number of defective items has been less than x or x_{th} defective item has been m_{th} inspected item. For the case $x \leq Y < m$, x_{th}

defective item has been Y_{th} inspected item thus in this case, the probability distribution function of Y follows a negative binomial distribution. The expected mean of the number of inspected items is determined as follows:

$$\begin{aligned} E[Y]_x &= m \sum_{z=0}^{x-1} \binom{m}{z} p^z (1-p)^{m-z} + \\ & m \binom{m-1}{x-1} p^x (1-p)^{m-x} + \\ & \sum_{Y=x}^{m-1} Y \binom{Y-1}{x-1} p^x (1-p)^{Y-x} \\ &= m \sum_{z=0}^{x-1} \binom{m}{z} p^z (1-p)^{m-z} + \\ & \sum_{Y=x}^m Y \binom{Y-1}{x-1} p^x (1-p)^{Y-x} \end{aligned} \quad (2)$$

$$\text{Since } \Pr\{Y\} = \binom{Y-1}{x-1} p^x (1-p)^{Y-x} \quad x \leq Y < m$$

is a negative binomial distribution thus using the approximation method of estimating negative binomial probabilities with Poisson distribution [11], following is concluded,

$$\Pr\{Y\} = \text{Poisson}(\lambda) = \frac{e^{-\lambda} \lambda^{Y-x}}{\Gamma(Y-x+1)} \quad (3)$$

where $\lambda = x \frac{1-p}{p}$ is the parameter of Poisson

distribution. In order to improve the accuracy of this approximation, m and x should be sufficiently large numbers. Using the above approximation method, following is concluded,

$$\begin{aligned} E[Y]_x &= \\ & m \sum_{z=0}^{x-1} \binom{m}{z} p^z (1-p)^{m-z} + \sum_{Y=x}^m Y \frac{e^{-\lambda} \lambda^{Y-x}}{\Gamma(Y-x+1)} \end{aligned} \quad (4)$$

Now, let P_x denotes the probability of rejecting the batch. The batch is rejected if the number of defective items is more than or equal to x thus the value of P_x is determined by the following equation,

$$P_x = \sum_{z=x}^m \binom{m}{z} p^z (1-p)^{m-z} \quad (5)$$

In order to calculate the total cost, including the cost of rejecting the batch, the cost of inspection and the cost of defective items, assume R is the cost of rejecting the batch, c is the inspection cost of one item and c' is the cost of one defective item, so the total cost, C_x , is determined by conditioning C_x on two events of rejecting or accepting the batch, thus the objective function is written as follows:

$$\begin{aligned} C_x &= E(C_x | \text{Reject the batch}) \\ &P(\text{Reject the batch}) + \\ &E(C_x | \text{Accept the batch}) \\ &P(\text{Accept the batch}) \\ &= P_x (R + cE[Y]_x) + \\ &(npc' + cE[Y]_x)(1 - P_x) \\ &= P_x R + npc'(1 - P_x) + cE[Y]_x \end{aligned} \quad (6)$$

Thus we have,

$$\begin{aligned} C_x &= P_x R + npc'(1 - P_x) + \\ &mc \sum_{z=0}^{x-1} \binom{m}{z} p^z (1-p)^{m-z} + c \sum_{Y=x}^m Y \frac{e^{-\lambda} \lambda^{Y-x}}{\Gamma(Y-x+1)} \\ &= R \sum_{z=x}^m \binom{m}{z} p^z (1-p)^{m-z} + \\ &\sum_{z=0}^{x-1} \binom{m}{z} p^z (1-p)^{m-z} (npc' + mc) + \\ &c \sum_{Y=x}^m Y \frac{e^{-\lambda} \lambda^{Y-x}}{\Gamma(Y-x+1)} \end{aligned}$$

In equation (6), $cE[Y]_x$ is the total cost of inspection and npc' is the total cost of defective items. The optimal value of x is determined by minimizing the value of objective function C_x . Using the optimization methods, it is concluded that,

$$\begin{aligned} \Delta C_x &= C_x - C_{x-1} = R \sum_{z=x}^m \binom{m}{z} p^z (1-p)^{m-z} + \\ &(mc + npc') \sum_{z=0}^{x-1} \binom{m}{z} p^z (1-p)^{m-z} + \\ &c \sum_{Y=x}^m Y \frac{e^{-\lambda} \lambda^{Y-x}}{\Gamma(Y-x+1)} = \\ &\left(R \sum_{z=x-1}^m \binom{m}{z} p^z (1-p)^{m-z} + \right. \\ &\left. - (mc + npc') \sum_{z=0}^{x-2} \binom{m}{z} p^z (1-p)^{m-z} + \right. \\ &\left. c \sum_{Y=x-1}^m Y \frac{e^{-\lambda} \lambda^{Y-(x-1)}}{\Gamma(Y-(x-1)+1)} + \right) \end{aligned} \quad (7)$$

To evaluate above equation, following equality is considered,

$$\begin{aligned} \sum_{Y=x}^m Y \frac{e^{-\lambda} \lambda^{Y-x}}{\Gamma(Y-x+1)} - \sum_{Y=x-1}^m Y \frac{e^{-\lambda} \lambda^{Y-(x-1)}}{\Gamma(Y-(x-1)+1)} &= \\ \sum_{Y=x}^m \frac{e^{-\lambda} \lambda^{Y-x}}{\Gamma(Y-x+1)} - m \frac{e^{-\lambda} \lambda^{m-(x-1)}}{\Gamma(m-(x-1)+1)} \end{aligned} \quad (8)$$

Since m is a sufficiently large number thus the value of $m \frac{e^{-\lambda} \lambda^{m-(x-1)}}{\Gamma(m-(x-1)+1)}$ is approximately equal to zero therefore it is concluded that,

$$\begin{aligned} \Delta C_x &= -R \binom{m}{x-1} p^{x-1} (1-p)^{m-(x-1)} + \\ &(mc + npc') \binom{m}{x-1} p^{x-1} (1-p)^{m-(x-1)} + \\ &c \sum_{Y=x}^m \frac{e^{-\lambda} \lambda^{Y-x}}{\Gamma(Y-x+1)} \\ &= (mc + npc' - R) \binom{m}{x-1} p^{x-1} (1-p)^{m-(x-1)} \\ &+ c \sum_{Y=x}^m \frac{e^{-\lambda} \lambda^{Y-x}}{\Gamma(Y-x+1)} \end{aligned} \quad (9)$$

To ensure that x minimizes the objective function (6), it is necessary to find the value of x that satisfies following inequalities:

$$\begin{aligned}\Delta C_{x+1} &= C_{x+1} - C_x > 0 \\ \Delta C_x &= C_x - C_{x-1} < 0\end{aligned}\quad (10)$$

Hence,

$$\begin{aligned}\Delta C_{x+1} &= \\ (mc + npc' - R) \binom{m}{x} p^x (1-p)^{m-x} \\ + c \sum_{Y=x+1}^m \frac{e^{-\lambda} \lambda^Y}{\Gamma(Y - (x+1) + 1)} > 0 \\ \Delta C_x &= \\ (mc + npc' - R) \binom{m}{x-1} p^{x-1} (1-p)^{m-(x-1)} \\ + c \sum_{Y=x}^m \frac{e^{-\lambda} \lambda^Y}{\Gamma(Y - x + 1)} < 0\end{aligned}\quad (11)$$

Since m is a sufficiently large number thus following is concluded,

$$\sum_{Y=x}^m \frac{e^{-\lambda} \lambda^Y}{\Gamma(Y - x + 1)} = \sum_{Y=x+1}^m \frac{e^{-\lambda} \lambda^Y}{\Gamma(Y - (x+1) + 1)} \quad (12)$$

Now If $mc + npc' < R$, then,

$$\begin{aligned}\binom{m}{x-1} p^{x-1} (1-p)^{m-(x-1)} &> \frac{c \sum_{Y=x}^m \frac{e^{-\lambda} \lambda^Y}{\Gamma(Y - x + 1)}}{(R - (mc + npc'))} \\ &> \binom{m}{x} p^x (1-p)^{m-x}\end{aligned}\quad (13)$$

Since with increasing the value of x the value of binomial distribution with parameters m and p decreases thus according to the properties of binomial distribution, it is concluded that $x > (m+1)p$ therefore, the optimal value of x is determined using the following formula,

$$x = \text{Min} \left\{ \begin{aligned} &x; x > (m+1)p; \\ &\frac{c \sum_{Y=x}^m \frac{e^{-\lambda} \lambda^Y}{\Gamma(Y - x + 1)}}{(R - (mc + npc'))} \\ &\binom{m}{x} p^x (1-p)^{m-x} \end{aligned} \right\} \quad (14)$$

Also The objective function, C_x , should be minimized regarding two constraints on Type-I and Type-II errors associated with the acceptance sampling plans. Type-I error is the probability of rejecting the batch when the nonconformity proportion of the batch is acceptable. Type-II error is the probability of accepting the batch when the nonconforming proportion of the batch is not acceptable. Then, in one hand, if $p = \delta_1$, the probability of rejecting the batch should be less than ε_1 . On the other hand, in case where $p = \delta_2$, the probability of accepting the batch should be less than ε_2 where δ_1 is the maximum acceptable level of the batch quality (Accepted Quality Level (AQL)) and δ_2 is the minimum allowable level of the batch quality (Lot Tolerance Proportion Defective (LTPD)) and ε_1 is the probability of Type-I error and ε_2 is the probability of Type-II error in making a decision, therefore, the optimal value of x is determined using the following formula,

$$x = \text{Min} \left\{ \begin{aligned} &x; x > (m+1)p; \\ &\frac{c \sum_{Y=x}^m \frac{e^{-\lambda} \lambda^Y}{\Gamma(Y - x + 1)}}{(R - (mc + npc'))} > \\ &\binom{m}{x} p^x (1-p)^{m-x} \\ &\sum_{z=x}^m \binom{m}{z} \delta_1^z (1-\delta_1)^{m-z} \leq \varepsilon_1 \\ &\sum_{z=0}^{x-1} \binom{m}{z} \delta_2^z (1-\delta_2)^{m-z} \leq \varepsilon_2 \end{aligned} \right\} \quad (15)$$

When $mc + npc' > R$, It is concluded that equation (16) is positive for all values of x so $x = 0$. In this case, if one defective item is found in an inspected sample then the batch would be rejected. In this case, the rejection cost R is less than the total cost of inspecting m items and the cost of defective items, hence rejecting the batch would be the optimal decision. However, in practice the rejection cost R is usually big enough so that, we overlooked that case.

$$\Delta C_x = (mc + npc' - R) \binom{m}{x-1} p^{x-1} (1-p)^{m-(x-1)} + c \quad (16)$$

3. Numerical Example

In this section, we give a numerical example to illustrate the formulas for calculating a constant control level of x that minimizes the total cost, including the cost of rejecting the batch, the cost of inspecting and the cost of defective items. In this example, we take the

batch size, batch rejection cost, inspection cost for one item and the cost of one defective item $n = 200$, $R = 1000$, $c = 7$, $c' = 30$ respectively where

$\delta_1 = 0.05$, $\delta_2 = 0.2$, $\varepsilon_1 = 0.05$, and $\varepsilon_2 = 0.1$. The optimal values of x for different values of m, p are shown in Table 1.

Tab. 1. The optimal value of x for different values of (m, p)

(m, p)	optimal value of x	Optimal Design	Type-I error Probability	Type-II error Probability	Lower Bound for the Expected Number of Inspected Items
(50,0.1)	7	(50,7)	0.011	0.01	38.80
(55,0.1)	7	(55,7)	0.056	0.024	28.84
(45,0.1)	No feasible solution	-	-	-	-
(40,0.1)	No feasible solution	-	-	-	-
(80,0.05)	8	(80,8)	0.046	0.005	76.27
(60,0.05)	7	(60,7)	0.029	0.03	58.11
(70,0.05)	8	(70,8)	0.023	0.02	68.36
(90,0.05)	9	(90,9)	0.036	0.003	86.11

Based on the result shown in Table 1, as the value of m increases, the optimal value of control threshold, x , increases. The optimal acceptance thresholds are presented in column 3. For instant, the design in row 7 of column 3 indicates that, where $m = 60, p = 0.05$, the batch size of 60 will be rejected if the number of defective items is 7 or more.

For $m = 70, p = 0.05$ (row 8), if 8 items out of 70 inspected items were defective then the batch will be rejected.

It means that when at least 11 percent of the batch was defective then the batch should be rejected. For $m = 90, p = 0.05$ (row 9), if 9 items out of 90 inspected items were defective, then the batch will be rejected, it means that when at least 10 percent of the

batch was defective then the batch should be rejected. It is concluded that selecting the value of sample size for inspection strongly affects on the final result of the proposed model. Figure 1 shows $\frac{x}{m}$ as a function of

m where $p = 0.05$. It is inferred from Figure 1 that by increasing the value of sample size, the value of $\frac{x}{m}$ converges to a constant number also since the derivation of the model was based on assuming a sufficiently large number for the sample size thus increasing the value of sample size could increase the accuracy of the model.

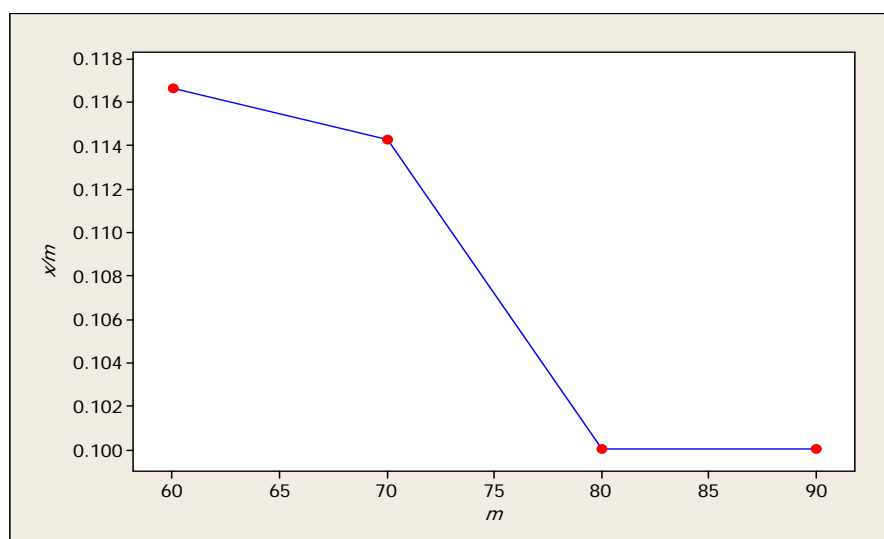


Fig. 1. Effect of changing sample size on $\frac{x}{m}$ where $p = 0.05$

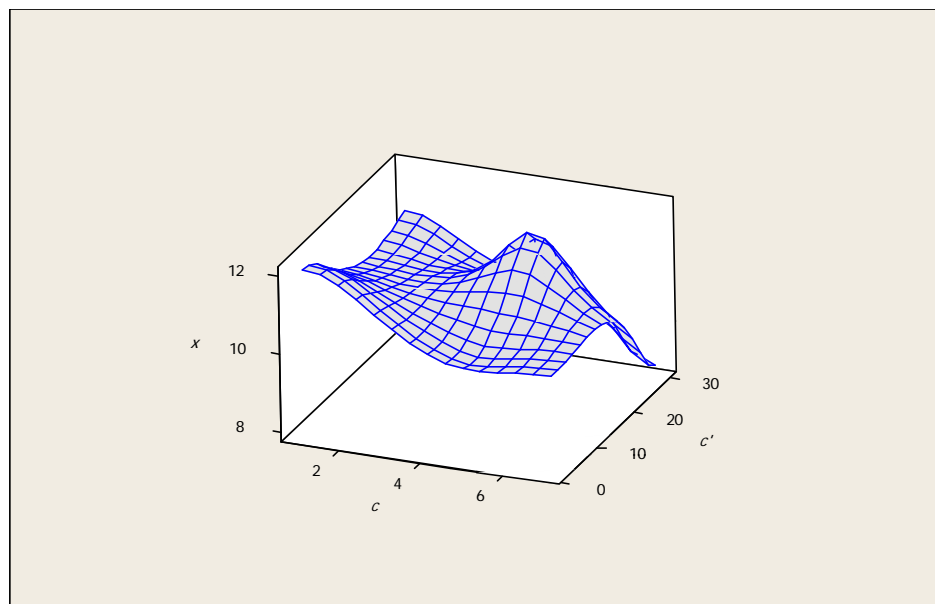
Tab. 2. The optimal value of x for different values of c and c' where $m = 80, p = 0.05$

(c, c')	Optimal value of x	Type-I error Probability	Type-II error Probability	Lower Bound for the Expected Number of Inspected Items
(1,2)	12	0	0.1	79
(1,30)	11	0	0.056	79
(5,5)	10	0.006	0.02	79
(5,30)	9	0.018	0.013	78
(5,20)	12	0	0.1	79
(4,30)	10	0.006	0.02	79
(7,30)	8	0.046	0.005	76

3.1. Sensitivity Analysis

A sensitivity analysis on the different values of c, c' for $m = 80, p = 0.05$ is carried out in this section. The results are shown in Table 2.

As it is clear in Table 2, by increasing the cost of a defective item and the cost of inspection, the optimal value of the control level, x decreases, as it was expected. Figure 2 shows the optimal value of control threshold as a function of c and c' .

Fig. 2. Effect of changing c and c' on x where $m = 80, p = 0.05$

3.2. Comparative Study

In this section, the performance of the proposed methodology is compared to the one using an acceptance-sampling plan, in which n items of a batch are inspected. In this plan, if the number of nonconforming items is less than a lower control threshold c_1 , the batch is accepted. If this number is more than a control threshold c_2 , the batch is rejected, and if the number of nonconforming items lies within the thresholds c_1 and c_2 , the process of inspecting the batch continues (Fallahnezhad and Niaki [12]).

Proposed sampling plan is compared with above sampling plan with sample size $n = 50$ and $p = 0.1$, based on the measures such as probabilities of Type-I and Type-II error and the expected number of inspected items.

Since $\sum_{Y=x}^m Y \frac{e^{-\lambda} \lambda^{Y-x}}{\Gamma(Y-x+1)} \leq m$ thus the lower and upper bound for the expected number of inspected items in proposed sampling method is determined by the following equations,

$$m \sum_{z=0}^{x-1} \binom{m}{z} p^z (1-p)^{m-z} + m \geq \quad (17)$$

$$E[Y]_x \geq m \sum_{z=0}^{x-1} \binom{m}{z} p^z (1-p)^{m-z}$$

Table 3 shows 12 different alternative combination values of c_1 and c_2 together with their probability of rejecting or accepting the batch, of which the ones in bold are feasible.

Based on the results, the best combination value is $c_1 = 4$ and $c_2 = 6$ with the minimum value of the expected number of inspected items of 75.64. Based on the results of Table 1, the lower bound and upper bound for the expected number of inspected items in proposed sampling method is 28.84 and 83.84 when $m = 55, p = 0.1$ that is a satisfactory performance with regards to the results of the second sampling plan.

Tab. 3. The probabilities of rejecting and accepting the batch in the second sampling plan

Expected Number of Inspected Items	Probability of rejecting the batch when $\delta_2 = 0.2$	Probability of accepting the batch when $\delta_1 = 0.05$	c_2	c_1
63.81686	1.00	0.54	3	1
119.7138	1.00	0.88	5	1
320.6548	1.00	0.99	7	1
73.47211	1.00	0.84	4	2
146.4121	1.00	0.98	6	2
294.8183	1.00	1.00	8	2
75.64622	0.98	0.99	6	4
102.2358	0.97	1.00	8	4
113.4937	0.96	1.00	10	4
60.37962	0.87	1.00	8	6
64.13698	0.80	1.00	10	6
64.83138	0.64	1.00	12	6

The results in Table 3 show that the second sampling plan is capable to provide sampling designs with sufficiently small probabilities of Type-I and Type-II error with a reasonable sample size but since the performance of proposed method is satisfactory and considering this fact that proposed sampling plan is a single stage sampling plan thus the results of the comparison studies are in favor of the proposed methodology.

4. Conclusion

A new model for the selection of cost minimizing single stage acceptance sampling plans has been presented. The relationship between the cost model and a decision theory model with binomial utilities has been investigated. However, the acceptance sampling plan, which is derived from the optimization of this model, may differ substantially from the plans that other economic approaches suggest but optimization of the model is simple and efficient, with negligible computational requirements.

Sensitivity analysis on the different values of sample size and rejection costs has shown that the optimal value of the control level decreases along with

increasing the cost of defective item and the cost of inspection. Also the results of the comparison studies are in favor of the proposed methodology.

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