A New Mathematical Approach based on Conic Quadratic Programming for the Stochastic Time-Cost Tradeoff Problem in Project Management

[1] M. Reza Peyghami, [2] Abdollah Aghaie, and [2] Hadi Mokhtari

1- Department of Applied Mathematics, K.N. Toosi University of Technology, P.O.Box 16315-1618, Tehran, Iran
and
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O.Box 19395-5746, Tehran, Iran

2- Department of Industrial Engineering, K.N. Toosi University of Technology, Tehran, Iran

Keywords

Project Management;
Conic Optimization;
Time-Cost Tradeoff Problem;
Interior Point Methods;
Monte Carlo Simulation

Abstract

In this paper, we consider a stochastic Time-Cost Tradeoff Problem (TCTP) in PERT networks for project management, in which all activities are subjected to a linear cost function and assumed to be exponentially distributed. The aim of this problem is to maximize the project completion probability with a pre-known deadline to a predefined probability such that the required additional cost is minimized. A single path TCTP is constructed as an optimization problem with decision variables of activity mean durations. We then reformulate the single path TCTP as a cone quadratic program in order to apply polynomial time interior point methods to solve the reformulation. Finally, we develop an iterative algorithm based on Monte Carlo simulation technique and conic optimization to solve general TCTP. The proposed approach has been tested on some randomly generated test problems. The results illustrate the good performance of our new approach.

1. Introduction

The Time-Cost Tradeoff Problem (TCTP) concerns a project scheduling problem where project total cost and project completion time are considered together. In scheduling a project, it is often important to expedite the duration of some activities through expending extra resources and therefore reduce the project duration with additional costs. This procedure can be conducted under either a fixed, available budget or a desirable threshold of project completion time. This problem is known as time/cost tradeoff problem, project crashing/compression problem and project expediting problem, in the project management literature. The main objective of the TCTP is to determine the optimal duration and cost that should be assigned to the activities such that the overall cost is minimized. This leads to a balance between the project completion time and the project total cost. For example, using additional resources, more productive equipments, highly skilled human resources or hiring more workers can save the time, but project direct cost could be increased. On the other hand, completing a project at a date greater than a desired due date may save some of budget or resources, but a penalty cost may be included. Equivalently, crashing an activity saves time but increases the activity's cost.

Many articles have investigated the deterministic TCTP, under various behaviors of the cost function such as discrete cost function [1,12,13,25,33], linear continuous cost function [10,17,29], nonlinear convex cost function [6,21], nonlinear concave cost function [15] and linear piecewise cost function [20,35].

Each pattern of the TCTP uses its own objective function for model formulation. Some studies, under deterministic assumptions, have tried to determine the economical duration of project completion time via minimizing project total cost, including direct, indirect, and penalty cost functions [9,11]. On the other hand, the importance of on-time project delivery has lead to the proposed model of TCTP which minimizes the project completion time [8,18,30]. The objective of these models is to determine the optimal allocation of limited budget and resources to the activities. Moreover, some authors have evaluated the TCTP, considering the multi criteria measures time and cost, concurrently [35].

There are also some other studies related to the application of optimal control theory in the stochastic multi objective resource allocation problem for PERT networks [2,3]. Using the time discretization process, Azaron *et al.* [4] developed a new analytical procedure based on the multi objective TCTP in order to achieve the minimum total direct costs, the minimum mean of project completion time and the minimum variance of project completion time. Eshtehardian *et al.* [14] applied a hybrid approach based on the Fuzzy logic and GA to solve a new pattern of Time Cost Optimization (TCO) in a non-deterministic environment by means of the Pareto front. Recently, Li and Wang [22] have proposed the application of Radial Basis Function (RBF) neural network to solve the multi-objective time-cost tradeoff problem considering the risk element, in dynamic PERT networks.

A number of research efforts have focused on TCTP modeling, under various assumptions, and applied an array of classic and computational methods to solve this important content of project management. But, in all of them, the conic formulation counterpart, which could be solved by polynomial time interior point methods, has not been investigated. The conic quadratic programming or Second-Order Cone Programming (SOCP) problem is to minimize or maximize a linear function over the intersection of an affine space with the Cartesian product of a finite number of second-order (Lorentz) cones. Linear programs, convex quadratic programs and quadratically constrained convex quadratic programs can all be formulated as SOCP problems, as can many other problems that do not fall into these three categories. Recently, this problem has received considerable attention for its wide range of applications (see e.g. [5,19,23,36]) and for being "easily" solvable in polynomial time via interior-point algorithms [26,27,28].

In this paper, we develop a new approach for solving a new stochastic model of TCTP based on SOCP formulation and Monte Carlo (MC) simulation technique. In our new model, all of the activities are subjected to linear cost functions and assumed to be exponentially distributed and the objective is to improve the project completion probability in a predefined due date based on a predefined probability. Our proposed model helps the project planner to manage the project completion time more accurately and prevent the project tardiness.

A new structure of the general MC simulation technique has been employed to solve our new model. In every iteration of the MC technique, by considering the mean durations of activities as decision variables, a nonlinear optimization formulation of the problem would be constructed. This optimization problem has been reformulated as SOCP model in order to solve it in polynomial time interior point methods. Therefore, the following hybrid procedure based on MC simulation and SOCP problem has been applied to allocate the optimal cost to the activities:

- Choosing the Most Critical Path (MCP) by using MC simulation technique.
- Using SOCP formulation to maximize the path completion probability for the selected MCP in previous step.

The paper is organized as follows: The mathematical model is presented in Section 2.

Section 3 introduces the SOCP reformulation of the proposed single path TCTP. A general algorithm based on the SOCP problem and Monte Carlo simulation technique is proposed in Section 4 in order to apply the proposed approach for all paths. Section 5 includes the characteristics of illustrative example and the results of applying the proposed approach to it. Some randomly generated examples have also been considered in Section 5. Finally, concluding remarks are presented in Section 6.

2. Mathematical model

We use an Activity On Arrow (AOA) representation of project scheduling networks. Let $G=(V,A)$ be an acyclic AOA graph with arrow set A and node set V, where the source and sink node are denoted by s and t, respectively. In a project PERT network with m nodes and n activities, $V=(v_1,v_2,\ldots,v_m)$ represents the set of events and $A=(a_1,a_2,\ldots,a_n)$ represents the set of activities. The following notations are used throughout the paper:

(i,j): Activity with head node i and tail node j

t_{ij} : Random variable of activity (i,j) duration

x_{ij} : The parameter of exponential distribution of activity (i,j) duration (decision variable)

μ_{ij} : Mean duration of activity (i,j)

σ_{ij} : Standard deviation of activity (i,j)

s_{ij} : Cost slope of activity (i,j)

T_d : Project completion due date

α : Desired amount of project completion probability

u_{ij} : Upper limit of mean duration of activity (i,j)

l_{ij} : Lower limit of mean duration of activity (i,j)

L : Total number of paths.

n_r : Number of activities on path r

T_r : Random variable of path r duration

μ_r : The mean duration of path r

σ_r : The standard deviation of path r

Assuming sufficiently large number of activities on paths, using Central Limit Theorem, the probability of falling the completion time of a certain path r in interval

$(0, T_d)$, is

$$\phi_r(Z) = \Pr(T_r < T_d) = \Pr\left(Z < \frac{T_d - \mu_r}{\sigma_r}\right)$$

where

$$\mu_r = \sum_{ij \in r} \mu_{ij}$$

$$\sigma_r = \sqrt{\sum_{ij \in r} \sigma_{ij}^2}$$

According to the assumed distribution on activity durations, we have:

$$\mu_{ij} = \sigma_{ij} = x_{ij}$$

Thus,

$$\mu_r = \sum_{ij \in r} x_{ij}$$

$$\sigma_r = \sqrt{\sum_{ij \in r} x_{ij}^2}$$

It is assumed that the distributions of activity durations are exponential with mean and standard deviation of x_{ij}, for $ij \in A$. Adding extra cost to the activity (i, j) decreases the mean and variance of activities x_{ij} with a linear cost function and cost slopes of s_{ij}.

After allocating the additional cost to the activities which lie on path r, new expected value of activities will be x_{ij} and new probability of meeting the predefined α is:

$$\phi(Z) = \Pr\left(Z < \frac{T_d - \mu_r}{\sigma_r}\right) = \Pr\left(Z < \frac{T_d - \sum_{ij \in r} x_{ij}}{\sqrt{\sum_{ij \in r} x_{ij}^2}}\right) \geq \alpha \tag{1}$$

We have to note that the equation (1) is key constraint of proposed model. It warrants prevention of tardiness by increasing the probability of meeting the predefined deadline. The following lemma provides an optimization model for increasing project completion probability on a certain path with minimum cost.

Lemma 1. For a given path r, the project completion probability on path r in a predefined due date T_d based on a predefined probability α with minimum cost can be improved by solving the following optimization problem:

$$\min \quad z = \sum_{ij \in r} s_{ij}\left(u_{ij} - x_{ij}\right)$$

$$\text{s.t.} \quad Z_{1-\alpha} \left\| \left(x_{ij}\right)_{ij \in r} \right\| \leq T_d - \sum_{ij \in r} x_{ij}$$

$$l_{ij} \le x_{ij} \le u_{ij} \qquad ij \in r$$

where $Z_{1-\alpha}$ denotes a point of normal standard distribution which covers a probability with amount of α in its left side, and $\|.\|$ denotes the 2-norm (Euclidean vector norm).

Proof. Using the probability principles, we can rewrite the probability of meeting the predefined α through the path r, which is described as a probabilistic inequality in (1), as a non probabilistic inequality in the following form:

$$\frac{T_d - \sum_{ij \in r} x_{ij}}{\sqrt{\sum_{ij \in r} x_{ij}^2}} \ge Z_{1-\alpha} \tag{2}$$

where $Z_{1-\alpha}$ denotes a point of normal standard distribution which covers a probability with amount of α in its left side. Thus, in order to minimize the additional costs allocated to the activities on path r, the cost function $z = \sum_{ij \in r} s_{ij}(u_{ij} - x_{ij})$ should be minimized in presence of the inequality (2) and the upper and lower bound limitation on activities. Therefore, the proposed model for the path r can be formulated as:

$$\min \quad z = \sum_{ij \in r} s_{ij}(u_{ij} - x_{ij})$$

$$\text{s.t.} \quad \frac{T_d - \sum_{ij \in r} x_{ij}}{\sqrt{\sum_{ij \in r} x_{ij}^2}} \ge Z_{1-\alpha} \tag{3}$$

$$l_{ij} \le x_{ij} \le u_{ij} \qquad ij \in r$$

By rewriting the inequality (3) as

$$(T_d - \sum_{ij \in r} x_{ij}) \ge Z_{1-\alpha} \sqrt{\sum_{ij \in r} x_{ij}^2}$$

the main model for the path r can be reformulated as:

$$\min \quad z = \sum_{ij \in r} s_{ij}(u_{ij} - x_{ij}) \tag{4}$$

$$\text{s.t.} \quad Z_{1-\alpha} \sqrt{\sum_{ij \in r} x_{ij}^2} \le T_d - \sum_{ij \in r} x_{ij} \tag{5}$$

$$l_{ij} \le x_{ij} \le u_{ij} \qquad ij \in r$$

This completes the lemma using the definition of the Euclidean vector norm.

Remark 1. It can be easily seen that the problem (4)-(5) is infeasible when the parameter T_d satisfies the following inequality:

$$T_d < l_{\min}\left(Z_{1-\alpha}\sqrt{n_r} - n_r\right)$$

where $l_{\min} = \min\{l_{ij} \mid ij \in r\}$. Thus, in order to have feasible problem on path r, we assume that at least $x = l$ satisfies the inequality (5), i.e.,

$$Z_{1-\alpha}\sqrt{\sum_{ij\in r} l_{ij}^2} + \sum_{ij\in r} l_{ij} \le T_d \tag{6}$$

Of course this assumption is not restrictive from an engineering view, since we want to minimize the cost function (4) where its worst case happens in $x = l$, when it is feasible.

3. SOCP reformulation of model for a single path TCTP

The proposed model of TCTP in (4)-(5) is a quadratically constrained program which attempts to maximize the path completion probability for path r using the minimum amount of resources. In this section we propose a SOCP reformulation of this problem in order to apply polynomial time interior point methods to solve it.

Let us briefly describe the SOCP problem and its dual (see more details in [5]). A second-order cone programming problem is defined as follows:

$$\text{(CP)} \qquad \begin{array}{ll} \min & c^T x \\ s.t. & Ax - b \succeq_K 0 \end{array}$$

Where the cone K is a direct product of several ice-cream (second order or Lorentz) cones, i.e.,

$$K = L^{m_1} \times L^{m_2} \times \cdots \times L^{m_k}$$

where the ice cream cone L^m is defined as follows:

$$L^m = \left\{ x \in R^m \mid \sqrt{\sum_{i=1}^{m-1} x_i^2} \le x_m \right\},$$

and the notation $x \succeq_K 0$ stands for $x \in K$. The ice-cream cone L^m is self-dual, i.e., $\left(L^m\right)^* = L^m$ (see [5]), where the dual cone K^* is defined as follows:

$$K^* = \left\{ y \in R^m \mid y^T x \ge 0, \quad \forall x \in K \right\}.$$

Therefore, the problem dual to (CP) is:

$$\text{(DCP)} \qquad \begin{aligned} \max \quad & \sum_{j=1}^{k} b_j^T y_j \\ s.t. \quad & \sum_{j=1}^{k} A_j^T y_j = c \\ & y_j \succeq_{L^{m_j}} 0, \quad j = 1, \ldots, k \end{aligned}$$

The SOCP problem is a direct extension of the linear programming problem. The weak duality theorem for the SOCP problem and its dual still holds as the linear programming case, but for the strong duality theorem, the SOCP problem and its dual need to be satisfied in Slater regularity conditions (see [5]).

Now, let $x = (x_1, \ldots, x_{n_r})^T$ denotes the parameter of exponential distribution of all activity durations on path r. We also denote the vectors of upper and lower limit mean duration of all activities on path r by $u = (u_1, \ldots, u_{n_r})^T$ and $l = (l_1, \ldots, l_{n_r})^T$, respectively, and the vector of cost slope of activities on this path by $s = (s_1, \ldots, s_{n_r})^T$. Assume that $e \in R^{n_r}$ is the all one vector. Then, the constraint (5) can be written as follows:

$$Z_{1-\alpha} \|x\| \le T_d - e^T x$$

or equivalently

$$\|x\| \le \frac{1}{Z_{1-\alpha}} \left(T_d - e^T x \right) \tag{7}$$

Therefore, we can rewrite the inequality (7) as follows:

$$\begin{pmatrix} x \\ \frac{1}{Z_{1-\alpha}} \left(T_d - e^T x \right) \end{pmatrix} \in L^{n_r+1}$$

or equivalently

$$\begin{pmatrix} x \\ \frac{1}{Z_{1-\alpha}} \left(T_d - e^T x \right) \end{pmatrix} \succeq_{L^{n_r+1}} 0 .$$

Using these notations and removing the constant $s^T u$ from the objective function (4), the optimization problem (4) can be rewritten as the following second order conic programming form:

$$\begin{aligned} \min \quad & z = -s^T x \\ \text{s.t.} \quad & \begin{pmatrix} x \\ \frac{1}{Z_{1-\alpha}} \left(T_d - e^T x \right) \end{pmatrix} \succeq_{L^{n_r+1}} 0 \\ & l \le x \le u \end{aligned} \tag{8}$$

Let

$$M = \begin{pmatrix} I_{n_r} \\ -Z_{1-\alpha}^{-1} e^T \end{pmatrix}_{(n_r+1)\times n_r}, \quad q = \begin{pmatrix} 0 \\ -Z_{1-\alpha}^{-1} T_d \end{pmatrix} \in R^{n_r+1}$$

Thus, the problem (8) and its dual are as follows:

$$\begin{aligned} \min \quad & z = -s^T x \\ s.t. \quad & Mx - q \succeq_{L^{n_r+1}} 0 \\ & l \leq x \leq u \end{aligned} \tag{9}$$

$$\begin{aligned} \max \quad & w = q^T y_0 - u^T y_1 + l^T y_2 \\ s.t. \quad & M^T y_0 - y_1 + y_2 = -s \\ & y_1, y_2 \geq 0, \quad y_1, y_2 \in R^{n_r} \\ & y_0 \succeq_{L^{n_r+1}} 0 \end{aligned} \tag{10}$$

Now, we have the following theorem for solvability of the problem (9).

Theorem 1. Suppose that the parameter T_d is given so that it satisfies (6). Then, the dual problem (10) is bounded above and strictly feasible, and therefore the primal problem (9) is solvable and the optimal values of both problems are equal.

***Proof*.** Using (6), one can easily see that $x = l$ is a feasible solution for the problem (9). Thus, using weak duality theorem, the dual problem (10) is bounded above. Now, we construct a strict feasible solution for the dual problem (10). Let ς be a positive constant and

$$y_0 = \begin{pmatrix} 0 \\ \varsigma \end{pmatrix} \in R^{n_r+1}, \quad y_1 = s \in R^{n_r}, \quad y_2 = Z_{1-\alpha}^{-1} \varsigma e \in R^{n_r}.$$

It can be easily verified that the vector $\left(y_0^T, y_1^T, y_2^T\right)^T$ is a feasible solution for the dual problem (9). Moreover,

$$y_0 \succ_{L^{n_r+1}} 0, \quad y_1 > 0, \quad y_2 > 0$$

i.e., the vector $\left(y_0^T, y_1^T, y_2^T\right)^T$ is a strictly feasible solution of the dual problem (10). Therefore, using Conic Duality Theorem, the primal problem (9) is solvable (i.e., it attains to its minimum value) and the optimal values of the both problems are equal. Due to the solvability of the problem (8), it can be easily solved by using available SOCP computer packages such as CVX, SeDuMi, CPLEX, etc in the polynomial time iteration complexity.

4. Determining priority of paths

In Section 3, the SOCP formulation of single path TCTP has been developed. Using this model, we can improve the completion probability of paths individually, while improving the project completion probability, all paths should be improved. Therefore, it is necessary to develop a procedure to rank the paths with respect to their criticality. By using this procedure, the SOCP formulation would be applied to the MCPs before other paths via a general iterative algorithm. For this purpose, a measure based on criticality concept has been applied. The Path Criticality Index (PCI) is the probability that the duration of path is greater than or equal to the duration of the other paths [24]. In other words, PCI gives the probability that the path is the Most Critical Path (MCP). The maximum amount of the PCI corresponds to the MCP.

We have to note that in PERT networks, numerous paths have the potential of becoming critical. In other words, there is not a unique critical path in stochastic networks, but we have "most critical path" at a given setting of network. The classical approach ignores this fact and uses a critical path that results in an extremely optimistic estimate for the probability of completion time. In general, this path is not the most critical path in the sense that it does not provide the smallest estimate for the probability of completing the project on time. Hence, in this paper, we shorten the paths in order of their criticalities until the required project completion probability is satisfied for that path. In each iteration of the algorithm, the most critical path is selected and the optimal activities are chosen on it to be assigned additional resources. This procedure continued until all the paths meet the predefined completion probability. During crashing one path, some of the activities of other paths may be crashed (because of joint activities), which may decrease the criticality of those paths.

Recently, some researchers attempted to present some methods for calculating the PCI. Martin [24] defined concept of the path criticality in PERT networks. But he did not present any method to compute its value. Van Slyke [34] calculated the criticality indices of the PERT networks by using the Monte Carlo simulation technique. Some researchers proposed application of conditional Monte Carlo simulation to compute the PCI [7,31]. Soroush [32] described an exact solution approach for determining the MCP based on concept of stochastic domination. Fatemi Ghomi and Teimouri [16] proposed a new analytical method for computing the PCI and activity criticality index for PERT networks with discrete random variables of activity durations. Here, we employ the Monte Carlo simulation technique for computing the MCP. Therefore, we need a large number of simulation runs to distinguish the MCP in each iteration. In order to compute the PCI, in each run, we first assign a sample value to every activity from its related distribution and then estimate the PCI using statistical analysis of obtained information (e.g. see this procedure in Table 3 related to the illustrative example in Section 5).

Assume that we have N simulation runs and Ψ_r represents a random variable corresponding to path r. It represents the number of simulation runs in which path r is the longest path. Therefore, Ψ_h follows a binomial distribution with parameters of (N, P_r), where P_r represents the probability of the criticality for path r. According to the above mentioned assumptions, the MCP can be computed by estimating the P_r using $\hat{P}_r$ which is defined as follows:

$$PCI_r = \hat{P}_r = \frac{\Psi_r}{N}$$

Therefore, the maximum value of PCIs defines the MCP, i.e.,

$$\text{MCP} = \{r \mid \text{PCI}_r = \max_{h=1,2,\ldots,L} \text{PCI}_h\}$$

Now, we can outline the flowchart of our proposed SOCP approach for solving the stochastic TCTP as it is shown in Figure 1.

5. Illustrative example

This section is organized in two parts. First, a numerical example is described step by step to illustrate the process of proposed SOCP approach. Detailed information is presented in the form of tables and figures to clear the method. Then, 20 randomly generated test problems of different size are presented to investigate the performance of the presented SOCP approach for stochastic TCTP. All computational results were obtained using MATLAB 7.6.0. We also use SeDuMi for solving the SOCP problem in each step.

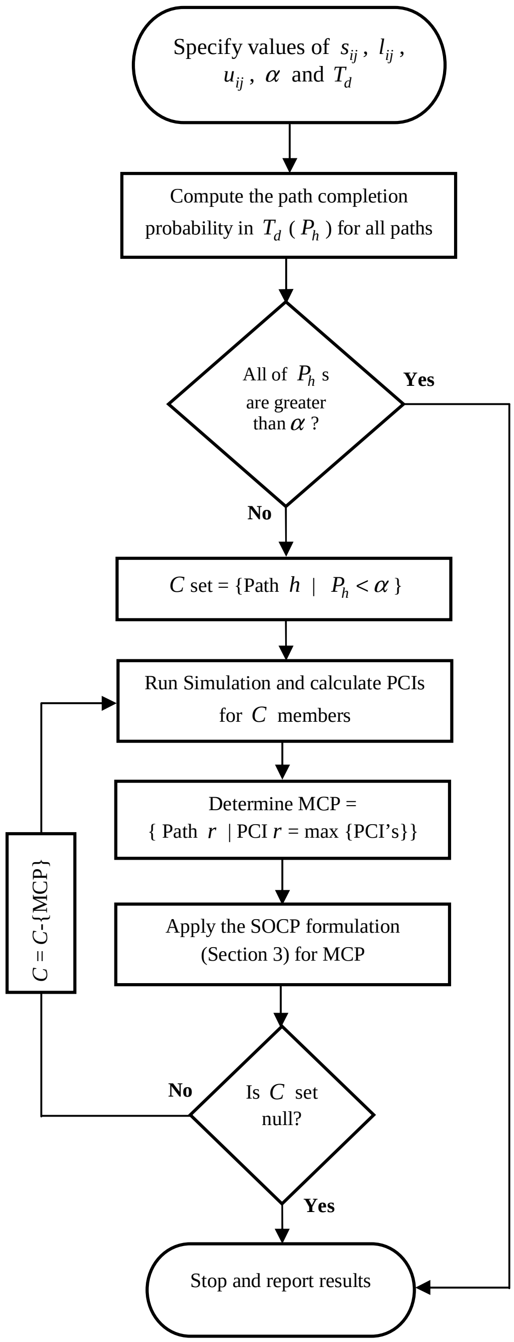

Figure 1. Flowchart of the proposed approach for solving the developed TCTP

We consider a PERT network with 10 nodes, 14 independent activities and 10 interrelated paths to demonstrate how the presented approach optimally improves the project completion probability. Figure 2 shows the AOA format of example network and characteristics of its activities and paths are presented in Tables 1 and 2.

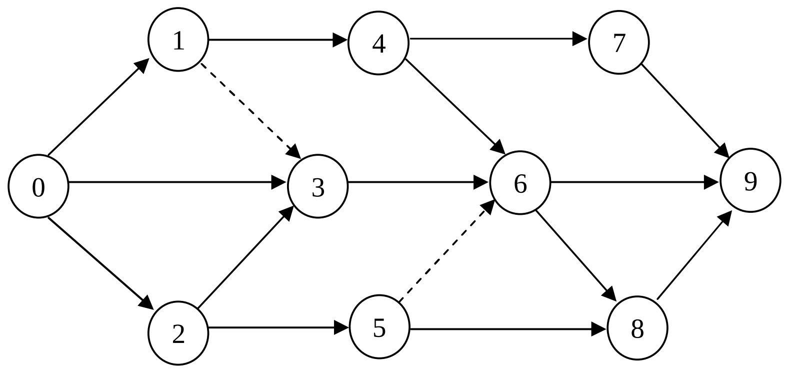

Figure 2. AOA network of example

Table 1. Characteristics of example

Activity No	Activity	s_{ij}	l_{ij}	u_{ij}
1	0-1	225	8	20
2	0-3	202	12	20
3	0-2	214.5	12	20
4	1-4	189	0.5	28
5	2-3	253.25	10	20
6	2-5	198.25	28.5	65
7	3-6	285.20	42	60
8	4-7	291	13	30
9	4-6	280	11.5	30
10	5-8	232.20	24	30
11	6-9	232	9.5	20
12	6-8	161.50	17	30
13	7-9	209.20	13.5	40
14	8-9	321.20	8.5	20

Table 2. Paths definition of example

Path No.	Activity sequences
1	1-4-8-13
2	1-4-9-11
3	1-4-9-12-14

4	2-7-11
5	2-7-12-14
6	3-5-7-11
7	3-5-7-12-14
8	3-6-11
9	2-6-12-14
10	3-6-10-14

The objective is to obtain the optimal allocated budget to activities for improving project completion probability from a risky value to a predefined confident level. It is assumed that the time unit is in weeks and the cost is in thousands. According to the presented characteristics of the example, the initial value of project completion probability at $T_d = 165$ is equal to 0.578, which can be approximately computed using the Central Limit Theorem. This value is concerned with the situation that all activities are planned in the upper bound of their distribution parameters (u_{ij}). It is also assumed that the desired amount of project completion probability (α) is equal to 0.90. The proposed method attempts to improve the initial probability (0.578) to a desired value of probability (0.90). The results of proposed model for presented example are organized in Tables 3 and 4.

Table 3. Results of applying proposed SOCP approach to considered example (10,000 simulation runs)

Path No.	Iteration 1		Iteration 2		Iteration 3		Iteration 4	
	PCI	P_h	PCI	P_h	PCI	P_h	PCI	P_h
1	0.1624	0.780	0.1688 *	0.900	-	0.900	-	0.917
2	0.0319	0.910	0.0316	0.934	0.0296	0.934	0.0334	0.949
3	0.1410	0.847	0.1417	0.879	0.1311	0.879	0.1338 *	0.900
4	0.0177	0.949	0.0175	0.949	0.0165	0.949	0.0167	0.949
5	0.0705	0.905	0.0681	0.905	0.0744	0.905	0.0761	0.905
6	0.0519	0.946	0.0552	0.946	0.0499	0.946	0.0558	0.946
7	0.2175 *	0.900	-	0.900	-	0.900	-	0.900
8	0.0314	0.837	0.0302	0.837	0.0299	0.982	0.0268	0.982
9	0.1112	0.778	0.1136	0.778	0.1111	0.958	0.1082	0.958
10	0.1645	0.708	0.1582	0.708	0.1713 *	0.900	-	0.900
MCP	7		1		10		3	
σ^*	13090		12976		14144		11365	

The MCPs have been marked by symbol * in Table 3. As it seems, the MCP has been selected in each replication of algorithm, using 10,000 Monte Carlo simulation runs. Then, the SOCP model (developed in Section 3) has been applied to the selected paths to improve the amount of P_h (path h completion probability) from its primary value to α. According to Table 3, the paths 7, 1, 10 and 3 are selected as MCP, respectively, in 4 iterations of algorithm. After improving the selected MCP (applying the developed SOCP model for the MCP), it is eliminated from unimproved paths list (the C set) in each step. This procedure is repeated until all of the paths satisfy the desired predefined probability (0.90). The optimum objective function (additional direct cost) and obtained project completion probability are computed 21.745 thousands and 0.90, respectively. As we see, Table 3 is organized for 4 above mentioned paths, only. This case may result from the following potential reasons.

(1) Other paths satisfy the α in their primary state (u_{ij}) and do not need for improvement (initial $P_h > \alpha$).

(2) Other paths do not satisfy α in their primary state, but interrelation between them and above mentioned paths (7, 1, 10 or 3) led to their appropriate improvement, formerly.

Table 4 shows the optimum decision variables in each step.

Table 4. The optimum decision variables obtained by the proposed SOCP

Activity	MCP				x_{ij}^*
	7	1	10	3	
0-1	20	20	20	20	20
0-3	20	20	20	20	20
0-2	12	12	12	12	12
1-4	28	21.73	21.73	18.66	18.66
2-3	10	10	10	10	10
2-5	65	65	37.30	37.30	37.30
3-6	42	42	42	42	42
4-7	30	30	30	30	30
4-6	30	30	30	30	30
5-8	30	30	30	30	30
6-9	20	20	20	20	20
6-8	17	17	17	17	17
7-9	40	28.28	28.28	28.28	28.28
8-9	18.27	18.27	18.27	18.27	18.27
CPU time (SeDuMi)	0.4524	0.4992	0.4516	0.4212	-------
CPU time (LINGO)	0.8726	2.6381	1.3892	0.9175	-------

There are not any standard test problems for the developed model of stochastic TCTP in the published literature to compare our results. Nonetheless, 20 randomly generated problems have been considered and the results of the proposed SOCP's objective function are presented in Table 5.

The problems 17-20 in the Table 5 are single path problems with random data, and therefore, there is no need to run Monte Carlo simulation to detect MCP in these problems. These problems are given to show the effect of conic reformulation (8). We solve the problems 17-20 by LINGO and SeDuMi software, simultaneously. The CPU time for these problems by LINGO are 3240, 951, 3325 and 2525 seconds, respectively, while the SeDuMi solves all of those in less than 4 seconds.

Table 5. The SOCP's objective function for 20 randomly generated problems

Problem No.	Number of Activities	Number of Nodes	Number of Paths	α	T_d	Optimum Objective
1	5	4	3	0.70	50	2562.34
2	5	4	3	0.78	80	1985.48
3	5	4	3	0.88	100	2358.01
4	5	4	3	0.95	120	2405.53
5	8	5	5	0.45	50	3872.02
6	8	5	5	0.70	80	4004.22
7	8	5	5	0.85	100	3959.04
8	8	5	5	0.90	120	4016.95
9	14	10	10	0.63	100	6123.62
10	14	10	10	0.78	120	5984.00
11	14	10	10	0.88	140	6008.35
12	14	10	10	0.95	180	5893.16
13	40	25	64	0.36	80	12364.45
14	40	25	64	0.58	100	13131.49
15	40	25	64	0.75	120	13002.98
16	40	25	64	0.95	150	12165.57
17	80	81	1	0.9	3500	4975.10
18	85	86	1	0.9	4200	3325.00
19	100	101	1	0.9	5500	5368.30
20	120	121	1	0.9	7200	9790.90

6. Conclusion

This paper proposes the application of Second Order Cone Programming (SOCP) and Monte Carlo simulation for solving the stochastic Time-Cost Tradeoff Problem (TCTP), where activities are subjected to linear cost function and assumed to be exponentially distributed. The main objective of the proposed model is to improve the project completion probability to a predefined desired value. First, the developed model reformulates the primary nonlinear formulation of single path TCTP to the compatible-form of SOCP problem; in addition a general algorithm based on path criticality index has been developed using Monte Carlo simulation to apply the SOCP approach for all paths, in order of their criticality indices. A numerical example has been discussed to illustrate the details of proposed SOCP process. Also a study has been conducted using several test problems to investigate the performance of SOCP approach. Our computations indicate that the proposed SOCP approach is applicable, reliable and also time benefit for the developed TCTP.

Acknowledgment: This research was in part supported by a grant from IPM (No. 88900027) for the first author. The authors would like to thank the research council of K.N. Toosi University of Technology for supporting this research.

References

[1] Aghaie, A and H Mokhtari (2009). Ant colony optimization algorithm for stochastic project crashing problem in PERT networks using MC simulation. *International Journal of Advanced Manufacturing Technology,* **45**:1051–1067.

[2] Azaron, A, C Perkgoz and W Sakawa (2005). A genetic algorithm approach for the time-cost tradeoff in PERT networks. *Applied Mathematics and Computation*, **168:**1317-1339.

[3] Azaron, A, and R Tavakkoli-Moghaddam (2006). A multi objective resource allocation problem in dynamic PERT networks. *Applied Mathematics and Computation*, **181:**163-174.

[4] Azaron, A, H Katagiri and M Sakawa (2007). Time-cost tradeoff via optimal control theory in Markov PERT networks. *Annals of Operations Research*, **150:**47-64.

[5] Ben-Tal, A and A Nemirovskii (2001). *Lectures on Modern Convex Optimization*. SIAM, Philadelphia, PA.

[6] Berman, EB (1964). Resource allocation in PERT network under activity continuous time-cost functions. *Management Science*, **10:**734–745.

[7] Bowman, RA (1995). Efficient estimation of arc criticalities in stochastic activity networks. *Management Science*, **41:**58-67.

[8] Buddhakulsomsiri, J and D Kim (2006). Properties of multi-mode resource-constrained project scheduling problems with resource vacations and activity splitting. *European Journal of Operational Research*, **175:**279-295.

[9] Burns, SA, L Liu and C Weifeng (1996). The Lp/Ip hybrid method for construction time cost trade off analysis. *Construction Management and Economics*, **14:**265-275.

[10] Cohen, I, B Golany and A Shtub (2007). The stochastic time–cost trade off problem: A robust optimization approach. *Networks*, **49:** 175-188.

[11] Deckro, RF, JE Hebert, WA Verdini, PH Grimsurd and E Venkateshwar (1995). Nonlinear time-cost trade off models in project management. *Computers & Industrial Engineering*, **28:**219–229.

[12] Demeulemeester, E, SE Elmaghraby and W Herroelen (1996). Optimal procedures for the discrete time/cost tradeoff problem in project networks. *European Journal of Operational Research,* **88:**50–68.

[13] Demeulemeester, E, B De Reyck, B Foubert, W Herroelen and M Vanhoucke (1998). New computational results on the discrete time/cost tradeoff problem in project networks. *Journal of the Operational Research Society*, **49:**1153–1163.

[14] Eshtehardian, E, A Afshar and R Abbasnia (2008). Time-cost optimization: using GA and fuzzy sets theory for uncertainties in cost. *Construction Management and Economics*, **26:**679-691.

[15] Falk, J and J Horowitz (1972). Critical path problems with concave cost-time curves. *Management Science*, **19:**446-455.

[16] Fatemi Ghomi, SM and TE Teimouri (2002). Path critical index and activity critical index in PERT networks. *European Journal of operational Research*, **141:**147-152.

[17] Fulkerson, DR (1961). A network flow computation for project cost curves. *Management Science*, **7:**167–178.

[18] Goyal, SK (1975). A note on "A simple CPM time-cost trade off algorithm". *Management Science*, **21:**718-722.

[19] Jarre, F, M Kocvara and J Zowe (1995). Optimal truss topology design by interior point methods. *Cooperative Research Report of the Institute of Statistical Mathematics*, **77:**236–252.

[20] Kelley, JR (1961). Critical-path planning and scheduling: Mathematical basis. *Operations Research*, **9:**296–320.

[21] Lamberson, LR and RR Hocking (1970). Optimum time compression in project scheduling. *Management Science*, **16:**B597–B606.

[22] Li, C and K Wang (2009). The risk element transmission theory research of multi-objective risk-time-cost tradeoff. *Computers and Mathematics with Applications*, **57(11-12)**, 1792-1799.

[23] Lobo, M, L Vandenberghe, S Boyd and H Lebret (1998). Applications of second-order cone programming. *Linear Algebra and its Applications*, **284:**193–228.

[24] Martin, JJ (1965). Distribution of the time through a directed acyclic network. *Operations Research*, **13:**46-66.

[25] Mokhtari, H, A Aghaie, J Rahimi and A. Mozdgir (2009). Project time-cost tradeoff scheduling: a hybrid optimization approach. *International Journal of Advanced Manufacturing Technology*, online first, DOI 10.1007/s00170-010-2543-4.

[26] Nesterov, Yu and A Nemirovskii (1988). Polynomial barrier methods in convex programming. *Ekonomika i Matem. Metody*, **24:**1084–1091.

[27] Nesterov, Yu and A Nemirovskii (1994). *Interior Point Polynomial Methods in Convex Programming*. SIAM Publications, Philadelphia, Pennsylvania.

[28] Nesterov, Yu and M Todd (1998). Primal-dual interior-point methods for self-scaled cones. *SIAM Journal on Optimization*, **8:**324–364.

[29] Pulat, PS and SJ Horn (1996). Time-resource trade off problem. *IEEE Transactions on Engineering Management*, **43:**411-417.

[30] Siemens, N (1971). A simple CPM time-cost trade off algorithm. *Management Science*, **17:**B354-B363.

[31] Sigal, CE, AAB Pritsker and JJ Solberg (1979). The use of cutsets in Monte Carlo analysis of stochastic networks. *Mathematics and Computers in Simulation*, **21:**376-384.

[32] Soroush, HM (1994). The most critical path in a PERT network. *Journal of the Operational Research Society*, **45:**287-300.

[33] Vanhoucke, M and D Debels (2007). The discrete time/cost tradeoff problem: extensions and heuristic procedures. *Journal of Scheduling*, **10:**311–326.

[34] Van Slyke, RM (1963). Monte Carlo methods and the PERT problem. *Operations Research*, **11:**839-860.

[35] Vrat, P and C Kriengkrairut (1986). A goal programming model for project crashing with piecewise linear time-cost tradeoff. *Engineering Costs and Production Economics*, **10:**161–172.

[36] Xue, G and Y Ye (1997). Efficient algorithms for minimizing a sum of Euclidean norms with applications. *SIAM Journal on Optimization*, **7:**1017–1036.