A New Approach to Solve Multiple Objective Programming Problems

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ABSTRACT

Multiple Objective Programming (MOP) problems have become famous among many researchers due to more practical and realistic implementations. There have been a lot of methods proposed especially during the past four decades. In this paper, we develop a new algorithm based on a new approach to solve MOP problems by starting from a utopian point (which is usually infeasible) and moving towards the feasible region via stepwise movements and a plain continuous interaction with Decision Maker (DM). We consider the case where all objective functions and constraints are linear. The implementation of the proposed algorithm is demonstrated with two numerical examples.

1. Introduction

During the past four decades, many methods and algorithms have been developed to solve Multiple Objective Programming (MOP) problems, in which some objectives are conflicting and the utility function of the Decision Maker (DM) is imprecise or fuzzy in nature. MOP is believed to be one of the fastest growing areas in management science and operations research, in that many decision making problems can be formulated in this domain. Decision making problems with several conflicting objectives are common in practice. Hence, a single objective function is not sufficient to guide the search for the optimum solution for such problems. Because of this limitation, a MOP method is needed to solve many real world applications (J. Kim and K. Kim, 2006).

Although different solution procedures have been introduced, the interactive approaches are generally believed to be the most promising ones, in which the preferred information of the DM is progressively articulated during the solution process and incorporated into it. The purpose of MOP problems in the mathematical programming framework is to optimize r

different objective functions, subject to a set of systematic constraints. A mathematical formulation of a MOP problem is also known as the vector maximization (or minimization) problem. Generally, MOP problems can be divided into four different categories. The first group of MOP problems does not need to get any information from DM during the process of finding an efficient solution. These types of algorithms rely solely on the pre-assumptions about DM's preferences. In this category, L-P Metric methods are noticeable, algorithms whose objectives are the minimization of deviations of the objective functions from the ideal solution. Since different objectives are different in nature, they must be normalized before the process of minimization of deviations starts. Therefore, a new problem is minimized which has no scale (Zeleny, 1982). The second group of MOP problems includes gathering cardinal or ordinal preferred information before the solving process initiates. In the method of Utility Function (Kenney and Raiffa, 1976), for example, we have to determine DM's utility as a function of objective functions and then we maximize the overall function under the initial constraints. The other method in this group, which is extensively used by many researchers, is Goal Programming (GP) (Romero, 1991; Gibbs, 1973) in which DM determines the least (the most) acceptable level of Max (Min) functions. Since attaining these values might lead to an infeasible

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point, the constraints are allowed to exceed, but we try to minimize these weighted deviations.

The third group of MOP problems provides a set of efficient solutions in which DM has an opportunity to choose his preferred solution among the efficient ones (Hwang and Masud, 1979). The set of all efficient feasible solutions in a Multiple Objective Linear Programming (MOLP) problem can be represented by convex combination of efficient extreme points and efficient extreme rays in the feasible region. Therefore, the set of efficient extreme points and efficient extreme rays can be regarded as the solution set for a MOLP problem (Ida, 2005). Finally, the forth group of MOP problems provides solutions based on a continuous interaction with DM and tries to reach the preferred solution at the end of the algorithm. Based on this sound idea, there are many developed methods categorized in this group. Homburg (1998), for instance, proposed a hierarchical procedure which consists of two levels, a top-level and a base-level. The main idea is that the top-level only provides general preference information from DM. Taking this information into account, the base-level then determines a compromise solution via interaction with DM by using an interactive procedure.

As another instance, Tchebycheff metric based approaches have become popular in this category for sampling the set of efficient solutions in an continuous interaction with DM to narrow his choices down to a single most preferred efficient solution. These approaches systematically reduce the set of efficient solutions which remain available for identification and selection from one iteration to the next. The only requirement on the part of the DM is to select a single most preferred solution from among a more and more concentrated set of efficient solutions at each iteration (Reeves and MacLeod, 1999). The interaction with DM proceeds by generating smaller subsets of the efficient set until a final solution is located.

To see another works in the group of interactive methods, interested reader can refer to (Geoffrion, 1972; Reeves and Franz, 1985; Zionts and Wallenius, 1983; Benayoun et al., 1971; Hwang and Masud, 1979; Tabucanon, 1988; Steuer and Choo, 1983; Steuer et al., 1995; Sun et al., 2000; Gardiner and Steuer, 1994; Malakooti and Alwani, 2002; Kaliszewski and Michalowski, 1999; Sun et al., 2000; Chen and Lin, 2003). There are many advantages on using interactive methods such as:

- there is no need to get any information from DM before the solving process initiates,
- the solving process helps DM learn more about the nature of the problem,
- only minor preferred information are needed during the solving process,
- since DM continuously contributes via analyst to the problem, he is more likely to accept the final solution,

- there are fewer restricting assumptions involved in these types of problems in comparison with other groups of MOP methods.
- However, there are some drawbacks associated with these types of algorithms that the most important ones are as follows:
- the accuracy of the final solution depends entirely upon DM's precise answers. In other words, if DM does not carefully interact with the analyst, the outcome(s) of the final solution may be undesirable,
- there is no guarantee to reach a desirable solution after a finite number of iterations,
- DM needs to make more effort during the process of these algorithms in comparison with other groups.

During the past decades, many researchers have tried to review or to discuss the strengths, the weaknesses, and the comparative studies on the existing methods. The main goals of these papers are to introduce some criteria to measure the efficiency of various algorithms and to introduce the characteristics of a good method (Aksoy, 1996; Buchanan and Daellenbach, 1987; Gibson et al., 1987; Lotfi et al., 1997; Mote et al., 1988; Reeves and Franz, 1985). In this domain Borges and Antunes (2002) dealt with the sensitivity analysis of the weights in MOLP problems. Sun (2005) examined some issues in measuring and reporting solution quality when value functions are used in computational experiments of interactive MOP procedures. He discussed value functions used, weights assigned to the objective functions in the value functions, the size of the efficient set, the number of objective functions, the feasibility of the ideal and nadir points, and existence of the ideal and nadir points.

Alves and Climaco (2007) made a review of interactive methods devoted to solve Multiple Objective Integer Programming (MOIP) and Multiple Objective Mixed-Integer Programming (MOMIP) problems. Their focus is on interactive MOIP and MOMIP methods, including their characterization according to the type of preference information required from the DM, the computing process used to determine efficient solutions and the interactive protocol used to communicate with the DM. Reeves and Franz (1985), introduced the characteristics of a proper interactive algorithm as follows:

1. Minimum amount of information be required from DM,
2. The nature of decision making be simple,
3. If DM provides his answers improperly in some interactions, he has had an opportunity to compensate it in the following interactions,
4. The number of iterations to reach the final solution be reasonable,
5. DM be familiar with the nature of judgments he is asked for,

6. The algorithm be suitable for solving large scale problems.

In this paper, we propose a new algorithm which is mainly in the group of interactive methods. However, we also need to get some information from DM before problem solving initiates; therefore, this algorithm is neither a pure interactive method nor a pure method in the second category. In addition, the proposed algorithm is based upon a novel approach to the problem, starting from an infeasible utopian point and moving towards the feasible region and then the final efficient point. The remaining of this paper is organized as follows. Section 2 provides some of the necessary definitions we need to use in this paper. In section 3, the problem statement and the proposed algorithm are explained. Two numerical examples are demonstrated in section 4 to illustrate the proposed algorithm. Finally, the conclusion remarks appear in section 5 to summarize the contribution of the paper.

2. Definitions

The best results will be obtained if your MS-Word 2003 application has several font sizes. The main font used throughout the document is **Times New Roman.** Try to follow the font sizes specified in Table 1, as best as you can.

Consider a MOLP problem defined as follows,

$$\max\{Z = f_k(X) = C_k^T X; k = 1,2,...,r\}$$
$$s.t. \qquad\qquad\qquad (1)$$
$$M = \{X \in R^n \mid A_i X \le b_i; X \ge 0; i = 1,2,...,m\}$$

where,

$f_k(X)$: is the kth objective function,

C_k: is the vector of coefficients in the kth objective function,

X: is an n-dimensional vector of decision variables,

A_i: is the ith row of technological coefficients,

b_i: is the RHS of the ith constraint, and

M: is the feasible region.

A solution $\overline{X} \in M$ is efficient if and only if there does not exist another $X \in M$ such that $f_k(X) \ge f_k(\overline{X})$ for all $k = 1,2,...,r$ and $f_k(X) > f_k(\overline{X})$ for at least one k. Then, the vector,

$$\overline{Z} = \{f_k(\overline{X}); k = 1,2,...,r\} \qquad (2)$$

is called a non-dominated criterion vector. All efficient solutions in M form the efficient set E. Although some interactive algorithms search the entire feasible region M, the majority of them are designed to search only the efficient set E. The vector,

$$Z^* = \{f_k(X^*) \mid f_k(X^*) = \max f_k(X); k = 1,2,...,r\} \qquad (3)$$

is called the ideal point or the ideal criterion vector. It should be mentioned that the ideal criterion vector, and so the ideal solution X^*, does not usually exist. The vector,

$$Z^{**} = \{f_k(X^{**}) \mid f_k(X^{**}) \ge \max f_k(X); k = 1,2,...,r\} \qquad (4)$$

is called a utopian vector or a utopian point. Unlike the ideal criterion vector, there exist many utopian vectors. Nevertheless, their corresponding X^{**}'s are most likely infeasible.

3. Problem Statement

The majority of methods proposed in the domain of interactive procedures search the feasible region M or the efficient set E through interaction with DM in order to attain the final solution. Here, we develop a new algorithm to solve MOLP problems by starting from a utopian point X^{**} (which is usually infeasible) and moving towards the feasible region M and then the efficient set E via stepwise movements and a plain continuous interaction with DM in order to be in line with his preferences. Since there are many utopian points outside the M, we choose the closest X^{**} to M as the start point, by considering the least sum of weighted deviations from the constraints.

3.1. The Proposed Algorithm

The proposed algorithm attains an efficient solution of a MOLP through the following steps:

Begin*:*

Step 1. Ask DM to determine a_k, the maximum acceptable reduction in the amount of f_k in any interaction. Also, ask him to determine w_i, a penalty for deviation of each unit from the ith constraint. In the next step, we find a utopian point allowing some deviations from the constraints $x_j \ge 0$, in that the utopian point maybe a point with some negative x_j's. However, we also consider a big penalty, w', for each unit of such deviations.

Step 2. Maximize each $f_k(X)$ with consideration of the feasible set M as follows,

$$\max f_k(X) = C_k^T X$$
$$s.t. \qquad\qquad\qquad (5)$$
$$M = \{X \in R^n \mid A_i X \le b_i; X \ge 0; i = 1,2,...,m\}$$

Step 3. Let $f_k(X^*)$ be the optimal solution for each $f_k(X); k = 1,2,...,r$. Solve the following GP problem,

$$\min\{\sum_{i=1}^{m} w_i d_i + w' \sum_{j=1}^{n} d'_j \mid f_k(X) \ge f_k(X^*); A_i X \le b_i + d_i$$
$$; x_j \ge -d'_j; d \ge 0; i = 1,2,...,m, j = 1,2,...,n; k = 1,2,...,r\} \qquad (6)$$

where, d_i represents the deviation from the ith constraint. In this step, we allow our solution to go

outside the feasible region. Suppose X is the solution for (6). Set $X^0 = X$ and go to step 4.

Step 4. Let θ_{ik} be the angle between f_i and f_k. Calculate $\sin\theta_{ik}$ as follows,

$$\sin\theta_{ik} = \sqrt{1 - \frac{C_i.C_k}{|C_i|.|C_k|}} \qquad (7)$$

Now, we can determine a small step δ by which we move towards the feasible region in each iteration as,

$$\delta = \min\{\frac{a_i}{|C_i|\sin\theta_{ik}}; i, k = 1,2,...,r; i \neq k\} \qquad (8)$$

Step 5. Consider constraints $f_k(X^0) \geq f_k(X^*)$ which remain active. Now ask DM to see which active objective has the least desirability. Let l be the index for the f_l which has the least desirability.

Step 6. Solve the following optimization problem in which we take a step δ from X^0 towards the feasible region while we hold the amount of f_l,

$$\min\{\sum_{i=1}^{m} w_i d_i + w' \sum_{j=1}^{n} d'_j \mid f_l(X) \geq f_l(X^0); A_i X \leq b_i + d_i \\ ; |X - X^0| = \delta; x_j \geq -d'_j; d \geq 0; i = 1,2,...,m; j = 1,2,...,n\} \qquad (9)$$

where, |.| is the 2-norm. In this step there is no change in the value of f_l but we usually expect that the other objective functions get worse, but not necessarily. In other words, we might encounter a situation in which the values of some active or inactive f_k get better.

Step 7. If $\sum_{i=1}^{m} w_i d_i + w' \sum_{j=1}^{n} d'_j = 0$ then go to step 8, otherwise set $X^0 = X$, calculate the new values of $f_k(X^0)$, and go to step 5.

Step 8. $\sum_{i=1}^{m} w_i d_i + w' \sum_{j=1}^{n} d'_j = 0$ implies that we are inside the feasible region, but most likely not on the boundary. Therefore, we take a smaller step to be stopped on the boundary by solving,

$$\min\{|X - X^0| \mid f_k(X) \geq f_k(X^0); A_i X \leq b_i \\ ; x_j \geq 0; i = 1,2,...,m; j = 1,2,...,n; k = 1,2,...,r\} \qquad (10)$$

There is no guarantee that the solution of step 8 is a non-dominated one. So, we move on the boundary to reach a non-dominated solution. Set $X^0 = X$, $S = \{1,2,...,r\}$, and go to step 9.

Step 9. Ask DM to see which objective in S has the least desirability. Let l be the index for the f_l which has the least desirability. Solve the following optimization problem in which we take a step at most with the amount of δ from X^0 on the boundary of the feasible region while we hold the amounts of $f_k; k = 1,2,...,r$,

$$\max\{f_l(X) \mid f_k(X) \geq f_k(X^0); A_i X \leq b_i; |X - X^0| \leq \delta \\ ; x_j \geq 0; i = 1,2,...,m; j = 1,2,...,n; k = 1,2,...,r\} \qquad (11)$$

Step 10. If $f_l(X) > f_l(X^0)$ then set $S = \{1,2,...,r\}$ and go to step 9, otherwise set $S = S - l$ and go to step 11.

Step 11. If $S = \phi$ then choose X as the final efficient solution, otherwise set $X^0 = X$ and go to step 9.
End.

It should be noted that steps 1-8 helps us to reach to the feasible region M by starting from the closest utopian point in line with DM's preferences, whereas steps 9-11 guarantee that the final solution is an efficient one, i.e., the final solution is in E.

3.2. Some Lemmas to Determine δ
    Here, we show how to choose δ in Step 4 of the proposed algorithm with the following three lemmas.
Lemma 1: Any step δ along gradient vector C_k will result a decrease (or increase) of $\delta|C_k|$ in f_k.
Proof: Let α_{kj} be the angle between C_k and axis x_j. Therefore,

$$\cos\alpha_{kj} = \frac{C_k.x_j}{|C_k||x_j|} = \frac{(c_{k1},...,c_{kj},...,c_{kn}).(0,...,1,...,0)}{|C_k|\times 1} = \frac{c_{kj}}{|C_k|} \quad (12)$$

where, x_j is the jth unique vector in an n-dimensional space. The angle between C_k and x_j helps us to compute the projection of C_k over the axis x_j, i.e., if we take a step δ along vector C_k, the amount of change in each element of x_j is $\delta\cos\alpha_{kj}$ or $\delta\cos(\pi - \alpha_{kj})$ depending on the direction we choose. Fig.1 depicts the gradient vector C_k and its projection in a 2-dimensional space.
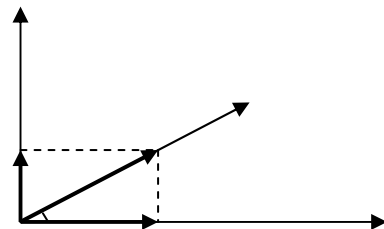

Fig. 1 The gradient vector C_k and its projection

Therefore,

$$\Delta x_j = \delta \cos \alpha_{kj} = \delta \frac{c_{kj}}{|C_k|} \qquad (13)$$

or

$$\Delta x_j = \delta \cos(\pi - \alpha_{kj}) = -\delta \cos \alpha_{kj} = -\delta \frac{c_{kj}}{|C_k|} \qquad (14)$$

Therefore, we can compute the change in the amount of f_k as follows,

$$|\Delta f_k| = \sum_{j=1}^{n} c_{kj}.\Delta x_j = \sum_{j=1}^{n} c_{kj}.\delta \frac{c_{kj}}{|C_k|}$$
$$= \frac{\delta}{|C_k|} \sum_{j=1}^{n} c_{kj}^2 = \frac{\delta}{|C_k|}.|C_k|^2 = \delta.|C_k| \qquad (15)$$

We now present a generalized form of Lemma (1).

Lemma 2: Any step δ along C_l which makes the angle θ_{lk} with C_k will result a decrease (increase) of $\delta |C_k| \cos \theta_{lk}$ in f_k.
Proof: It is clear that taking a step δ along C_l which makes the angle θ_{lk} with C_k is the same as taking a step $\delta \cos \theta_{lk}$ along C_k. Using the results of Lemma (1) yields,

$$|\Delta f_k| = \delta \cos \theta_{lk} |C_k| \qquad (16)$$

Lemma 3: Let H_l be a hyperplane which is orthogonal on C_l and C_l makes the angle θ_{lk} with C_k. Any step δ on the hyperplane H_l in any direction will result a decrease (increase) of $\delta \sin \theta_{lk} |C_k|$ in f_k.
Proof: We prove this lemma in two steps. In the first step, let $0 \le \theta_{lk} \le \pi/2$, then taking any step δ on H_l in any direction is the same as taking a step δ in the direction whose angle with C_l is $\pi/2$ and therefore makes the angle $\pi/2 \pm \theta_{lk}$ with C_k. Fig. 2(a) demonstrates the situation in a 2-dimensional space.
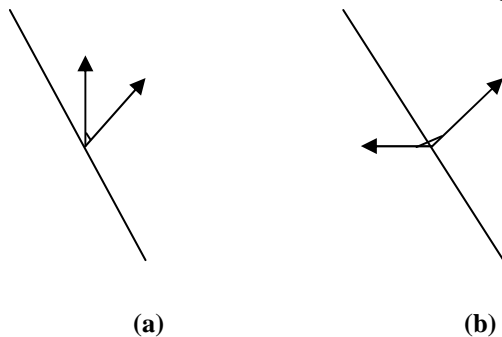

(a)   **(b)**
Fig. 2. Demonstration of taking a step δ on H_l in a 2-dimensional space

According to lemma (2), taking any step δ along the direction which makes the angle $\pi/2 + \theta_{lk}$ or $\pi/2 - \theta_{lk}$ with C_k will result a change with the amount of $\delta \cos(\pi/2 + \theta_{lk})|C_k|$ or $\delta \cos(\pi/2 - \theta_{lk})|C_k|$ in f_k. Since $0 \le \theta_{lk} \le \pi/2$, we have,

$$\delta \cos(\pi/2 + \theta_{lk})|C_k| = -\delta \sin \theta_{lk} |C_k| \qquad (17)$$

or

$$\delta \cos(\pi/2 - \theta_{lk})|C_k| = \delta \sin \theta_{lk} |C_k| \qquad (18)$$

Finally, we have,

$$|\Delta f_k| = \delta \sin \theta_{lk} |C_k| \qquad (19)$$

Now, in the second step, suppose $\pi/2 \le \theta_{lk} \le \pi$. Taking any step δ on H_l in any direction is the same as taking a step δ in the direction whose angle with C_k is $\theta_{lk} - \pi/2$ or $3\pi/2 - \theta_{lk}$. Fig. 2(b) demonstrates the situation in a 2-dimensional space. Using similar argument used in the first step yields,

$$\delta \cos(\theta_{lk} - \pi/2)|C_k| = \delta \sin \theta_{lk} |C_k| \qquad (20)$$

or

$$\delta \cos(3\pi/2 - \theta_{lk})|C_k| = -\delta \sin \theta_{lk} |C_k| \qquad (21)$$

Finally, we have,

$$|\Delta f_k| = \delta \sin \theta_{lk} |C_k| \qquad (22)$$

Now, we are ready to determine the amount of δ properly. Suppose DM determines that he wouldn't expect any reduction more than a_k in the amount of f_k in any interaction. When we perform step (4) in the algorithm, actually we keep f_l unchanged. In order to achieve this goal, we have to take step δ on H_l. According to lemma (3), the step leads to an increase (decrease) $\delta \sin \theta_{lk} |C_k|$ in f_k.
There is no problem in our approach in case f_k increases. However, we must ensure that the step δ would not worsen f_k more than a_k, which suggest to keep the following condition,

$$\delta \sin \theta_{lk} |C_k| \le a_k; k = 1,...,r; k \ne l \qquad (23)$$
or

$$\delta \leq \frac{a_k}{|C_k| \sin \theta_{lk}} ; k = 1,...,r; k \neq l \qquad (24)$$

Holding (19) in all interactions throughout the algorithm guarantees that there would be no reduction in any $f_k; k \neq l$ more than a_k. Since DM is entitled to keep the amount of any f_k, the following condition must be hold in order to obtain an appropriate δ,

$$\delta \leq \frac{a_k}{|C_k| \sin \theta_{lk}} ; k,l = 1,...,r; k \neq l \qquad (25)$$

Finally, we are about to determine the best amount of δ with consideration of DM's intentions and concurrently reaching to the feasible solution by implementing the algorithm as fewer interactions as possible. Thus, we have,

$$\delta = \min\{\frac{a_k}{|C_k| \sin \theta_{lk}} ; k,l = 1,...,r; k \neq l\} \qquad (26)$$

4. Numerical Examples

In this section we demonstrate implementation of the proposed method using two numerical examples.

4.1. Example 1

Consider the following MOLP problem with two objective functions,

$$Max z_1 = x_1 + 6x_2$$
$$Max z_2 = 5x_1 + 2x_2$$
$$ST.$$
$$-x_1 + 4x_2 \leq 20$$
$$7x_1 + 9x_2 \leq 63 \qquad (27)$$
$$22x_1 + 15x_2 \leq 165$$
$$x_1 \leq 6.5$$
$$x_1, x_2 \geq 0$$

We first ask DM to specify his sensitivity about the constraints and the objectives.

As we already defined, w_i's are the penalties associated with the constraints and a_k's are the permitted amounts of reduction on the objective functions in each iteration. For the sake of simplicity suppose that all constraints have equal sensitivity, i.e., $w_i = 1; i = 1,...,4$. Next, we have to determine the acceptable amount of reduction on the objectives z_1 and z_2.

For this example, suppose DM specifies 2 and 3 for a_1 and a_2, respectively. The optimal value for δ can be determined as the following,

$$C_1 = (1,6) \Rightarrow |C_1| = \sqrt{1^2 + 6^2} = \sqrt{37}$$

$$C_2 = (5,2) \Rightarrow |C_2| = \sqrt{5^2 + 2^2} = \sqrt{29}$$

$$\cos \theta_{12} = \frac{C_1.C_2}{|C_1|.|C_2|} = \frac{(1,6).(5,2)}{\sqrt{37}.\sqrt{29}} = 0.52$$

$$\sin \theta_{12} = \sqrt{1 - (0.52)^2} = 0.85$$

$$\delta = \min\{\frac{a_1}{|C_1| \sin \theta_{12}}, \frac{a_2}{|C_2| \sin \theta_{21}}\}$$
$$= \min\{\frac{2}{\sqrt{37}(0.85)}, \frac{3}{\sqrt{29}(0.85)}\} = 0.38$$

Then, we must find z_1^* and z_2^*. Solving two distinct LP problems with consideration of z_1 and z_2 yields $(x_1, x_2) = (1.95, 5.49)$ with $z_1^* = 34.86$ and $(x_1, x_2) = (6.50, 1.47)$ with $z_2^* = 35.43$, respectively.

In the next step, we obtain the utopian point in which both objectives are satisfied at least with their optimal values, while we reach to a common point. Hence, we have,

$$Min D = d_1 + d_2 + d_3 + d_4 + 1000(d_5 + d_6)$$
$$ST.$$
$$-x_1 + 4x_2 \leq 20 + d_1$$
$$7x_1 + 9x_2 \leq 63 + d_2$$
$$22x_1 + 15x_2 \leq 165 + d_3$$
$$x_1 \leq 6.5 + d_4 \qquad (28)$$
$$x_1 + 6x_2 \geq 34.86$$
$$5x_1 + 2x_2 \geq 35.43$$
$$x_1 \geq -d_5$$
$$x_2 \geq -d_6$$
$$x_1, x_2 : free\ in\ sign$$
$$d_i \geq 0; i = 1,...,6$$

The optimal solution for (28) is $(x_1^{**}, x_2^{**}) = (5.10, 4.96)$ with $(z_1^{**}, z_2^{**}) = (34.86, 35.43)$ and $D^{**} = 39.02$. In the next step, the DM is asked to select the objective which has the least desirability for him. Suppose in the first

interaction the DM adopts z_2. Therefore, we must solve the following problem,

$$MinD = d_1 + d_2 + d_3 + d_4 + 1000(d_5 + d_6)$$
$$ST.$$
$$-x_1 + 4x_2 \leq 20 + d_1$$
$$7x_1 + 9x_2 \leq 63 + d_2$$
$$22x_1 + 15x_2 \leq 165 + d_3$$
$$x_1 \leq 6.5 + d_4 \tag{29}$$
$$x_1 \geq -d_5$$
$$x_2 \geq -d_6$$
$$5x_1 + 2x_2 \geq 35.43$$
$$\sqrt{(x_1 - 5.10)^2 + (x_2 - 4.96)^2} = 0.38$$
$$x_1, x_2 : free\ in\ sign$$
$$d_i \geq 0; i = 1, ..., 6$$

The optimal solution for (29) is $(x_1, x_2) = (5.24, 4.61)$ with $(z_2, z_2) = (32.89, 35.43)$ and $D = 34.65$. Table 1 summarizes the results of implementation of the proposed algorithm during the next iterations.

Tab. 1. The detailed information for implementation of the proposed method for example 1

Iter.	Objec.	x_1	x_2	D	z_1	z_2
0	max z_1	1.95	5.49	0	34.86	20.73
0	max z_2	6.5	1.47	0	15.32	35.43
0	Utopian	5.1	4.96	39.02	34.86	35.43
1	Hold z_2	5.24	4.61	34.65	32.89	35.43
2	Hold z_2	5.38	4.25	30.27	30.88	35.43
3	Hold z_1	5.01	4.32	20.9	30.88	33.69
4	Hold z_2	5.17	3.91	15.87	28.63	33.69
5	Hold z_1	4.8	3.97	6.5	28.63	31.94
6	Hold z_1	4.42	4.04	4.28	28.63	30.18
7	Hold z_1	4.04	4.1	2.14	28.63	28.38
8	Hold z_2	4.18	3.75	0	26.65	28.38
9	min δ	4.17	3.75	0	26.69	28.38
10	max z_1	4.17	3.75	0	26.69	28.38
11	max z_2	4.17	3.75	0	26.69	28.38

As one can observe, we have reached to the feasible region after 8 iterations.
The final step by which we reach to the feasible region is from $(x_1, x_2) = (4.04, 4.10)$ to $(x_1, x_2) = (4.18, 3.75)$ with feasible amounts $(z_1, z_2) = (26.65, 28.38)$. So, in order to reach to the feasible region by a smaller step we solve,

$$MinD = \sqrt{(x_1 - 4.04)^2 + (x_2 - 4.10)^2}$$
$$ST.$$
$$x_1 + 6x_2 \geq 26.65$$
$$5x_1 + 2x_2 \geq 28.38$$
$$-x_1 + 4x_2 \leq 20 \tag{30}$$
$$7x_1 + 9x_2 \leq 63$$
$$22x_1 + 15x_2 \leq 165$$
$$x_1 \leq 6.5$$
$$x_j \geq 0; j = 1, 2$$

Problem (30) leads to $(x_1, x_2) = (4.17, 3.75)$, with $(z_1, z_2) = (26.69, 28.38)$ and $\delta = 0.37$, which is the first feasible point on the boundary of the feasible region. Then, the DM is asked to determine the objective function which has the least desirability. Suppose he adopts z_1, so we solve,

$$Max z_1 = x_1 + 6x_2$$
$$ST.$$
$$\sqrt{(x_1 - 4.17)^2 + (x_2 - 3.75)^2} \leq 0.38$$
$$5x_1 + 2x_2 \geq 28.38 \tag{31}$$
$$-x_1 + 4x_2 \leq 20$$
$$7x_1 + 9x_2 \leq 63$$
$$22x_1 + 15x_2 \leq 165$$
$$x_1 \leq 6.5$$
$$x_j \geq 0; j = 1, 2$$

Problem (31) leads to $(x_1, x_2) = (4.17, 3.75)$ with $(z_1, z_2) = (26.69, 28.38)$. As one can see, z_1 cannot be improved by moving from $(x_1, x_2) = (4.17, 3.75)$. So, we have $S = \{2\}$ and z_2 is chosen to get improved. We solve,

$$Max z_2 = 5x_1 + 2x_2$$
$$ST.$$
$$\sqrt{(x_1 - 4.17)^2 + (x_2 - 3.75)^2} \leq 0.38$$
$$x_1 + 6x_2 \geq 26.69 \tag{32}$$
$$-x_1 + 4x_2 \leq 20$$
$$7x_1 + 9x_2 \leq 63$$
$$22x_1 + 15x_2 \leq 165$$
$$x_1 \leq 6.5$$
$$x_j \geq 0; j = 1, 2$$

Problem (32) leads to $(x_1, x_2) = (4.17, 3.75)$ with $(z_1, z_2) = (26.69, 28.38)$. As one can see, z_2 cannot be improved by moving from $(x_1, x_2) = (4.17, 3.75)$. So, $S = \phi$ and $(x_1, x_2) = (4.17, 3.75)$ with $(z_1, z_2) = (26.69, 28.38)$ is the final efficient feasible solution.

4.2. Example 2

Consider the following MOLP problem with three objective functions,

$$Max z_1 = 10x_1 + 80x_2 + 25x_3 + 16x_4$$
$$Max z_2 = 6x_1 + 7x_2 + 25x_3 + 8x_4$$
$$Max z_3 = 8x_1 - 5x_2 + 12x_3 + 4x_4$$
$$ST.$$
$$-6x_1 + 7x_2 + 5x_3 + 3x_4 \le 142 \qquad (33)$$
$$2x_1 + 7x_2 + 25x_3 + 9x_4 \le 320$$
$$20x_1 + 13x_2 + 40x_3 + 16x_4 \le 800$$
$$3x_1 - 10x_2 + x_3 - 24x_4 \le 15$$
$$16x_1 + 5x_2 - 2x_3 + 80x_4 \ge 228$$
$$x_1,...,x_4 \ge 0$$

Suppose that the values 12, 5, 45, 2, and 6 are specified by the DM for w_1, w_2, w_3, w_4, and w_5, respectively and we consider $w^l = 1000$.

Also, 300, 50, and 30 are determined as the acceptable amount of reduction for z_1, z_2, and z_3. The optimal value for δ is determined as follows,

$$C_1 = (10,80,25,15) \Rightarrow |C_1| = \sqrt{7350}$$

$$C_2 = (6,7,25,8) \Rightarrow |C_2| = \sqrt{774}$$

$$C_3 = (8,-5,12,4) \Rightarrow |C_2| = \sqrt{249}$$

$$\cos\theta_{12} = \frac{C_1.C_2}{|C_1|.|C_2|} = \frac{(10,80,25,15).(6,7,25,8)}{\sqrt{7350}.\sqrt{774}} = 0.57$$
$$\Rightarrow \sin\theta_{12} = \sqrt{1-(0.57)^2} = 0.82$$

$$\cos\theta_{13} = \frac{C_1.C_3}{|C_1|.|C_3|} = \frac{(10,80,25,15).(8,-5,12,4)}{\sqrt{7350}.\sqrt{249}} = 0.03$$
$$\Rightarrow \sin\theta_{13} = \sqrt{1-(0.03)^2} = 1$$

$$\cos\theta_{23} = \frac{C_2.C_3}{|C_2|.|C_3|} = \frac{(6,7,25,8).(8,-5,12,4)}{\sqrt{774}.\sqrt{249}} = 0.79$$
$$\Rightarrow \sin\theta_{23} = \sqrt{1-(0.79)^2} = 0.62$$

$$\delta = \min\{\frac{a_1}{|C_1|\sin\theta_{12}}, \frac{a_1}{|C_1|\sin\theta_{13}}, \frac{a_2}{|C_2|\sin\theta_{21}}$$
$$, \frac{a_2}{|C_2|\sin\theta_{23}}, \frac{a_3}{|C_3|\sin\theta_{31}}, \frac{a_3}{|C_3|\sin\theta_{32}}\}$$
$$= \min\{4.27, 3.50, 2.19, 2.90, 1.90, 3.07\} = 1.90$$

Now, z_1^*, z_2^*, and z_3^* must be found. Solving three LP problems with consideration of z_1, z_2, and z_3 separately yields

$(x_1, x_2, x_3, x_4) = (17.22, 35.05, 0, 0)$ with $z_1^* = 2975.87$,
$(x_1, x_2, x_3, x_4) = (16.66, 4.52, 10.20, 0)$ with $z_2^* = 386.64$, and

$(x_1, x_2, x_3, x_4) = (36.82, 0, 0, 3.98)$ with $z_3^* = 310.45$, respectively.

Then, we obtain the utopian point in which three objectives are satisfied at least with their optimal values while we reach to a common point. Therefore, we have,

$$Min D = 12d_1 + 5d_2 + 45d_3 + 2d_4 + 6d_5$$
$$+1000(d_6 + d_7 + d_8 + d_9)$$
$$ST.$$
$$-6x_1 + 7x_2 + 5x_3 + 3x_4 \le 142 + d_1$$
$$2x_1 + 7x_2 + 25x_3 + 9x_4 \le 320 + d_2$$
$$20x_1 + 13x_2 + 40x_3 + 16x_4 \le 800 + d_3$$
$$3x_1 - 10x_2 + x_3 - 24x_4 \le 15 + d_4$$
$$16x_1 + 5x_2 - 2x_3 + 80x_4 \ge 228 - d_5$$
$$10x_1 + 80x_2 + 25x_3 + 16x_4 \ge 2975.87$$
$$6x_1 + 7x_2 + 25x_3 + 8x_4 \ge 386.64$$
$$8x_1 - 5x_2 + 12x_3 + 4x_4 \ge 310.45$$
$$x_1 \ge -d_6$$
$$x_2 \ge -d_7$$
$$x_3 \ge -d_8$$
$$x_4 \ge -d_9$$
$$x_1,...,x_4 : free\ in\ sign \qquad (34)$$
$$d_i \ge 0; i = 1,...,9$$

The optimal solution is $(x_1^{**}, x_2^{**}, x_3^{**}, x_4^{**}) = (57.56, 30, 0, 0)$ with $(z_1^{**}, z_2^{**}, z_3^{**}) = (2975.60, 555.36, 310.48)$ and $D^{**} = 333803$. In the next step, the DM is asked to select the objective which has the least desirability for him. Since the constraint associated with z_2 is not active, the DM is allowed to select one of the objectives z_1 or z_3 to keep its value. Suppose in the first iteration the DM adopts z_3. Therefore, the following problem should be solved,

$$Min D = 12d_1 + 5d_2 + 45d_3 + 2d_4 + 6d_5$$
$$+1000(d_6 + d_7 + d_8 + d_9)$$
$$ST.$$
$$-6x_1 + 7x_2 + 5x_3 + 3x_4 \le 142 + d_1$$
$$2x_1 + 7x_2 + 25x_3 + 9x_4 \le 320 + d_2$$
$$20x_1 + 13x_2 + 40x_3 + 16x_4 \le 800 + d_3$$
$$3x_1 - 10x_2 + x_3 - 24x_4 \le 15 + d_4$$
$$16x_1 + 5x_2 - 2x_3 + 80x_4 \ge 228 - d_5$$
$$x_1 \ge -d_6$$
$$x_2 \ge -d_7$$
$$x_3 \ge -d_8$$
$$x_4 \ge -d_9$$
$$8x_1 - 5x_2 + 12x_3 + 4x_4 \ge 310.48$$
$$\sqrt{(x_1 - 57.56)^2 + (x_2 - 30)^2 + (x_3 - 0)^2 + (x_4 - 0)^2} = 1.9$$
$$x_1,...,x_4 : free\ in\ sign \qquad (35)$$
$$d_i \ge 0; i = 1,...,9$$

The optimal solution for (35) is $(x_1, x_2, x_3, x_4) = (56.74, 28.29, -0.17, 0)$ with $(z_1, z_2, z_3) = (2826.35, 534.22, 310.43)$ and $D = 31484.82$.

Table 2 summarizes the results of implementation of the proposed algorithm for example 2. Note that the constraint associated with z_2 is not active till iteration 8. Therefore, he is allowed to choose z_2 as the objective whose desirability is the least amount from iteration 8.

According to Table 2, we reach to the feasible region in iteration 22. So, solving the following problem helps us to attain the boundary of the feasible region,

$$MinD = \sqrt{(x_1 - 32.26)^2 + (x_2 - 12.53)^2 + (x_3 + 0.2)^2 + (x_4 - 0)^2}$$

$ST.$

$$10x_1 + 80x_2 + 25x_3 + 16x_4 \geq 1320.02$$
$$6x_1 + 7x_2 + 25x_3 + 8x_4 \geq 274.99$$
$$8x_1 - 5x_2 + 12x_3 + 4x_4 \geq 182.73 \qquad (36)$$
$$-6x_1 + 7x_2 + 5x_3 + 3x_4 \leq 142$$
$$2x_1 + 7x_2 + 25x_3 + 9x_4 \leq 320$$
$$20x_1 + 13x_2 + 40x_3 + 16x_4 \leq 800$$
$$3x_1 - 10x_2 + x_3 - 24x_4 \leq 15$$
$$16x_1 + 5x_2 - 2x_3 + 80x_4 \geq 228$$
$$x_j \geq 0; j = 1, ..., 4$$

Tab. 2. The detailed information for implementation of the proposed method for example 2

Iter.	Objec.	x_1	x_2	x_3	x_4	D	z_1	z_2	z_3
0	max z_1	17.22	35.05	0	0	0	2976.2	348.67	-37.49
0	max z_2	16.66	4.52	10.2	0	0	783.2	386.6	233.08
0	max z_3	36.82	0	0	3.98	0	431.88	252.76	310.48
0	Utopian	57.56	30	0	0	33380.43	2975.6	555.36	310.48
1	hold z_3	56.74	28.29	-0.17	0	31484.82	2826.35	534.22	310.48
2	hold z_3	55.92	26.58	-0.33	0	29613.44	2677.35	513.33	310.48
3	hold z_1	54.4	27.09	-1.35	0	27726.82	2677.35	482.28	283.55
4	hold z_3	53.58	25.38	-1.52	0	25860.25	2528.2	461.14	283.55
5	hold z_3	52.76	23.67	-1.68	0	23988.88	2379.2	440.25	283.55
6	hold z_3	51.94	21.96	-1.85	0	22118.36	2229.95	419.11	283.55
7	hold z_3	51.12	20.25	-2.02	0	20244.01	2080.7	397.97	283.55
8	hold z_1	49.6	20.76	-3.04	0	18351.77	2080.7	366.92	256.52
9	hold z_1	48.08	21.27	-4.06	0	16465.06	2080.7	335.87	229.57
10	hold z_3	47.26	19.56	-4.23	0	14599.55	1931.65	314.73	229.57
11	hold z_3	46.44	17.85	-4.39	0	12728.18	1782.65	293.84	229.57
12	hold z_3	45.62	16.14	-4.56	0	10857.66	1633.4	272.7	229.57
13	hold z_1	44.1	16.65	-5.58	0	8965.42	1633.4	241.65	202.59
14	hold z_1	42.58	17.16	-6.6	0	7078.71	1633.4	210.6	175.64
15	hold z_3	41.12	16.21	-5.83	0	5830.92	1562.25	214.44	177.95
16	hold z_3	39.75	15.32	-4.86	0	4855.56	1501.6	224.24	183.08
17	hold z_3	38.38	14.43	-3.88	0	3882.43	1441.2	234.29	188.33
18	hold z_2	37.01	13.54	-2.91	0	2906.67	1380.55	244.09	193.46
19	hold z_2	35.64	12.65	-1.93	0	1933.53	1320.15	254.14	198.71
20	hold z_1	33.95	12.59	-1.07	0	1066.54	1320.15	265.08	195.81
21	hold z_1	32.26	12.53	-0.2	0	203.27	1320.15	276.27	193.03
22	hold z_1	30.45	12.59	0	0.52	0	1320.15	274.99	182.73
23	min δ	31.86	12.52	0	0	0	1320.2	278.8	192.28
24	max z_1	31.86	12.52	0	0	0	1320.2	278.8	192.28
25	max z_2	31.85	12.52	0	0	0	1320.2	278.8	192.28
26	max z_3	31.86	12.52	0	0	0	1320.2	278.8	192.28

The optimal solution for (36) is $(x_1,x_2,x_3,x_4)=(31.86,12.52,0,0)$ with $(z_1,z_2,z_3)=(13202,2788,19228)$ and $\delta=0.44$. Suppose the DM adopts z_1 as the objective to get improved. Hence,

$$Max z_1 = 10x_1 + 80x_2 + 25x_3 + 16x_4$$

$$s.t.$$

$$\sqrt{(x_1-32.26)^2+(x_2-12.53)^2+(x_3+0.2)^2+(x_4-0)^2} \leq 1.9$$

$$6x_1 + 7x_2 + 25x_3 + 8x_4 \geq 27499$$

$$8x_1 - 5x_2 + 12x_3 + 4x_4 \geq 18273$$

$$-6x_1 + 7x_2 + 5x_3 + 3x_4 \leq 142 \qquad\qquad (37)$$

$$2x_1 + 7x_2 + 25x_3 + 9x_4 \leq 320$$

$$20x_1 + 13x_2 + 40x_3 + 16x_4 \leq 800$$

$$3x_1 - 10x_2 + x_3 - 24x_4 \leq 15$$

$$16x_1 + 5x_2 - 2x_3 + 80x_4 \geq 228$$

$$x_j \geq 0; j = 1,...,4$$

The optimal solution for (37) is $(x_1,x_2,x_3,x_4)=(31.86,12.52,0,0)$ with $(z_1,z_2,z_3)=(13202,2788,19228)$. Since z_1 does not change, we have $S=\{2,3\}$. Then, z_2 is adopted by the DM to get improved, which leads to,

$$Max z_2 = 6x_1 + 7x_2 + 25x_3 + 8x_4$$

$$s.t.$$

$$\sqrt{(x_1-32.26)^2+(x_2-12.53)^2+(x_3+0.2)^2+(x_4-0)^2} \leq 1.9$$

$$10x_1 + 80x_2 + 25x_3 + 16x_4 \geq 13202$$

$$8x_1 - 5x_2 + 12x_3 + 4x_4 \geq 18273$$

$$-6x_1 + 7x_2 + 5x_3 + 3x_4 \leq 142 \qquad\qquad (38)$$

$$2x_1 + 7x_2 + 25x_3 + 9x_4 \leq 320$$

$$20x_1 + 13x_2 + 40x_3 + 16x_4 \leq 800$$

$$3x_1 - 10x_2 + x_3 - 24x_4 \leq 15$$

$$16x_1 + 5x_2 - 2x_3 + 80x_4 \geq 228$$

$$x_j \geq 0; j = 1,...,4$$

The optimal solution for (38) is $(x_1,x_2,x_3,x_4)=(31.86,12.52,0,0)$ with $(z_1,z_2,z_3)=(13202,2788,19228)$. Obviously, z_2 remains unchanged; so, $S=\{3\}$. The only remaining objective is z_3 and we have,

$$Max z_3 = 8x_1 - 5x_2 + 12x_3 + 4x_4$$

$$s.t.$$

$$\sqrt{(x_1-32.26)^2+(x_2-12.53)^2+(x_3+0.2)^2+(x_4-0)^2} \leq 1.9$$

$$10x_1 + 80x_2 + 25x_3 + 16x_4 \geq 13202$$

$$6x_1 + 7x_2 + 25x_3 + 8x_4 \geq 2788$$

$$-6x_1 + 7x_2 + 5x_3 + 3x_4 \leq 142 \qquad\qquad (39)$$

$$2x_1 + 7x_2 + 25x_3 + 9x_4 \leq 320$$

$$20x_1 + 13x_2 + 40x_3 + 16x_4 \leq 800$$

$$3x_1 - 10x_2 + x_3 - 24x_4 \leq 15$$

$$16x_1 + 5x_2 - 2x_3 + 80x_4 \geq 228$$

$$x_j \geq 0; j = 1,...,4$$

The optimal solution for (39) is $(x_1,x_2,x_3,x_4)=(31.86,12.52,0,0)$ with $(z_1,z_2,z_3)=(13202,2788,19228)$. Since similar to z_1 and z_2, the amount of z_3 remains unchanged, we have $S=\phi$.

Therefore, the final efficient feasible solution is $(x_1,x_2,x_3,x_4)=(31.86,12.52,0,0)$ with $(z_1,z_2,z_3)=(13202,2788,19228)$.

5. Conclusion

We have proposed a new interactive algorithm to solve MOLP problems in which we need some initial information about DM's preferences. Unlike the majority of interactive methods, we have started from the utopian point, where it's usually infeasible, and have moved towards the feasible region and the efficient set. Based on the results of some proved lemmas, we have been able to specify the amount of steps towards the feasible region. Our method satisfies most of the characteristics that a good interactive method needs, such as simplicity of the nature of judgments for DM, having opportunity to compensate improper decisions in previous interactions, and handling his nonlinear utility. The implementation of the proposed method has been demonstrated by using two numerical examples.

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