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RESEARCH PAPER

The New Distribution of Exponential Singh-Maddala, Some Properties, and Its Application in Reliability

Zahra Karimi Ezmareh[1] & Gholamhossein Yari[2*]

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ABSTRACT
In this paper, a new distribution that is highly applicable in the fields of reliability and economics is introduced. The parameters of this distribution are estimated by using two methods of Maximum Likelihood and Bayes with two prior distributions, Weibull and Uniform. These two methods are compared using Monte-Carlo simulation. Finally, this new model is fitted on the real data (with the failure time of 84 aircraft), and some of comparative criteria are calculated to confirm the superiority of the proposed model to others.

KEYWORDS *Failure time, Exponential distribution, Singh-Maddala distribution, Estimation parameters, Monte-Carlo simulation, Fitting the model.*

1. Introduction

Reliability is used in various fields of science, insurance, economics, medicine, engineering, etc. In the past, reliability was a concern for sensitive and complex industries such as military, nuclear, and aerospace industries, whose lack of reliability could exert irreparably damage; however, now, this has become a general concern.
The history of reliability growth can be referenced to the period before the 1930s. During that period, due to concerns about the proper functioning of the products, studies were conducted on the designing of systems with parallel components. The exact history of reliability was stated by Knight (1991) [10] and Andrew [2].
Now, two important numerical quantities are defined here to measure non-repairable reliability.

Definition 1-1. The mean residual life ($\boldsymbol{MRL}$) function of a lifetime random variable $\boldsymbol{X}$ is given by

$$\mu(x) = \frac{1}{\bar{F}(x)}\int_x^{\infty} tf(t)dt - x\,, x > 0. \tag{1}$$

* Corresponding author: *Gholamhossein Yari*
yari@iust.ac.ir

1. *School of Mathematics, Iran University of Science and Technology, Tehran, Iran.*
2. *School of Mathematics, Iran University of Science and Technology, Tehran, Iran.*

Definition 1-2. The mean time to failure ($\boldsymbol{MTTF}$) of a lifetime random variable is defined as:

$$MTTF = E(T) = \int_0^{\infty} tf(t)dt. \tag{2}$$

A commonly used distribution in the analysis of lifetime data is the Generalized Beta distribution of second kind (II) (GB(II)). In addition, by increasing the skewness in the income data, in order to achieve more flexible distribution in fit, four-parameter distributions with more shape parameters are introduced in economic modeling by increasing the skewness in the income data. One of these distributions is the GB(II) distribution, which was first proposed by McDonld (1984) [14]. The probability density function (pdf) of this distribution is as follows:

$$f(x) = \frac{\alpha x^{\delta\alpha - 1}}{\beta^{\delta\alpha} B(\delta,\lambda)[1 + (\frac{x}{\beta})^{\alpha}]^{\lambda+\delta}}\,, x, \alpha, \beta, \lambda, \delta > 0. \tag{3}$$

This distribution involves many statistical distributions as special or limited. One of the most important distributions, which is very useful in the fields of reliability, economics, and finance, is the distribution of the three parameters of Singh Maddala (SM), obtained by placing $\boldsymbol{\delta = 1}$. McDonld (1984)[14] showed that the SM distribution provided better fits than gamma and lognormal. Shahzad and Asghar (2013) [16] used the L-moments and TL-moments methods to derive the point estimators of the parameters for SM distribution. The family of distributions proposed by Singh and Maddala (1976) [17], whose core distribution was a generalized beta distribution, became a popular distribution for fitting the distribution on income and expenditure.

Definition 1-3. The random variable $\boldsymbol{X}$ has Singh-Maddala ($\boldsymbol{SM}$) distribution with parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, and $\boldsymbol{\mu}$ and is represented by the symbol $\boldsymbol{X} \sim \boldsymbol{SM}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\mu})$ if the cumulative distribution function ($\boldsymbol{cdf}$) is as follows:

$$G(x) = 1 - \left[1 + \left(\frac{x-\mu}{\beta}\right)^{\frac{1}{\gamma}}\right]^{-\alpha}, \; x \geq \mu \tag{4}$$

The SM distribution is also known as Pareto(IV)(Arnold (1983)[3]), Beta-P (Mielke and Johnson (1974)[15]), and generalized log-logestic (El-Saidi et al. (1990)[7]) distributions. This distribution is a common distribution in the modeling of lifetime data and personal income data. Its positive features include the closed form of the distribution function and its quantile function, as well as the relatively simple estimation of its parameters based on nonlinear regression.

Since this distribution has been used extensively in the economics, reliability, and analysis of lifetime data, it has been studied by many authors. Among these studies, we can mention the following: Devendra Kumar (2017)[5] investigated the involved properties and estimated the parameters of this distribution. Ojo and Olapade (2003)[11] studied the moments of this distribution. Srinivasa et al. (2013)[8] carried out an economic reliability test plan for this distribution. Vytaras Brazauskas (2003)[18] reviewed the information Matrix for SM and related distributions.

In the theory of distributions, using some transformations and mathematical relations, new distributions are obtained, which are known as generalized distributions. In addition, the generalizations of SM distribution have been studied recently. Ayman and Ghosh (2016)[4] introduced Gamma-SM distribution and its applications. Rosaiah1 et al. (2016) [9] studied the Odd Generalized SM distribution. Marcelo Bourguignon studied a new type of this distribution with application in reliability and income data [12]. However, in this paper, a new generalization of SM distribution is introduced. The transformation of this generalization is expressed by Abed Al-Kadim and Boshi (2013)[1], and its definition is as follows:

Definition 1-4. If $\boldsymbol{X}$ has a cdf $\boldsymbol{G(x)}$, then the function

$$F(x) = \int_0^{\frac{1}{1-G(x)}} \lambda e^{-\lambda t} dt, \; t \in [0, \infty), \tag{5}$$

is called Exponential distribution [1] and [20].

The rest of this article is organized as follows: Section 2 introduces the new Exponential Singh-Maddala (ESM) distribution. Section 3 is concerned with some of the statistical properties of this distribution. Section 4 is dedicated to the estimation of the parameters of this distribution using two methods of Maximum Likelihood (MLE) and Bayes (with two prior distribution Weibull and Uniform, which are shown with the symbol BW and BU respectively). Then, in Section 5, we compare these methods using the Monte Carlo simulation. Finally, Section 6 shows the agreement of distribution with the real data (the failure time of 84 aircraft) and proves the superiority of this distribution to other distributions.

2. New Model

To introduce a new distribution by substituting (4) into (5), the Exponential Singh-Maddala(ESM) distribution is obtained as follows:

$$F(x) = \int_0^{\frac{1}{[1+(\frac{x-\mu}{\beta})^{\frac{1}{\gamma}}]^{-\alpha}}} \lambda e^{-\lambda t} dt = 1 - e^{-\lambda\left[1+\left(\frac{x-\mu}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}} \tag{6}$$

The pdf, the Survival function (S), and the Hazard rate function (H) of this distribution are as follows:

$$f(x) = \frac{\lambda\alpha}{\gamma\beta^{\frac{1}{\gamma}}}(x-\mu)^{\frac{1}{\gamma}-1}\left[1 + \left(\frac{x-\mu}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha-1} e^{-\lambda\left[1+\left(\frac{x-\mu}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}}, \tag{7}$$

$$s(x) = \bar{F}(x) = e^{-\lambda\left[1+\left(\frac{x-\mu}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}}, \tag{8}$$

$$H(x) = \frac{f(x)}{\bar{F}(x)} = \frac{\lambda\alpha}{\gamma\beta^{\frac{1}{\gamma}}}(x-\mu)^{\frac{1}{\gamma}-1}\left[1+\left(\frac{x-\mu}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha-1} \tag{9}$$

where $x \geq \mu$ and $\alpha, \beta, \gamma, \lambda > 0$.
Quantiles, skewness, and kurtosis of ESM in Table (1) and cdf (F), pdf(f), S, and H of ESM in Table (2) for selected values of $\alpha = 2, \beta = 3, \gamma = 4$ and different values of λ are shown. In addition, the graph of cdf (F), pdf (f), S, and H functions for ESM for different values of parameters is shown in Figure (1).

Tab. 1. Quantiles, skewness, and kurtosis of ESM for selected values of $\alpha = 2, \beta = 3, \gamma = 4$, and $x = 0.6$

Parameter	Median	Q_1	Q_3	Skewness	Kurtosis
$\lambda = 0.5$	0.0030	0.0102	0.5871	1.0302	5.6507
$\lambda = 1.2$	0.0100	0.2035	0.0001	-0.952	2.4728
$\lambda = 2$	0.0858	0.4454	0.0024	-0.8169	1.6257
$\lambda = 3.2$	0.2450	0.7210	0.0409	-0.7601	1.2992

Tab. 2. Cdf(F), pdf(f), S, and H of ESM for selected values of $\alpha = 2, \beta = 3, \gamma = 4$, and $x = 0.6$

Parameter	F	f	H	S
$\lambda = 0.5$	0.7515	0.0788	0.3170	0.2485
$\lambda = 1.2$	0.9646	0.0269	0.7608	0.0354
$\lambda = 2$	0.9962	0.0048	1.2680	0.0038
$\lambda = 3.2$	0.9999	0.0003	2.0287	0.0001

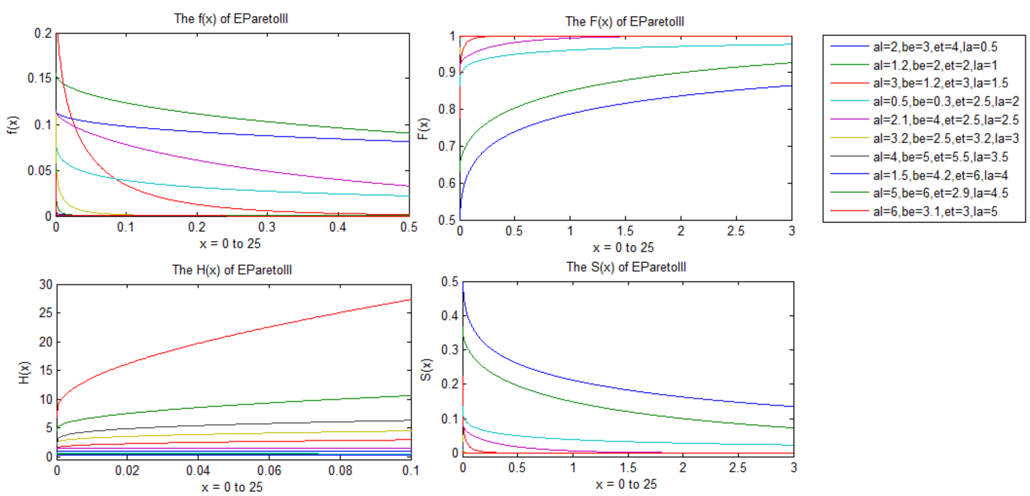

Fig. 1. Figures of cdf, pdf, S, and H of ESM for different values of the parameters

2-1. Sub-models

In distribution $F(x) = 1 - e^{-\lambda\left[1+\left(\frac{x-\mu}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}}$, the sub models are as follows:
when $\mu = 0, \gamma = 1$, we have the Exponential-Lomax distribution.
when $\alpha = 1, \mu = 1, \beta = 1, \gamma = 1$, we have the Exponential distribution.
when $\mu = \beta, \gamma = 1, \lambda = 1$, we have the Weibull distribution.

3. Some Statistical Properties

This section studies some of the statistical properties of the ESM distribution. The following article assumes that $\mu = 0$.

Theorem 3.1. If X has $ESM(\alpha, \beta, \gamma, \lambda)$ distribution, then the quantile of a random variable X is given by

$$x_p = \left(\left(\frac{-\log(1-p)}{\lambda}\right)^{\frac{1}{\alpha}} - 1\right)^{\gamma}. \tag{10}$$

Proof: Given that the quantile of any distribution is given by solving Equation $F(x_p) = p, 0 < p < 1$, as a result, the quantile of ESM distribution is calculated as follows:

$$F(x_p) = p \Leftrightarrow 1 - e^{-\lambda\left[1+\left(\frac{x_p}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}} = p \Leftrightarrow \log(1-p) = -\lambda\left[1+\left(\frac{x_p}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}, \Leftrightarrow \left(\frac{-\log(1-p)}{\lambda}\right)^{\frac{1}{\alpha}} = 1 + \left(\frac{x_p}{\beta}\right)^{\frac{1}{\gamma}}, \Leftrightarrow x_p = \left(\left(\frac{-\log(1-p)}{\lambda}\right)^{\frac{1}{\alpha}} - 1\right)^{\gamma}. \tag{11}$$

Theorem 3.2. If X has $ESM(\alpha, \beta, \gamma, \lambda)$ distribution, then the rth moments of a random variable X are given by

$$E(X^r) = \beta^r \sum_{j=0}^{\infty} \binom{\gamma r}{j} \frac{(-1)^{\gamma r-j}}{\lambda^{\frac{j}{\alpha}}} \left[1 - \Gamma\left(\frac{j}{\alpha}+1, \lambda\right)\right]. \tag{12}$$

Proof: Using the transform $u = -\lambda(1 + (\frac{x}{\beta})^{\frac{1}{\gamma}})^{\alpha}$, Binomial expansion $(1+x)^p = \sum_{i=0}^{\infty}\binom{p}{i} x^i, p \in R$, and definition expectation, we have:

$$E(X^r) = \int_0^{\infty} x^r f(x)dx = \frac{\lambda\alpha}{\gamma\beta^{\frac{1}{\gamma}}} \int_0^{\infty} x^{r+\frac{1}{\gamma}-1}\left[1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha-1} e^{-\lambda\left[1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}} dx = \int_{\lambda}^{\infty} e^{-u}\beta^r\left(\left(\frac{u}{\lambda}\right)^{\frac{1}{\alpha}} - 1\right)^{\gamma r} du, = \beta^r \sum_{j=0}^{\infty}\binom{\gamma r}{j}\frac{(-1)^{\gamma r-j}}{\lambda^{\frac{j}{\alpha}}}\int_{\lambda}^{\infty} e^{-u}u^{\frac{j}{\alpha}}du = \beta^r \sum_{j=0}^{\infty}\binom{\gamma r}{j}\frac{(-1)^{\gamma r-j}}{\lambda^{\frac{j}{\alpha}}}\left[1 - \Gamma\left(\frac{j}{\alpha}+1, \lambda\right)\right]. \tag{13}$$

where $\Gamma(c, x) = \int_0^x t^{c-1}e^{-t}dt.$

Theorem 3.3. If X has $ESM(\alpha, \beta, \gamma, \lambda)$ distribution, then the moment-generating function(mgf) of a random variable X is given by

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\gamma i}{j}\frac{(\beta t)^i(-1)^{\gamma i-j}}{i!\lambda^{\frac{j}{\alpha}}}\left[1 - \Gamma\left(\frac{j}{\alpha}+1, \lambda\right)\right]. \tag{14}$$

Proof: Using the transform $u = -\lambda(1 + (\frac{x}{\beta})^{\frac{1}{\gamma}})^{\alpha}$, Binomial expansion, expansion $e^{tx} = \sum_{i=0}^{\infty}\frac{(tx)^i}{i!}$, and definition mgf, we have

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx}f(x)dx = \frac{\lambda\alpha}{\gamma\beta^{\frac{1}{\gamma}}}\int_0^{\infty} e^{tx}x^{\frac{1}{\gamma}-1}\left[1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha-1} e^{-\lambda\left[1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}} dx, =$$

$$\frac{\lambda\alpha}{\gamma\beta^{\frac{1}{\gamma}}}\int_0^{\infty}\sum_{i=0}^{\infty}\frac{t^i}{i!}x^{i+\frac{1}{\gamma}-1}\left[1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha-1} e^{-\lambda\left[1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}} dx, =$$

$$\sum_{i=0}^{\infty}\frac{t^i}{i!}\int_{\lambda}^{\infty} e^{-u}\beta^i\left(\left(\frac{u}{\lambda}\right)^{\frac{1}{\alpha}} - 1\right)^{\gamma i} du, \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\gamma i}{j}\frac{(\beta t)^i(-1)^{\gamma i-j}}{i!\lambda^{\frac{j}{\alpha}}}\int_{\lambda}^{\infty} e^{-u}u^{\frac{j}{\alpha}}du, =$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\gamma i}{j}\frac{(\beta t)^i(-1)^{\gamma i-j}}{i!\lambda^{\frac{j}{\alpha}}}\left[1 - \Gamma\left(\frac{j}{\alpha}+1, \lambda\right)\right]. \tag{15}$$

Theorem 3.4. If **X** has $ESM(\alpha, \beta, \gamma, \lambda)$ distribution, then the MRL function of a lifetime random variable X is given by

$$MRL = \frac{\beta}{\bar{F}(x)}\sum_{i=0}^{\infty}\binom{\gamma}{i}\frac{(-1)^{\gamma-i}}{\lambda^{\frac{i}{\alpha}}}\left[1 - \Gamma\left(\frac{i}{\alpha}+1, \lambda\left[1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}\right)\right]. \tag{16}$$

Proof: Using the transform $u=-\lambda(1+(\frac{x}{\beta})^{\frac{1}{\gamma}})^{\alpha}$, Binomial expansion, and definition MRL, we have:

$$MRL=\frac{1}{\bar{F}(x)}\frac{\lambda\alpha}{\gamma\beta^{\frac{1}{\gamma}}}\int_x^{\infty}t^{\frac{1}{\gamma}}\left[1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha-1}e^{-\lambda\left[1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}}dt-x=\frac{\beta}{\bar{F}(x)}\int_{\left(1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha}}^{\infty}e^{-u}\left(\left(\frac{u}{\lambda}\right)^{\frac{1}{\alpha}}-1\right)^{\gamma}du-x,=$$

$$\frac{\beta}{\bar{F}(x)}\sum_{i=0}^{\infty}\binom{\gamma}{i}\frac{(-1)^{\gamma-i}}{\lambda^{\frac{i}{\alpha}}}\int_{\left(1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha}}^{\infty}e^{-u}u^{\frac{i}{\alpha}}du-x=\frac{\beta}{\bar{F}(x)}\sum_{i=0}^{\infty}\binom{\gamma}{i}\frac{(-1)^{\gamma-i}}{\lambda^{\frac{i}{\alpha}}}\left[1-\Gamma\left(\frac{i}{\alpha}+1,\lambda\left[1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}\right)\right]. \quad (17)$$

Theorem3.5. Mean deviation from the mean of the ESM distribution is as follows:

$$D(\mu)=2(\mu-d)-2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\alpha i}{j}\left(\frac{-\lambda}{\beta^{\frac{1}{\gamma}}}\right)^{i}\frac{\beta^{\frac{j}{\alpha}}}{\left(\frac{\alpha i-j}{\gamma}+1\right)}\left(\mu^{\frac{\alpha i-j}{\gamma}+1}-d^{\frac{\alpha i-j}{\gamma}+1}\right). \quad (18)$$

Proof: According to the definition and using Binomial expansion, we have:

$$D(\mu)=E(|X-\mu|)=\int_d^{\infty}|X-\mu|\,f(x)dx=\int_d^{\mu}(\mu-x)\,f(x)dx+\int_{\mu}^{\infty}(x-\mu)\,f(x)dx=2\int_d^{\mu}(\mu-x)\,f(x)dx,=$$

$$2\mu\int_d^{\mu}f(x)\,dx-2\int_d^{\mu}x\,f(x)dx=2\mu f(\mu)-2\int_d^{\mu}x\,f(x)dx=2\int_d^{\mu}F(x)\,dx,=2(\mu-d)-2\int_d^{\mu}e^{-\lambda\left[1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}}dx=$$

$$2(\mu-d)-2\int_d^{\mu}\sum_{i=0}^{\infty}\frac{\left[-\lambda\left(1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha}\right]^{i}}{i!}dx,=2(\mu-d)-2\sum_{i=0}^{\infty}\left(\frac{-\lambda}{\beta^{\frac{1}{\gamma}}}\right)^{i}\int_d^{\mu}\left(\beta^{\frac{1}{\gamma}}+x^{\frac{1}{\gamma}}\right)^{\alpha i}dx,=$$

$$2(\mu-d)-2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\alpha i}{j}\left(\frac{-\lambda}{\beta^{\frac{1}{\gamma}}}\right)^{i}\frac{\beta^{\frac{j}{\alpha}}}{\left(\frac{\alpha i-j}{\gamma}+1\right)}\left(\mu^{\frac{\alpha i-j}{\gamma}+1}-d^{\frac{\alpha i-j}{\gamma}+1}\right). \quad (19)$$

Theorem 3.6. Mean deviation from the median of the ESM distribution is as follows:

$$D(m)=\mu+m-2d-2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\alpha i}{j}\left(\frac{-\lambda}{\beta^{\frac{1}{\gamma}}}\right)^{i}\frac{\beta^{\frac{j}{\alpha}}}{\left(\frac{\alpha i-j}{\gamma}+1\right)}\left(m^{\frac{\alpha i-j}{\gamma}+1}-d^{\frac{\alpha i-j}{\gamma}+1}\right). \quad (20)$$

Proof: According to the definition and using Binomial expansion, we have:

$$D(m)=E(|X-m|)\int_d^{\infty}|X-m|\,f(x)dx=\int_d^{m}(m-x)\,f(x)dx+\int_m^{\infty}(x-\mu)\,f(x)dx,=$$

$$2mF(m)-2\int_d^{m}x\,f(x)dx+E(X-m)=2mF(m)-2\int_d^{m}x\,f(x)dx+\mu-m,=\mu-m+2\int_d^{m}F(x)\,dx,=\ \mu+$$

$$m-2d-2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\alpha i}{j}\left(\frac{-\lambda}{\beta^{\frac{1}{\gamma}}}\right)^{i}\frac{\beta^{\frac{j}{\alpha}}}{\left(\frac{\alpha i-j}{\gamma}+1\right)}\left(m^{\frac{\alpha i-j}{\gamma}+1}-d^{\frac{\alpha i-j}{\gamma}+1}\right). \quad (21)$$

Theorem 3.7. If X has $ESM(\alpha,\beta,\gamma,\lambda)$ distribution, then the kth order statistics pdf of a random variable X is given by

$$f_{X_{(k)}}(x)=\frac{n!}{(k-1)!(n-k)!}\times\frac{\lambda\alpha}{\gamma\beta^{\frac{1}{\gamma}}}x^{\frac{1}{\gamma}-1}\left[1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha-1}\left[e^{-\lambda\left[1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}}\right]^{n-k+1}\left[1-e^{-\lambda\left[1+\left(\frac{x}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}}\right]^{k-1}. \quad (22)$$

Proof: To prove it, it is sufficient to put the cdf and the pdf of ESM in the formula for the pdf of the kth order statistics as follows:

$$f_{X_{(k)}}(x)=\frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}[1-F(x)]^{n-k}f(x). \quad (23)$$

4. Estimation Parameters

4-1. Maximum likelihood estimation (MLE)

Let $X_1, \ldots, X_n$ denote random samples, each having the pdf of the ESM distribution with parameter vector $\Theta = (\alpha, \beta, \gamma, \lambda)$. Then, the likelihood function is given by

$$l(x_1, \ldots, x_n|\Theta) = \prod_{i=1}^{n} \frac{\lambda\alpha}{\gamma\beta^{\frac{1}{\gamma}}} x_i^{\frac{1}{\gamma}-1} \left[1 + \left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha-1} e^{-\lambda\left[1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}} = \left(\frac{\lambda\alpha}{\gamma\beta^{\frac{1}{\gamma}}}\right)^{n} \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{\gamma}-1} \prod_{i=1}^{n} \left[1 + \left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha-1} e^{-\lambda\sum_{i=1}^{n}\left[1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}}. \tag{24}$$

The logarithm of the likelihood function is as follows

$$L = \log l(x_1, \ldots, x_n|\Theta) = n\log\lambda + n\log\alpha - n\log\gamma - \frac{n}{\gamma}\log\beta + \left(\frac{1}{\gamma} - 1\right)\sum_{i=1}^{n}\log x_i + (\alpha - 1)\sum_{i=1}^{n}\log\left[1 + \left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right] - \lambda\sum_{i=1}^{n}\left[1 + \left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}. \tag{25}$$

By taking the partial derivatives of the above log-likelihood function with respect to $\alpha, \beta, \gamma,$ and λ, the maximum likelihood estimates of this parameters are obtained:

$$\frac{\partial L}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left[1 + \left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right]\left(1 - \lambda\left[1 + \left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha}\right) = 0, \tag{26}$$

$$\frac{\partial L}{\partial\beta} = -\frac{n}{\gamma\beta} + \frac{1}{\gamma}\left(\frac{1}{\beta}\right)^{\frac{1}{\gamma}+1}\sum_{i=1}^{n}\frac{x_i^{\frac{1}{\gamma}}}{1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}}\left\{1 - \alpha + \lambda\alpha\left(1 + \left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha}\right\} = 0, \tag{27}$$

$$\frac{\partial L}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\log\left[1 + \left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right]^{\alpha} = 0, \tag{28}$$

$$\frac{\partial L}{\partial\gamma} = -\frac{n}{\gamma} + \frac{n}{\gamma^2}\log\beta - \sum_{i=1}^{n}\frac{\log x_i}{\gamma^2} + \sum_{i=1}^{n}\frac{x_i^{\frac{1}{\gamma}}\log x_i}{\beta^{\frac{1}{\gamma}}\gamma^2}\left(1 + \left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{-1}\left\{1 - \alpha + \lambda\alpha\left(1 + \left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha}\right\} = 0. \tag{29}$$

In addition, by deriving the second derivative of the logarithm likelihood function, the elements of the information matrix are obtained as follows:

$$I(\Theta) = \begin{bmatrix} \frac{\partial^2 L}{\partial\alpha^2} & \frac{\partial^2 L}{\partial\alpha\partial\beta} & \frac{\partial^2 L}{\partial\alpha\partial\gamma} & \frac{\partial^2 L}{\partial\alpha\partial\lambda} \\ \frac{\partial^2 L}{\partial\beta\partial\alpha} & \frac{\partial^2 L}{\partial\beta^2} & \frac{\partial^2 L}{\partial\beta\partial\gamma} & \frac{\partial^2 L}{\partial\beta\partial\lambda} \\ \frac{\partial^2 L}{\partial\gamma\partial\alpha} & \frac{\partial^2 L}{\partial\gamma\partial\beta} & \frac{\partial^2 L}{\partial\gamma^2} & \frac{\partial^2 L}{\partial\gamma\partial\lambda} \\ \frac{\partial^2 L}{\partial\lambda\partial\alpha} & \frac{\partial^2 L}{\partial\lambda\partial\beta} & \frac{\partial^2 L}{\partial\lambda\partial\gamma} & \frac{\partial^2 L}{\partial\lambda^2} \end{bmatrix} \tag{30}$$

$$\frac{\partial^2 L}{\partial\alpha^2} = -\frac{n}{\alpha^2} - \lambda\sum_{i=1}^{n}\left(1 + \left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha}\left[\log\left(1 + \left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)\right]^2, \tag{31}$$

$$\frac{\partial^2 L}{\partial\lambda^2} = -\frac{n}{\lambda^2}, \tag{32}$$

$$\frac{\partial^2 L}{\partial \beta^2}=\frac{n}{\gamma\beta^2}-\frac{1-\alpha}{\gamma}\sum_{i=1}^{n}x_i^{\frac{1}{\gamma}}\left\{\left(\frac{1}{\beta}\right)^{\frac{1}{\gamma}+2}\left(\frac{1}{\gamma}+1\right)\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)-\left(\frac{1}{\beta}\right)^{\frac{2}{\gamma}+2}\frac{x_i^{\frac{1}{\gamma}}}{\gamma}\right\}-\frac{\lambda\alpha}{\gamma}\sum_{i=1}^{n}x_i^{\frac{1}{\gamma}}\left\{\left(\frac{1}{\beta}\right)^{\frac{1}{\gamma}+1}\left(\frac{1}{\gamma}+1\right)\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha-1}+\frac{\alpha-1}{\gamma}\left(\frac{1}{\beta}\right)^{\frac{2}{\gamma}+2}x_i^{\frac{1}{\gamma}}\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha-2}\right\}, \tag{33}$$

$$\frac{\partial^2 L}{\partial \gamma^2}=\frac{n}{\gamma^2}+\frac{2}{\gamma^3}\sum_{i=1}^{n}\log\left(\frac{x_i}{\beta}\right)+\frac{1-\alpha}{\gamma^4}\sum_{i=1}^{n}\frac{x_i^{\frac{1}{\gamma}}\log x_i}{\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^2}\left\{\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)\left(\frac{\log\beta}{\beta^{\frac{2}{\gamma}}}-\frac{2\gamma}{\beta^{\frac{1}{\gamma}}}-\frac{\log x_i}{\beta^{\frac{1}{\gamma}}}\right)+\frac{x_i^{\frac{1}{\gamma}}}{\beta^{\frac{2}{\gamma}}}\log\left(\frac{x_i}{\beta}\right)\right\}+\frac{\lambda\alpha}{\gamma^4}\sum_{i=1}^{n}x_i^{\frac{1}{\gamma}}\log x_i\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha-1}\left\{\frac{\log\beta}{\beta^{\frac{2}{\gamma}}}-\frac{2\gamma}{\beta^{\frac{1}{\gamma}}}-\frac{\log x_i}{\beta^{\frac{1}{\gamma}}}-\frac{\alpha-1}{\beta^{\frac{3}{\gamma}}}x_i^{\frac{1}{\gamma}}\log x_i\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{-1}\right\}, \tag{34}$$

$$\frac{\partial^2 L}{\partial \alpha\partial \gamma}=\frac{1}{\gamma^2}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\log\left(\frac{x_i}{\beta}\right)}{\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)}\left\{-1+\lambda\alpha\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha}\log\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)-\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha}\right\}, \tag{35}$$

$$\frac{\partial^2 L}{\partial \alpha\partial \lambda}=-\sum_{i=1}^{n}\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha}\log\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right), \tag{36}$$

$$\frac{\partial^2 L}{\partial \alpha\partial \beta}=\frac{1}{\gamma}\left(\frac{1}{\beta}\right)^{\frac{1}{\gamma}+1}\sum_{i=1}^{n}\frac{x_i^{\frac{1}{\gamma}}}{\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)}\left\{-1+\lambda\alpha\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha-1}\log\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)-\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha}\right\}, \tag{37}$$

$$\frac{\partial^2 L}{\partial \lambda\partial \beta}=\frac{\alpha}{\lambda}\left(\frac{1}{\beta}\right)^{\frac{1}{\gamma}+1}\sum_{i=1}^{n}x_i^{\frac{1}{\gamma}}\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha-1}, \tag{38}$$

$$\frac{\partial^2 L}{\partial \lambda\partial \gamma}=\frac{\alpha}{\gamma^2}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\log\left(\frac{x_i}{\beta}\right)\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha-1}, \tag{39}$$

$$\frac{\partial^2 L}{\partial \gamma\partial \beta}=\frac{n}{\beta\gamma^2}\sum_{i=1}^{n}\frac{x_i^{\frac{1}{\gamma}}\log x_i}{\gamma^3}\left\{\left(-\left(\frac{1}{\beta}\right)^{\frac{1}{\gamma}+1}\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)\right)\left(1-\alpha-\lambda\alpha\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha-2}\right)+\left(\left(\frac{1}{\beta}\right)^{\frac{2}{\gamma}+1}x_i^{\frac{1}{\gamma}}\right)\left(1-\alpha-\lambda\alpha(\alpha-1)\left(1+\left(\frac{x_i}{\beta}\right)^{\frac{1}{\gamma}}\right)^{\alpha-2}\right)\right\}. \tag{40}$$

The approximate $(1-\xi)100\%$ confidence intervals for the parameters α,β,γ, and λ are determined, respectively, as follows:

$$\hat{\alpha}\pm Z_{\frac{\xi}{2}}\sqrt{\frac{Var(\hat{\alpha})}{n}},\hat{\beta}\pm Z_{\frac{\xi}{2}}\sqrt{\frac{Var(\hat{\beta})}{n}},\hat{\gamma}\pm Z_{\frac{\xi}{2}}\sqrt{\frac{Var(\hat{\gamma})}{n}},\hat{\lambda}\pm Z_{\frac{\xi}{2}}\sqrt{\frac{Var(\hat{\lambda})}{n}}, \tag{41}$$

where $Var(\hat{\alpha}),Var(\hat{\beta}),Var(\hat{\gamma})$, and $Var(\hat{\lambda})$ are given by the diagonal elements of $I(\Theta)$, and $Z_{\frac{\xi}{2}}$ is the upper $\frac{\xi}{2}$ percentile of the standard normal distribution.

4-2. Bayes estimation

To estimate the parameters of ESM distribution using the Bayes method, it is assumed that the previous information of α,β,γ, and λ is independent of each other; therefore, $\pi(\alpha,\beta,\gamma,\lambda)=\pi(\alpha)\pi(\beta)\pi(\gamma)\Pi(\lambda)$. In this method, we use the two prior distributions Weibull (BW) and Uniform (BU) using the Monte Carlo simulation. Because the denominators of the posterior distributions have a four-integral, it is not easy to calculate them; therefore, "the important sampling method" (Rubinstein and Kroese(2008) is used [19]. In the relevant section, the results are presented.

5. Simulation Study with Using Monte-Carlo Method

In this section, using the Monte Carlo simulation, the ESM distribution parameters are estimated in two ways: MLE and Bayes (BW and BU). For this purpose, first, a random sample from this distribution (for selected values of $\alpha=2,\beta=$

$3, \gamma = 4$, and $\lambda = 0.5$) is generated by the following algorithm, which is known as the Inverse-transform method:

1. Generate U from $U(0,1)$.
2. Return $X = F^{-1}(U)$.

where $X = ((\frac{-\log(1-U)}{\lambda})^{\frac{1}{\alpha}} - 1)^{\gamma}$

Note that the number of replicates is $j = 1000$. Then, the estimation of the parameters is performed using the Monte-Carlo simulation and by means of the MATLAB software. Finally, for comparing these two methods, criteria such as Variance, Mean Squared Error (MSE), and ratio $\frac{\hat{\Theta}}{\Theta_{true}}$ are used. The estimated results of the sample sizes $n = 20,40,\ldots,100$ are summarized in Figures (2), (3), (4), and (5) and also in Tables (3), (4), (5), and (6).

According to Figures (2), (3), (4), and (5) and Tables (3), (4), (5), and (6), it can be concluded that the use of the Bayes, especially with the Uniform prior, provides estimates very close to the real value. In the MLE method, with increasing sample size, the estimators have appropriate values. The method of BU has a very small bias and its variance and MSE is less than MLE and BW.

Tab. 3. Results of estimation of α using MLE, BW, and BU methods.

n	Method	$\hat{\alpha}$	$Var(\hat{\alpha})$	$\frac{\hat{\alpha}}{\alpha_{True}}$	$MSE(\hat{\alpha})$
	MLE	2.3051	0.1215	1.1525	0.2146
20	BW	1.7659	0.4032	0.8829	0.4580
	BU	2.0202	0.3285	1.0101	0.3289
	MLE	2.2730	0.0472	1.1365	0.1217
40	BW	1.7928	0.4151	0.8964	0.4581
	BU	1.9786	0.3309	0.9893	0.3314
	MLE	2.2621	0.0311	1.1310	0.0997
60	BW	1.7842	0.4034	0.8921	0.4500
	BU	2.0250	0.3396	1.0125	0.3403
	MLE	2.2501	0.0255	1.1251	0.0880
80	BW	1.7629	0.4288	0.8814	0.4850
	BU	1.9771	0.3341	0.9885	0.3346
	MLE	2.2440	0.0191	1.1220	0.0786
100	BW	1.7613	0.4104	0.8806	0.4674
	BU	2.0040	0.3205	1.0020	0.3205

Tab. 4. Results of estimation of β using MLE, BW, and BU methods.

n	Method	$\hat{\beta}$	$Var(\hat{\beta})$	$\frac{\hat{\beta}}{\beta_{True}}$	$MSE(\hat{\beta})$
	MLE	1.7084	1.1776	0.5695	2.8458
20	BW	1.8155	0.2444	0.6052	1.6474
	BU	3.0084	0.3261	1.0028	0.3262
	MLE	1.6205	0.7029	0.5402	2.6059
40	BW	1.8285	0.2458	0.6095	1.6181
	BU	3.0187	0.3289	1.0062	0.3292
	MLE	1.5670	0.4595	0.5223	2.5131
60	BW	1.8359	0.2722	0.6120	1.6272
	BU	3.0252	0.3273	1.0084	0.3280
	MLE	1.5760	0.3743	0.5255	2.4009
80	BW	1.8228	0.2519	0.6076	1.6378
	BU	3.0034	0.3305	1.0011	0.3305
	MLE	1.5585	0.2791	0.5195	2.3571
100	BW	1.7792	0.2357	0.5931	1.7261
	BU	2.9827	0.3455	0.9942	0.3458

Tab. 5. Results of estimation of γ using MLE, BW, and BU methods.

n	Method	$\hat{\gamma}$	$Var(\hat{\gamma})$	$\frac{\hat{\gamma}}{\gamma_{True}}$	$MSE(\hat{\gamma})$
	MLE	4.7648	0.5888	1.1912	1.1736
20	BW	3.6692	0.7177	0.9173	0.8271
	BU	4.0285	0.3290	1.0071	0.3298
	MLE	4.7794	0.2851	1.1949	0.8926
40	BW	3.7220	0.7083	0.9305	0.7856
	BU	3.9917	0.3257	0.9979	0.3258
	MLE	4.7662	0.1925	1.1916	0.7796
60	BW	3.6772	0.7144	0.9193	0.8186
	BU	4.0005	0.3306	1.0001	0.3306
	MLE	4.7832	0.1548	1.1958	0.7682
80	BW	3.6624	0.7746	0.9156	0.8885
	BU	3.9961	0.3406	0.9990	0.3406
	MLE	4.7890	0.1185	1.1972	0.7409
100	BW	3.6664	0.6497	0.9166	0.7610
	BU	3.9665	0.3469	0.9916	0.3480

Tab. 6. Results of estimation of λ using MLE, BW, and BU methods.

n	Method	$\hat{\lambda}$	$Var(\hat{\lambda})$	$\frac{\hat{\lambda}}{\lambda_{True}}$	$MSE(\hat{\lambda})$
	MLE	0.3946	0.0033	0.7891	0.0144
20	BW	0.4689	0.2158	0.9378	0.2168
	BU	0.5476	0.0677	1.0952	0.0699
	MLE	0.3921	0.0016	0.7842	0.0132
40	BW	0.4912	0.2253	0.9824	0.2254
	BU	0.5427	0.0691	1.0952	0.0709
	MLE	0.3914	0.0011	0.7828	0.0129
60	BW	0.5158	0.2686	1.0316	0.2689
	BU	0.5375	0.0692	1.0749	0.0706
	MLE	0.3901	0.0009	0.7802	0.0129
80	BW	0.5351	0.2645	1.0701	0.2658
	BU	0.5566	0.0683	1.1132	0.0715
	MLE	0.3898	0.0007	0.7796	0.0128
100	BW	0.4937	0.2739	0.9873	0.2740
	BU	0.5583	0.0695	1.1166	0.0729

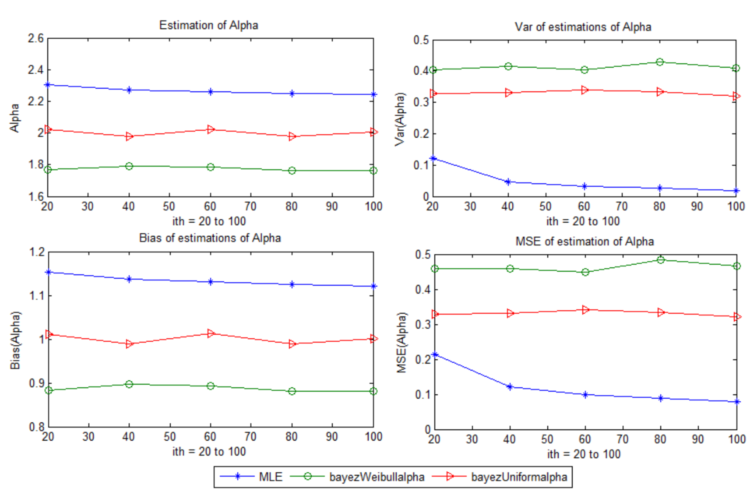

Fig. 2. Results of estimation of α using MLE, BW, and BU methods.

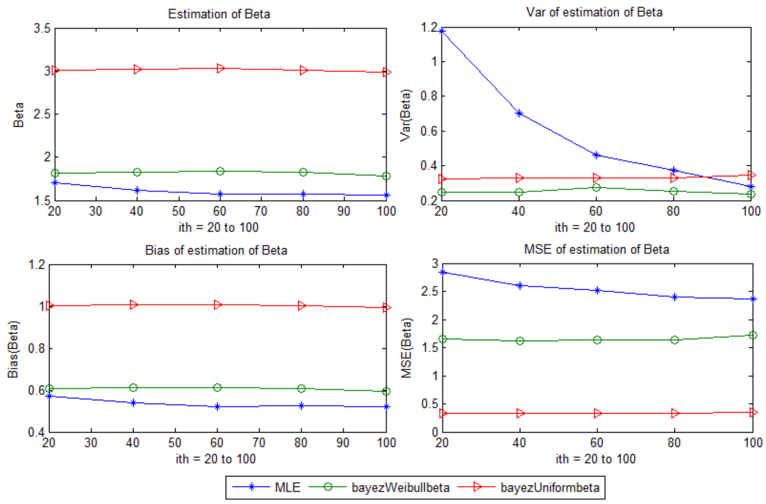

Fig. 3. Results of estimation of β using MLE, BW, and BU methods

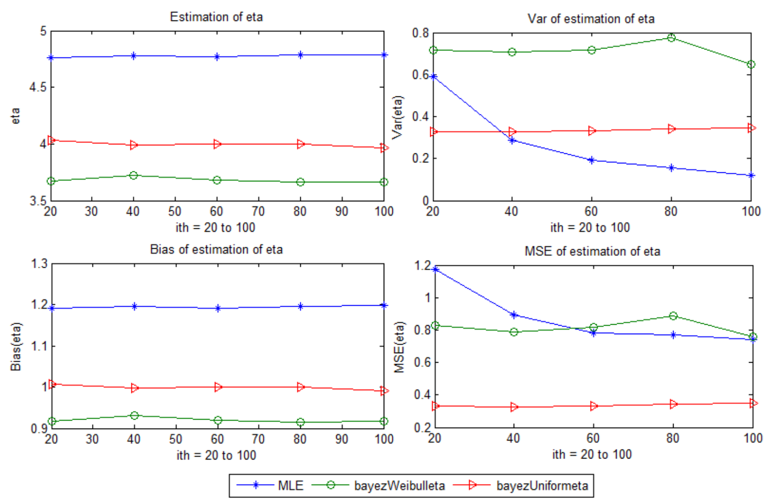

Fig. 4. Results of estimation of γ using MLE, BW, and BU methods

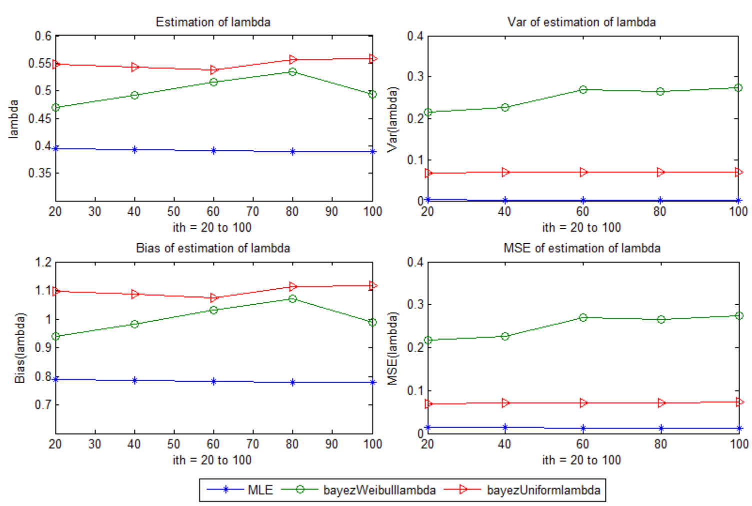

Fig. 5. Results of estimation of λ using MLE, BW, and BU methods

6. Goodness of Fit Tests and Model Selection Criteria

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be order statistics of random sample $X_1, X_2, \ldots, X_n$ from distribution with cdf $F(x)$ and let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be observations in ascending order, so that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$. The empirical distribution function is defined as follows:

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(x_{(i)} \leq x). \quad (42)$$

where $I(x_{(i)} \leq x) = 1$, if $x_{(i)} \leq x$ and 0 otherwise.

Kolmogorov-Smirnov test ($K-S$): We want to test the hypothesis

H_0 : The data follow the specified distribution with $cdf\ F(x)$,

H_1 : The data do not follow the specified distribution with $cdf\ F(x)$.

The $K-S$ test statistics can be computed by calculating

$$D_n = \max_{1 \leq i \leq n}\left\{\left|\frac{i}{n} - F(x_{(i)})\right|, \left|F(x_{(i)}) - \frac{i-1}{n}\right|\right\}. \quad (43)$$

In this study, $P-Value$ is used in hypothesis testing. The hypothesis H_0 is rejected at the chosen significance level α if $P-Value < \alpha$. While comparing two different distributions, the distribution with higher $p-value$ is likely to better fit regardless of the level of significance.

Akaike's information criterion (AIC), Bayesian information criterion (BIC), Consistent Akaike information criteria(CAIC), root mean square error (RMSE), and coefficient of determination (R^2) can be calculated as follows:

$$AIC = -2L + 2q, \quad (44)$$

$$BIC = -2L + q\log n, \quad (45)$$

$$CAIC = -2L + \frac{2qn}{n-q-1}, \quad (46)$$

$$RMSE = \left[\sum_{i=1}^{n} \frac{\left(\hat{F}(x_{(i)}) - \frac{i}{n+1}\right)^2}{n}\right]^{\frac{1}{2}}, \quad (47)$$

$$R^2 = 1 - \frac{\sum_{i=1}^{n}\left(\hat{F}(x_{(i)}) - \frac{i}{n+1}\right)^2}{\sum_{i=1}^{n}\left(\hat{F}(x_{(i)}) - \bar{\hat{F}}(x_{(i)})\right)^2}, \quad (48)$$

where L is the log likelihood function, q is the number of parameters, and n is the sample size. The AIC, the BIC, and the $CAIC$ are the measure of the goodness of fit for an estimated statistical model. However, these criteria do not represent tests of the model in the sense of the hypothesis testing; rather, they are tools for model selection. The model with smaller values of the AIC, BIC, and $CAIC$ is the preferred model. In addition, small values of the $RMSE$ and R^2 for a model indicate the suitability of that model.

7. Application with Real Data

In this section, using a real data set, which is the failure time of 84 aircraft (El-Bassiouny et al. (2015)[6]), we prove the superiority of the proposed model compared to other models such as Exponential Lomax (ELomax) and Lomax and Weibull. The pdf of these distributions is as follows:

$$f_{ELomax} = \frac{\alpha\lambda}{\beta}\left(\frac{\beta}{\beta+x}\right)^{-\alpha+1} e^{-\lambda\left(\frac{\beta}{\beta+x}\right)^{-\alpha}}, x \geq -\beta, \quad (49)$$

$$f_{Lomax} = \frac{\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{-\alpha+1}, x > 0, \quad (50)$$

$$f_{Weibull} = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, x > 0. \quad (51)$$

The failure times of 84 Aircraft Windshield are 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

Tab. 7. MLEs for real data

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$
ESM	$3.718e+01$	$1.410e+02$	$2.481e+00$	$8.143e-04$
$ELomax$	2.448	0.04	-	$2.763e-05$
$Weibull$	2.3744	2.8629	-	-
$Lomax$	51425	131789.78	-	-

Tab. 8. Goodness-of-fit statistics corresponding to data set

Distribution	L	AIC	BIC	CAIC	K-S	P-Value	RMSE	R^2
ESM	-126.9225	261.845	261.5683	262.3513	0.05367	0.9689	0.02277054	0.9937275
ELomax	-129.193	264.386	264.158	264.686	0.9454	0.9454	0.02422539	0.9929997
Weibull	-130.0533	264.1067	268.9683	264.2548	0.0877	0.5375	0.02459319	0.9925975
Lomax	-164.9884	333.9767	338.8620	334.1230	0.3022	4.35e-07	0.16271620	0.1333968

Based on these data, the following preliminary results are obtained:
$min = 0.04, max = 4.663, median = 2.3545, mean = 2.557452, estimated, sd = 1.118824,$
$estimated\ skewness = 0.1013122, estimated, kurtosis = 2.3821.$

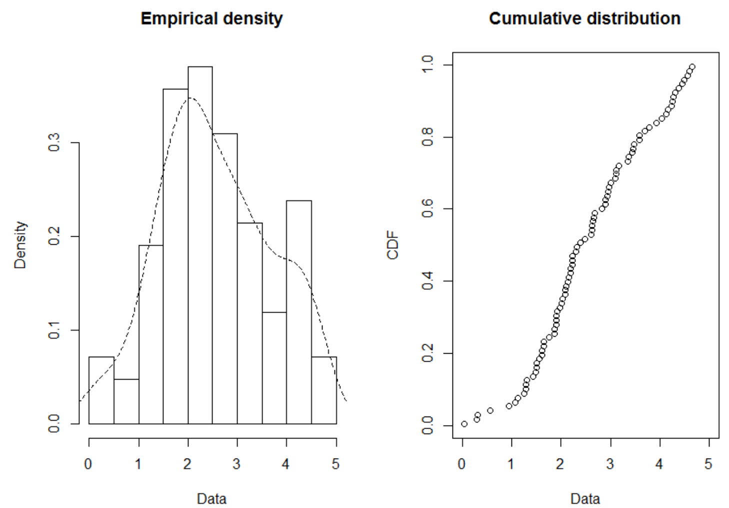

Fig. 6. Empirical density and cumulative distribution for real data

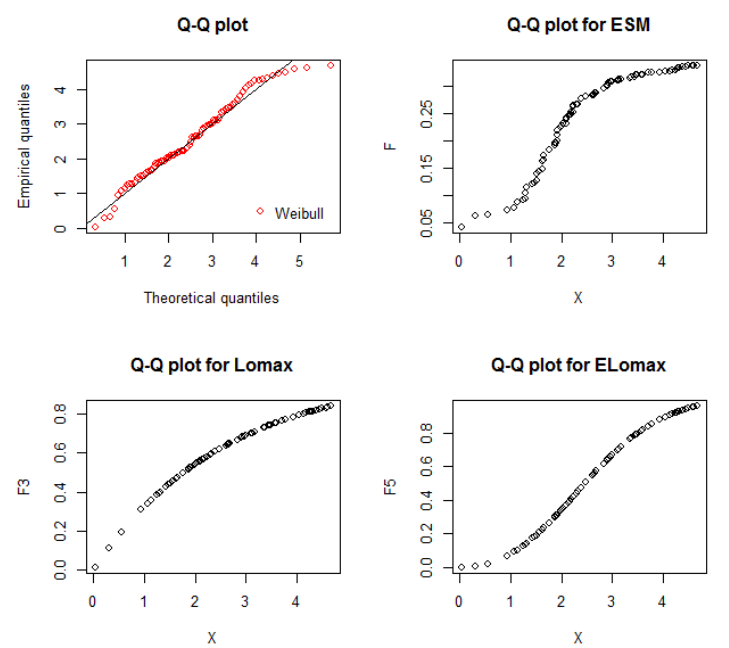

Fig. 7. The Q-Q plot of the selected distributions for the real data

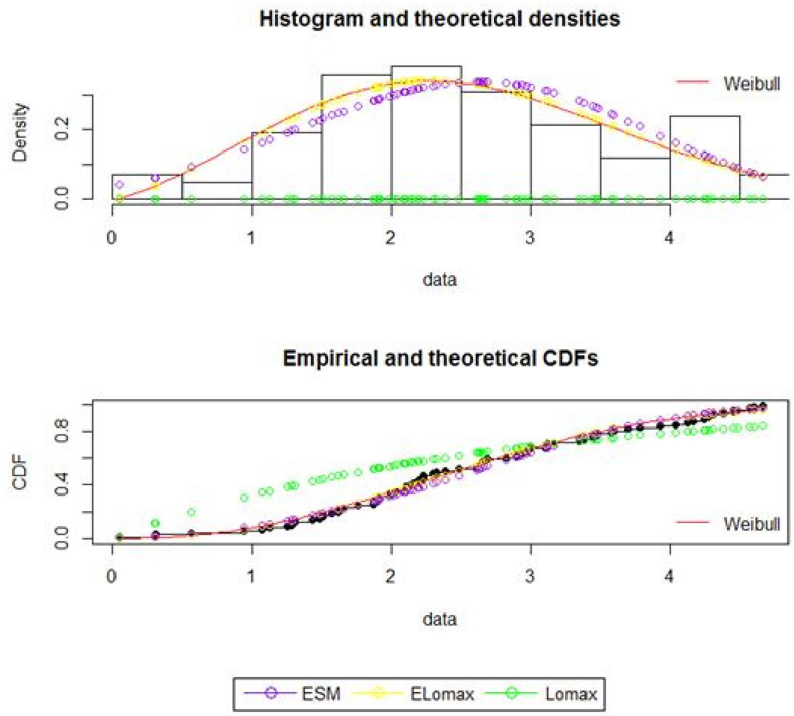

Fig. 8. Estimated densities, the empirical and fitted cumulative functions of the selected models

Figures (6) and (7) present respectively empirical density, cumulative distribution, and The Q-Q plot of the selected distributions for the real data. Table (7) presents the MLE of the parameters for the ESM, ELomax, Lomax, and Weibull distributions. This study used the **"Maxlik"** package in software **R** to calculate MLEs. The **"fitdistrplus"** package was used to fit the selected distributions on the real data, where the result of fitting is shown in Figure (8). Finally, to check the goodness of fit, the statistics of L, AIC, BIC, CAIC, K-S, RMSE, R^2, and P-Value using the **"gofstat"** package (Marie Laure (2018)[13]) is computed. The results are summarized in Table (8).

According to Figure (8), ESM distribution is well fitted to the data. On the other hand, according to Table (8) and with respect to the values of $P-Value$, ESM, ELomax and Weibull are suitable for fitting data. To select the most suitable model, it is enough to check the statistics $AIC, BIC, CAIC, L, KS, RMSE,$ and R^2. The values of $AIC, BIC, CAIC, KS,$ and $RMSE$ for ESM distribution are lower than other distributions. In addition, the values of L and R^2 for the ESM distribution are at maximum. Finally, it can be concluded that the ESM distribution for these data is more appropriate than other distributions. Figure (8) also confirms these results.

8. Conclusion

This study introduced a new ESM distribution. Some of the statistical properties of this distribution were studied. Then, the parameters of this distribution were estimated using two methods of MLE and Bayes. The results of the Monte-Carlo simulation showed that the use of the Bayes method with the Uniform prior was much more suitable and, for the large sample size, the use of the MLE method was recommended. Finally, this distribution was fitted to a real data set, indicating that the ESM distribution for these data is much more suitable than other distributions that have been recently fitted to these data.

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