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RESEARCH PAPER

A Novel Approach to Fuzzy Multi-Attribute Group Decision Making Based On Interval-Valued Intuitionistic Fuzzy Best-Worst Method

Seyed Erfan Mohammadi[1] & Emran Mohammadi[*2]

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ABSTRACT
Today, due to the globalization and competitive conditions of the market, decisions are generally made in group and in accordance with different attributes. In addition, when information is subject to uncertainty, one can expect inconsistency and contradiction. Therefore, the development and application of tools that can adequately address uncertainty in the decision-making process and be appropriate for group decision making is an important area of Multi-Criteria Decision Making (MCDM). Therefore, first, this study developed the traditional Best-Worst Method (BWM), proposed an Interval-Valued Intuitionistic Fuzzy Best-Worst Method (IVIFBWM), and introduced a novel approach to fuzzy multi-attribute group decision-making based on the proposed method. Finally, in order to demonstrate how the introduced approach can be applied in practice, it was applied to an Iranian investment company and the experimental results were examined. From the experimental results, it was found that the introduced approach is not only simple in calculation but also convenient in implementation, especially in interval-valued intuitionistic fuzzy environments.

KEYWORDS: *Multi-attribute group decision making; Interval-valued intuitionistic fuzzy sets; Interval-valued intuitionistic fuzzy best-worst method; Financial environments.*

1. Introduction

Today, it is clear to everyone that decision making is part of life and inevitable. This process is not a difficult task when only one criterion is considered in the problem. However, when decision-makers evaluate alternatives with multiple criteria, many problems such as weights of criteria, preference dependence, and conflicts among criteria seem to complicate the problems and it must be solved using more complex methods [1]. Based on the solution space of the studied issue, Multi-Criteria Decision Making (MCDM) can be divided into two classes: Multi-Attribute Decision Making (MADM) and Multi-Objective Decision Making (MODM). For MADM, the decision variables are discrete and the number of alternatives is limited, which can also be called as discrete MCDM. For MODM, it contains continuous decision variables and unlimited number of alternatives, which can also be called continuous MCDM. The MADM firstly evaluates the alternatives and arranges them from superior to inferior and then, selects the best one. Meanwhile, the MODM employs the vector-based optimization technique, which is a type of mathematical programming method. It is worth noting that in this paper, our focus will be on the MADM's topic. However, in order to adapt conventional MADM to the real-world problem, some issues should be considered such as making the decision in group and contributing uncertainty to information analysis.

One of the most common ways of contributing uncertainty to the real-world problems is the use of fuzzy sets. Fuzzy sets were introduced independently by Zadeh [2] as an extension of the classical notion of set. In classical set theory, the membership of elements in a set is assessed in binary terms and according to a bivalent condition, an element either belongs or does not belong to the set. Besides, fuzzy set theory allows the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued at the real unit interval [0, 1]. The fuzzy set theory can be used in a wide range of domains in which information

* Corresponding author: *Emran Mohammadi*
e_mohammadi@iust.ac.ir

1. *Ph.D Student Iran University of Science & Technology - - Department of Industrial Engineering, Iran University of Science & Technology, Tehran, Iran.*
2. *Assistant Professor Iran University of Science & Technology - - Department of Industrial Engineering, Iran University of Science & Technology, Tehran, Iran.*

is incomplete or imprecise. There are many mathematical constructions similar to or more general than fuzzy sets including Intuitionistic Fuzzy Sets (IFSs), as introduced by Atanassov [3], and they are characterized by the membership function, non-membership function, and hesitancy function. Given the strength and capacity of the IFSs in covering uncertainty in various problems Atanassov and Gargov [4] later introduced Interval-Valued Intuitionistic Fuzzy Sets (IVIFSs) as a generalization of the fuzzy sets and the IFSs that provide the membership function and non-membership function with intervals rather than exact numbers. Hereupon, Zhang et al. [5] suggested that it would be more appropriate to express an individual's opinion based on IVIFSs. Therefore, Interval-Valued Intuitionistic Fuzzy Numbers (IVIFNs) were increasingly used in different studies.

Further to the above, in recent years, based on IVIFSs, many approaches to multi-attribute group-decision making have been proposed. Wang et al. [6] proposed a multi-attribute group decision-making method based on IVIFSs, which would help determine the weights of decision-makers by applying the knowledge level of the experts to the decision-making problem. Wang and Dong [7] defined the possibility degree of comparing two IVIFNs using the notion of two-dimensional random vector and then, developed a new method to rank IVIFNs in multi-attribute group decision making problems. Makui et al. [8] presented a multi-attribute group decision making approach by which the preference relations and the judgment of decision makers could be formulated as an IVIFS; by using the proposed approach, they have attempted to consider the attitudes of decision-makers towards risk-taking in solving the problem. Liu et al. [9] proposed the Principal Component Analysis (PCA) model for interval-valued intuitionistic fuzzy variables and applied this model to the complex multi-attribute large group decision making problems. Also, Qi et al. [10] proposed a novel generalized cross-entropy measure for IVIFSs to enable decision makers to express their attitudes and then, developed a new method for solving multi-attribute group decision making problems based on incomplete attribute weight information. In the meantime, Azarnivand and Malekian [11] used a multi-attribute group decision making method based on IVIFSs for flood risk management in order to prioritize the strategies with consideration of sustainable development attributes. In the same vein, Mohammadi and Makui [12] presented a new multi-attribute group decision making approach based on IVIFSs and Evidential Reasoning Methodology (ERM), while their main goal was to prevent a condition would favor an unreasonable preference order of the alternatives. Büyüközkan et al. [13] presented a multi-attribute group decision-making approach based on IVIFSs to provide an effective framework for evaluating and selecting the most appropriate cloud computing technology by means of MULTIMOORA (Multi-objective Optimization by Ratio Analysis plus the Full Multiplicative Form). Liu [14] proposed some interval-valued intuitionistic fuzzy power Heronian aggregation (IVIFPHA) operators and then, presented a multi-attribute group decision making approach based on these operators. More recently, Joshi and Kumar [15] defined entropy measures for Interval-Valued Intuitionistic Hesitant Fuzzy Sets (IVIHFSs) and investigated their application to multi-attribute group decision-making problems. Also, Kong et al. [16] presented a threat assessment method that can be applied by group under the interval-valued intuitionistic fuzzy environment. However, in all of these methods, a very significant challenge comes from the lack of consistency of the pairwise comparison matrices that usually occurs in practice. When a comparison matrix is inconsistent, the recommended action is to revise the comparison such that the comparison matrix becomes consistent. Although this is a very common approach, this action will not be successful. Therefore, in order to overcome this defect, a new approach to multi-attribute group decision making based on IVIFSs should be developed.

On the other hand, lately, in [17], a new method called Best-Worst Method (BWM) was presented, which is a comparison-based method that makes comparisons in a particularly structured way such that not only less information is required, but the comparisons are also more consistent. By using this method, decision makers only need to identify the most and least desirable attributes and then, make pairwise comparisons between the best/worst attributes and the other attributes. Finally, a minimax model is constructed to determine the weights of different attributes and a new definition of consistency ratio is established to check the reliability of the method. Therefore, BWM can be used to resolve the lack of consistency of the pairwise comparison matrices mentioned above. However, it is worth noting that although Rezaei [18] attempted to develop his proposed method, BWM still has the following drawbacks that would limit the scope of its application:

- In the BWM, it is not easy to determine which attribute is the best or worst when the number of attributes and decision makers who participated in the decision-making process increases.
- With the rapid development of modern economy and society, uncertainty and fuzziness can always be found in modern decision-making problems; in this situation, the traditional BWM is not competent for solving the problem with uncertainty and ambiguity.

Therefore, in order to eliminate the limitations of the BWM, it is necessary to use a specific procedure to identify the best or worst attribute and also adjust this method to the environment associated with uncertainty and ambiguity.

Based on the mentioned issues, in this paper, firstly, we proposed an Interval-Valued Intuitionistic Fuzzy Best-Worst Method (IVIFBWM), developed the traditional BWM for solving the problem with uncertainty and ambiguity, and then introduced a novel approach to fuzzy multi-attribute group decision-making based on the proposed method. The main contributions of this paper can be summarized as follows:

- The traditional BWM was developed by IVIFSs and it was made applicable to environments with uncertainty and ambiguity.
- A specific procedure was proposed to identify the best or worst attribute based on graph set theory and determine the ordered attribute set;
- A mathematical framework was proposed for group decision making that remained the same regardless of the number of attributes, alternatives, and decision makers;
- The mathematical equations of the consistency ratio were proposed for the proposed approach;
- In order to demonstrate how the introduced approach can be applied in practice, it is implemented in an Iranian investment company and the experimental results are examined.

The reminder of this paper is organized as follows. In Section 2, some relevant concepts such as IFSs and IVIFSs are illustrated. In Section 3, IVIFBWM is given and the traditional BWM is developed for solving the problem with uncertainty and ambiguity. Section 4 introduces a novel approach to fuzzy multi-attribute group decision making based on the IVIFBWM. Section 5 applies the introduced approach to a real-world problem in order to illustrate the implementation steps and analysis of the results. Eventually, the conclusions and suggestions for future research are discussed in Section 6.

2. Preliminaries

As a preparation to introduce the new approach, some basic concepts are briefly reviewed in this section.

2.1. Intuitionistic fuzzy sets (IFSs)

IFSs take into account both the membership and non-membership functions for describing any x in X. The sum of these functions is less than or equal to 1. The basic definition of IFSs is given in the following part.

Definition:

[19]. An IFS A in X, where $X \neq \Phi$ be a given set, can be defined as follows:

$$A = \{(x, \mu_A(x), v_A(x)) : x \in X\} \quad (1)$$

where $\mu_A(x)$ and $v_A(x)$ denote the membership and non-membership functions of the element x to the set A, respectively. $\mu_A(x) \in [0,1]$ and $v_A(x) \in [0,1]$ satisfy the condition $0 \leq \mu_A(x) + v_A(x) \leq 1$.

This paper uses IVIFSs since they can represent the membership and non-membership functions based on the closed intervals that can provide a more reliable definition than the definition based on exact values.

2.2. Interval-valued intuitionistic fuzzy sets (IVIFSs)

Definition:

[20]. An IVIFS A in X, where $X \neq \Phi$ be a given set, can be defined as follows:

$$A = \{(x, [\mu_{AL}(x), \mu_{AU}(x)], [v_{AL}(x), v_{AU}(x)]) : x \in X\}, \quad (2)$$

where $\mu_{AL}(x)$, $\mu_{AU}(x)$, $v_{AL}(x)$, and $v_{AU}(x)$ denote the starting and ending points of $\mu_A(x)$ and $v_A(x)$ as the membership and non-membership functions of the element x to the set A, respectively. $\mu_{AU}(x) \in [0,1]$ and $v_{AU}(x) \in [0,1]$ satisfy the condition $0 \leq \mu_{AU}(x) + v_{AU}(x) \leq 1$.
For each element x, the uncertainty function in A can be defined as follows:

$$\pi_A(x) = 1 - \mu_A(x) - v_A(x) \\ = [1 - \mu_{AU}(x) - v_{AU}(x), 1 - \mu_{AL}(x) - v_{AL}(x)], \quad (3)$$

An IVIFS is denoted by $A = ([a,b],[c,d])$ for convenience.

Definition:
[21]. Let $\tilde{\alpha}_1 = ([a_1,b_1],[c_1,d_1])$ and $\tilde{\alpha}_2 = ([a_2,b_2],[c_2,d_2])$ be any two IVIFNs; then, their operational laws can be defined as follows:

$$\bar{\tilde{\alpha}}_1 = ([c_1,d_1],[a_1,b_1]) \quad (4)$$
$$\tilde{\alpha}_1 + \tilde{\alpha}_2 = ([a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2],[c_1c_2, d_1d_2]) \quad (5)$$
$$\tilde{\alpha}_1 . \tilde{\alpha}_2 = ([a_1a_2, b_1b_2],[c_1 + c_2 - c_1c_2, d_1 + d_2 - d_1d_2]) \quad (6)$$
$$\lambda\tilde{\alpha}_1 = \left([1-(1-a_1)^\lambda, 1-(1-b_1)^\lambda],[{c_1}^\lambda, {d_1}^\lambda]\right) \quad , \lambda \geq 0 \quad (7)$$

Definition:
[21]. Let $\tilde{\alpha} = ([a,b],[c,d])$ be an IVIFN. Then, the score function (S) is defined as follows:

$$S(\tilde{\alpha}) = 1/2(a - c + b - d) \quad (8)$$

where $S(\tilde{\alpha}) \in [-1,1]$ and the greater value of $S(\tilde{\alpha})$ denotes the greater IVIFN $\tilde{\alpha}$.
Definition:
[22]. Let $\tilde{\alpha}_j = \left([a_j,b_j],[c_j,d_j]\right)$, $j = (1,2,\dots,n)$ be a collection of IVIFNs. The Interval-Valued Intuitionistic Fuzzy Multiplicative Weighted Geometric Aggregation (IVIFMWGA) operator is defined as follows:

$$IVIFMWGA_\omega(\tilde{\alpha}_1,\tilde{\alpha}_2,\dots,\tilde{\alpha}_n) = \left(\left[\prod_{j=1}^{n} {a_j}^{\omega_j}, \prod_{j=1}^{n} {b_j}^{\omega_j}\right],\left[\prod_{j=1}^{n} {c_j}^{\omega_j}, \prod_{j=1}^{n} {d_j}^{\omega_j}\right]\right) \quad (9)$$

where $\omega = (\omega_1,\omega_2,\dots,\omega_n)^T$ is the weight vector of $\tilde{\alpha}_j (j = 1,2,\dots,n)$; $\omega_j \in [0,1]$ and $\sum_{j=1}^{n}\omega_j = 1$.

Also, Xu and Chen [23] proposed the Interval-Valued Intuitionistic Fuzzy Ordered Weighted Aggregation (IVIFOWA) operator to aggregate IVIFNs. The operator is characterized by reordering the IVIFNs in descending order. A weight, w_j is associated with a particular ordered position. The arguments are endowed with new weights w_j rather than the initial weights ω_j .

Definition:
[23]. Let $\tilde{\alpha}_j = \left([a_j,b_j],[c_j,d_j]\right)$, $j = (1,2,\dots,n)$ be a collection of IVIFNs and $\left(\tilde{\alpha}_{\sigma(1)},\tilde{\alpha}_{\sigma(2)},\dots,\tilde{\alpha}_{\sigma(n)}\right)$ be a permutation of $(\tilde{\alpha}_1,\tilde{\alpha}_2,\dots,\tilde{\alpha}_n)$ such that $\tilde{\alpha}_{\sigma(j-1)} \geq \tilde{\alpha}_{\sigma(j)}$ for all j's, and let $\tilde{\alpha}_{\sigma(j)} = \left([a_{\sigma(j)},b_{\sigma(j)}],[c_{\sigma(j)},d_{\sigma(j)}]\right)$. Then, the IVIFOWA operator can be defined as follows:

$$IVIFOWA_w(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \left(\left[1-\prod_{j=1}^{n}\left(1-a_{\sigma(j)}\right)^{w_j}, 1-\prod_{j=1}^{n}\left(1-b_{\sigma(j)}\right)^{w_j}\right], \left[\prod_{j=1}^{n} c_{\sigma(j)}{}^{w_j}, \prod_{j=1}^{n} d_{\sigma(j)}{}^{w_j}\right]\right) \quad (10)$$

where $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of the IVIFOWA operator, $w_j \in [0,1]$, and $\sum_{j=1}^{n} w_j = 1$. The weight vector of the IVIFOWA operator can be determined by Xu's method [24], which uses the perspective of normal distribution to gain weights. In this way, it can reduce the influence of unfair arguments in the final results by assigning low weights to the "optimistic" or "pessimistic" discretions.

Definition:
[25]. For two IVIFSs A and B in $X = \{x_1, x_2, \dots, x_m\}$, the normalized Hamming distance can be defined as follows:

$$d_h(A,B) = \frac{1}{2m}\sum_{i=1}^{m}(|\mu_A(x_i) - \mu_B(x_i)| + |v_A(x_i) - v_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|), \quad (11)$$

Definition:
[26]. Let $G = (H, O, Q)$ be a directed network, where H is a node set, O is an arc set, and Q is a weight set associated with all the preference information. The weight $([a_{ij}, b_{ij}], [c_{ij}, d_{ij}]) \in Q$ indicates the relative preference degree of the node i to the node j.

Definition:
[26]. The out-degree of the node i is defined as the number of all arcs whose arrow tails are the node i (denoted by D_i^{out}) and the in-degree of the node i is defined as the number of all arcs whose arrow heads are the node i (denoted by D_i^{in}).
Note: If there are two nodes with the same number of arrow tails, we should consider the number of arrow tails and the degrees of preferences.
Note: If there are two nodes $(j = k, t)$ in the node set H whose out-degrees are equal (i.e., $D_k^{out} = D_t^{out}$), we can rank them by their in-degrees. When $D_k^{out} = n$ and $D_k^{in} = 0$, the node k is called the best node (i.e., the source node in $G = (H, O, Q)$). In return, when $D_t^{out} = 0$ and $D_t^{in} = n$, the node t is called the worst node (i.e., the sink node in $G = (H, O, Q)$).

3. Interval-Valued Intuitionistic Fuzzy Best-Worst Method (IVIFBWM) for Determining The Weights of Attributes

As mentioned in the previous sections, in [17], Rezaei presented the BWM. However, the traditional BWM is not competent for solving the problem with uncertainty and ambiguity. Hence, in this section, in order to overcome the drawbacks of Rezaei's method, the strengths of the BWM merged with the representation capability of the IVIFSs to propose an IVIFBWM for determining the weights of attributes. Therefore, the proposed method involves the following steps:

Step1: Determine a set of decision attributes.
In this step, the decision-maker identifies n attributes $\{g_1, g_2, \dots, g_n\}$ that are used to make a decision.

Step2: Determine the best (e.g., most desirable, most important) and the worst (e.g. least desirable, least important) attributes.

Step3: Determine the preference for the best attribute over all the other attributes. The resulting best-to-others (BO) vector would be:

$$A_B = (\tilde{\alpha}_{B1}, \tilde{\alpha}_{B2}, \dots, \tilde{\alpha}_{Bn}),$$

where $\tilde{\alpha}_{Bj} = ([a_{Bj}, b_{Bj}], [c_{Bj}, d_{Bj}])$, $j = (1,2,\dots,n)$ indicates the preference of the best attribute B over attribute j. It is clear that $\tilde{\alpha}_{BB} = ([0.5,0.5], [0.5,0.5])$.

Step4: Determine the preference of all the other attributes over the worst attribute. The resulting Others-to-Worst (OW) vector would be:

$$A_W = (\tilde{\alpha}_{1W}, \tilde{\alpha}_{2W}, \dots, \tilde{\alpha}_{nW})^T,$$

where $\tilde{\alpha}_{jW} = ([a_{jW}, b_{jW}], [c_{jW}, d_{jW}])$, $j = (1,2,\dots,n)$ indicates the preference of the attribute j over the worst attribute W. It is clear that $\tilde{\alpha}_{WW} = ([0.5,0.5],[0.5,0.5])$.

Step5: Find the optimal weights of attributes $(\tilde{w}_1^*, \tilde{w}_2^*, \dots, \tilde{w}_n^*)$.
Suppose that the optimal weight vector is $(\tilde{w}_1^*, \tilde{w}_2^*, \dots, \tilde{w}_n^*)$, where $\tilde{w}_j^* = ([a_j, b_j], [c_j, d_j])$, $j = (1,2,\dots,n)$, $\tau_j^* = [a_j, b_j]$, and $\sigma_j^* = [c_j, d_j]$ are the membership functions and the non-membership functions of importance, respectively. For simplicity, firstly, the membership functions were only considered. The aim is to determine the optimal weights of the attributes such that the maximum absolute differences between $\left|\frac{\tau_B}{\tau_j} - \mu_{Bj}\right|$ and $\left|\frac{\tau_j}{\tau_W} - \mu_{jW}\right|$ for all j's is minimized as follows:

$$\begin{aligned}
&\min \max_j \left\{ \left|\frac{\tau_B}{\tau_j} - \mu_{Bj}\right|, \left|\frac{\tau_j}{\tau_W} - \mu_{jW}\right| \right\} \\
&S.t. \\
&\sum_{j=1}^{n} \tau_j = 1 \\
&\tau_j \geq 0, \quad j = 1,2,\dots,n.
\end{aligned} \tag{12}$$

Model (12) is equivalent to the following model:

$$\begin{aligned}
&\min \varphi \\
&S.t. \\
&\left|\frac{\tau_B}{\tau_j} - \mu_{Bj}\right| \leq \varphi, \quad j = 1,2,\dots,n. \\
&\left|\frac{\tau_j}{\tau_W} - \mu_{jW}\right| \leq \varphi, \quad j = 1,2,\dots,n. \\
&\sum_{j=1}^{n} \tau_j = 1 \\
&\tau_j \geq 0, \quad j = 1,2,\dots,n.
\end{aligned} \tag{13}$$

By solving Model (13), the first part of the optimal interval-valued intuitionistic fuzzy weights, $(\tau_1^*, \tau_2^*, \dots, \tau_n^*)$ and φ^* are obtained.
Similarly, if we consider the non-membership functions, Model (14) can be constructed as follows:

$$\begin{aligned}
&\min \max_j \left\{ \left|\frac{\sigma_B}{\sigma_j} - v_{Bj}\right|, \left|\frac{\sigma_j}{\sigma_W} - v_{jW}\right| \right\} \\
&S.t. \\
&\sum_{j=1}^{n} \sigma_j = 1 \\
&\sigma_j \geq 0, \quad j = 1,2,\dots,n.
\end{aligned} \tag{14}$$

Model (14) is equivalent to the following model:

$$\begin{aligned}
&\min \psi \\
&S.t. \\
&\left|\frac{\sigma_B}{\sigma_j} - v_{Bj}\right| \leq \psi, \quad j = 1,2,\dots,n. \\
&\left|\frac{\sigma_j}{\sigma_W} - v_{jW}\right| \leq \psi, \quad j = 1,2,\dots,n.
\end{aligned} \tag{15}$$

$$\sum_{j=1}^{n} \sigma_j = 1$$
$$\sigma_j \geq 0, \quad j = 1,2, \dots, n.$$

By solving Model (15), the second part of the optimal interval-valued intuitionistic fuzzy weights, $(\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)$ and ψ^* are obtained.
Thus, the optimal weight vector of the attributes can be easily elicited as follows:

$$W^* = (\widetilde{w}_1^*, \widetilde{w}_2^*, \dots, \widetilde{w}_n^*) = ((\tau_1^*, \sigma_1^*), (\tau_2^*, \sigma_2^*), \dots, (\tau_n^*, \sigma_n^*)) \quad (16)$$

Step 6: Calculate the consistency ratio (CR).
CR is an important indicator to check the inconsistency level of pairwise comparisons. In this step, the computation of CR for IVIFBWM is explained. The comparison result is fully consistent when $\mu_{Bj} \times \mu_{jW} = \mu_{BW}$ and $v_{Bj} \times v_{jW} = v_{BW}$ for all j 's, where (μ_{Bj}, v_{Bj}), (μ_{jW}, v_{jW}), and (μ_{BW}, v_{BW}) are the preference functions and the non-preference functions of the best attribute over the attribute j, the attribute j over the worst attribute, and the best attribute over the worst attribute, respectively.
When $\mu_{Bj} \times \mu_{jW} \neq \mu_{BW}$, or $v_{Bj} \times v_{jW} \neq v_{BW}$ which means $\mu_{Bj} \times \mu_{jW}$ or $v_{Bj} \times v_{jW}$ may be higher or lower than μ_{BW} or v_{BW}, the inconsistency of intuitionistic fuzzy pairwise comparisons occurd. When both μ_{Bj} and μ_{jW} are equal to μ_{BW} or both v_{Bj} and v_{jW} are equal to v_{BW}, inequalities will reach the highest value, which results in φ and ψ.
Considering the occurrence of the greatest inequalities, according to the equality relation $\left(\frac{\widetilde{w}_B}{\widetilde{w}_j}\right) \times \left(\frac{\widetilde{w}_j}{\widetilde{w}_W}\right) = \left(\frac{\widetilde{w}_B}{\widetilde{w}_W}\right)$, the following equations can be obtained:

$$\begin{aligned} (\mu_{Bj} - \delta) \times (\mu_{jW} - \delta) &= (\mu_{BW} + \delta) \\ (v_{Bj} + \varepsilon) \times (v_{jW} + \varepsilon) &= (v_{BW} - \varepsilon) \end{aligned} \quad (17)$$

For the maximum intuitionistic fuzzy inconsistency, $\mu_{Bj} = \mu_{jW} = \mu_{BW}$ and $v_{Bj} = v_{jW} = v_{BW}$. Then, Eq. (17) can be written as follows:

$$\begin{aligned} (\mu_{BW} - \delta) \times (\mu_{BW} - \delta) &= (\mu_{BW} + \delta) \\ (v_{BW} + \varepsilon) \times (v_{BW} + \varepsilon) &= (v_{BW} - \varepsilon) \end{aligned} \quad (18)$$

Derived from the above, Eq. (18) is formulated as follows:

$$\begin{aligned} \delta^2 - (1 + 2\mu_{BW})\delta + (\mu_{BW}^2 - \mu_{BW}) &= 0 \\ \varepsilon^2 + (1 + 2v_{BW})\varepsilon + (v_{BW}^2 - v_{BW}) &= 0 \end{aligned} \quad (19)$$

For $\mu_{BW} = [a_{BW}, b_{BW}]$, the highest possible intuitionistic fuzzy value is 1. It shows that the maximum value of a_{BW} and b_{BW} cannot exceed 1. In this case, if we use the upper boundary b_{BW}, we can find the maximum possible δ, because b_{BW} is the largest in the interval $[a_{BW}, b_{BW}]$, while δ can also be represented by a crisp value. Moreover, for $v_{BW} = [c_{BW}, d_{BW}]$, the lowest possible intuitionistic fuzzy value is 0. It shows that the minimum value of c_{BW} and d_{BW} cannot be less than 0. In this case, if we use the upper boundary d_{BW}, we can find the maximum possible ε, because d_{BW} is the largest in the interval $[c_{BW}, d_{BW}]$. ε can also be represented by a crisp value. Therefore, Eq. (19) can be transferred to the following equations:

$$\begin{aligned} \delta^2 - (1 + 2b_{BW})\delta + (b_{BW}^2 - b_{BW}) &= 0 \\ \varepsilon^2 + (1 + 2d_{BW})\varepsilon + (d_{BW}^2 - d_{BW}) &= 0 \end{aligned} \quad (20)$$

where $b_{BW} \in \{0, 0.1, 0.2, \dots, 1\}$ and $d_{BW} \in \{1, 0.9, 0.8, \dots, 0\}$.

By solving Eq. (20) for different values of b_{BW} and d_{BW}, we can obtain the smallest consistency and the corresponding maximum possible values δ^* (i.e., max δ) and ε^* (i.e., max ε). These maximum values (δ^* and ε^*) can be considered as the consistency index 1 (CI_1) and the consistency index 2 (CI_2), respectively, for IVIFBWM (Table 1). Then, the CR can be calculated as follows:

$$CR = max\left\{\frac{\varphi^*}{CI_1}, \frac{\psi^*}{CI_2}\right\} \quad (21)$$

Therefore, by maximizing the two ratios of the optimal values $\frac{\varphi^*}{CI_1}$ and $\frac{\psi^*}{CI_2}$, the CR can be obtained, as shown in Eq. (21). CR can be taken as a measure to check the reliability of the weights. The smaller the CR, the better the consistency.

Tab. 1. The consistency index 1 (CI_1, max δ) and the consistency index 2 (CI_2, max ε)

μ_{BW}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
v_{BW}	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
CI_1	1.27	1.51	1.72	1.92	2.12	2.30	2.48	2.66	2.83
CI_2	0.03	0.06	0.08	0.10	0.12	0.12	0.12	0.12	0.12

4. A novel Approach to Fuzzy Multi-Attribute Group Decision-Making Based On The IVIFBWM

In this section, based on the development of the traditional BWM proposed in the previous section, we have introduced a novel approach to fuzzy multi-attribute group decision making based on the IVIFBWM in order to facilitate the decision-making process and increase the consistency of comparisons. Let E be a set of decision-makers, where $E = \{e_1, e_2, \dots, e_t\}$, Y be a set of alternatives, where $Y = \{y_1, y_2, \dots, y_m\}$, and G be a set of attributes, where $G = \{g_1, g_2, \dots, g_n\}$. Therefore, our introduced approach is as follows:

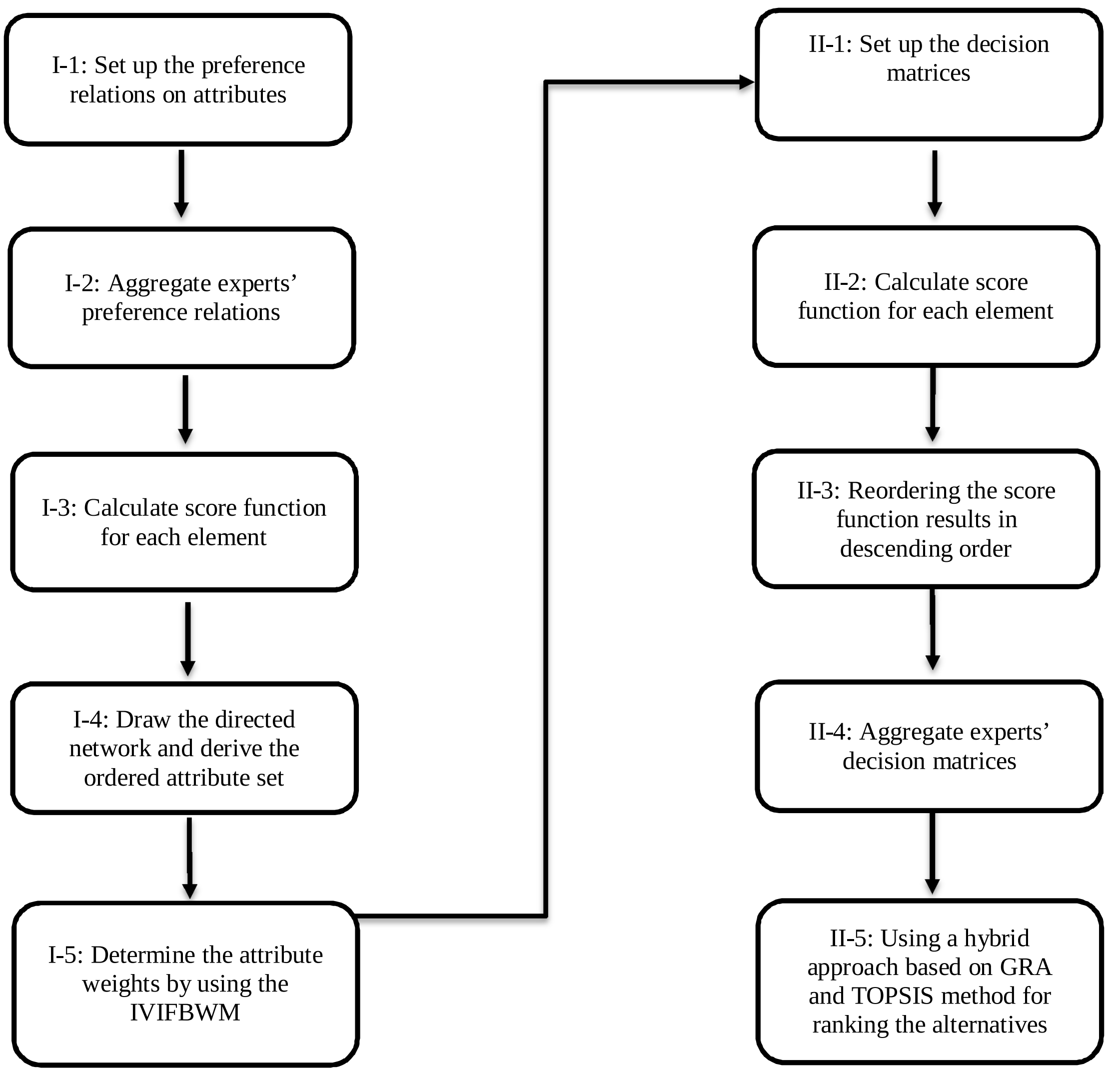


Fig 1. The diagram for the introduced approach.

In the first stage, the preference relation matrices for attribute weights are required. Experts have used IVIFSs to express their preferences. In the condition where the attribute weights are unknown, experts delivered the preference relations on attributes by pairwise comparison.

Step I-1: Set up the Interval-Valued Intuitionistic Fuzzy Preference Relation (IVIFPR) on attributes as follows:

$\tilde{G}_k = \left(\tilde{g}_{ij}{}^{(k)}\right)_{n\times n}$

where $\tilde{g}_{ij}{}^{(k)} = \left(\left[a_{ij}, b_{ij}\right]^{(k)}, \left[c_{ij}, d_{ij}\right]^{(k)}\right)$, $i, j = 1,2, \dots, n$; $k = 1,2, \dots, t$; is an IVIFS. $\left[a_{ij}, b_{ij}\right]^{(k)}$ indicates the expert e_k's interval-valued intuitionistic fuzzy preference degree for the attribute g_i when the attributes g_i and g_j are compared and attribute g_i is preferred over the other one; also, $\left[c_{ij}, d_{ij}\right]^{(k)}$ indicates the expert e_k's interval-valued intuitionistic fuzzy preference degree for the attribute g_i when the attributes g_i and g_j are compared and attribute g_j is preferred over the other one; $\left[a_{ij}, b_{ij}\right]^{(k)} \subset [0,1]$, $\left[c_{ij}, d_{ij}\right]^{(k)} \subset [0,1]$, $\left[a_{ji}, b_{ji}\right]^{(k)} = \left[c_{ij}, d_{ij}\right]^{(k)}$, $\left[c_{ji}, d_{ji}\right]^{(k)} = \left[a_{ij}, b_{ij}\right]^{(k)}$, $[a_{ii}, b_{ii}]^{(k)} = [c_{ii}, d_{ii}]^{(k)} = [0.5,0.5]$, and $b_{ij}{}^{(k)} + d_{ij}{}^{(k)} \le 1$, $i, j = 1,2, \dots, n$, and $k = 1,2, \dots, t$.

Step I-2: Apply the operator in (9) to integrate experts' opinions on preference relation and establish the aggregated preference relation matrix as follows:

$\dot{\tilde{G}} = \left(\dot{\tilde{g}}_{ij}\right)_{n\times n}$

where

$\dot{\tilde{g}}_{ij} = IVIFMWGA_{\omega}\left(\tilde{g}_{ij}{}^{(1)}, \tilde{g}_{ij}{}^{(2)}, \dots, \tilde{g}_{ij}{}^{(k)}\right); i, j = 1,2, \dots, n; k = 1,2, \dots, t.$

Step I-3: Use (8) to calculate the score function for each element in $\dot{\tilde{G}}$ as follows:

$S_{ij} = S\left(\dot{\tilde{g}}_{ij}\right), \quad i, j = 1,2, \dots, n.$

Step I-4: Draw the directed network and derive the ordered attribute set.

For this purpose, we select those IVIFSs that satisfy $\left[a_{ij}, b_{ij}\right] \ge \left[a_{ji}, b_{ji}\right]$. After introducing this condition, the decision-making process becomes much easier and the burden of calculation is alleviated. Thus, we need to calculate the number of arrow tails and rank them from the biggest to the smallest ones. The attribute with the largest number of arrow tails is the best attribute, while the one with the smallest number is the worst attribute. In this case, the membership function of IVIFSs is only considered.

Note that there are n^2 IVIFSs in the $\dot{\tilde{G}} = \left(\dot{\tilde{g}}_{ij}\right)_{n\times n}$. Obviously, the main diagonal elements are $\dot{\tilde{g}}_{ij} = \left(\left[a_{ij}, b_{ij}\right], \left[c_{ij}, d_{ij}\right]\right) = ([0.5,0.5], [0.5,0.5])$; $i, j = 1,2, \dots, n$; thus, the rest is $n^2 - n$. In addition, by considering the characteristic of preference relation, we only need $(n^2 - n)/2$ preference information to rank the n attributes.

Step I-5: Determine the attribute weights by using IVIFBWM.

Since the values of weights are given in the form of interval, it is of significance to attach greater attention to the corresponding linear programming solution. In this way, the Interval Linear Programming (ILP) model is transformed into two sub-models and form a solution area by solving these sub-models and obtaining their optimal solutions. For this purpose, Tong's method [27] is applied to properly interact with the liner programming problems with interval values.

In the second stage, the decision matrices of attribute values are another required input for our introduced approach. Experts have used the IVIFSs to express their opinions.

Step II-1: Set up decision matrices of attribute values as follows:

$\widetilde{D}_k = \left(\tilde{d}_{ij}{}^{(k)}\right)_{m\times n}$

where $\tilde{d}_{ij}{}^{(k)} = \left(\left[a_{ij}, b_{ij}\right]^{(k)}, \left[c_{ij}, d_{ij}\right]^{(k)}\right)$, $i = 1,2, \dots, m$; $j = 1,2, \dots, n$; $k = 1,2, \dots, t$ is an IVIFS. $\left[a_{ij}, b_{ij}\right]^{(k)}$ indicates the opinion of the expert e_k about the alternative y_i to what extent one can satisfy the attribute g_j for the fuzzy concept ''excellence''. In addition, $\left[a_{ij}, b_{ij}\right]^{(k)} \subset [0,1]$, $\left[c_{ij}, d_{ij}\right]^{(k)} \subset [0,1]$, $0 \le b_{ij}{}^{(k)} + d_{ij}{}^{(k)} \le 1$, $i = 1,2, \dots, m$, $j = 1,2, \dots, n$, $k = 1,2, \dots, t$.

Step II-2: Use (8) to calculate score function for each element in the decision matrices as follows:

$S_{ij}{}^{(k)} = S\left(\tilde{d}_{ij}{}^{(k)}\right),$
$i = 1,2, \dots, m;\ j = 1,2, \dots, n;\ k = 1,2, \dots, t.$

Step II-3: Reorder the score function results in descending order based on the previous step such that $\tilde{d}_{ij_{\sigma(\mathrm{j}-1)}} \ge \tilde{d}_{ij_{\sigma(\mathrm{j})}}$ for all, and let $\tilde{d}_{ij_{\sigma(j)}} = \left(\left[a_{ij_{\sigma(j)}}, b_{ij_{\sigma(j)}}\right], \left[c_{ij_{\sigma(j)}}, d_{ij_{\sigma(j)}}\right]\right)$.

Step II-4: Apply the operator in (10) to integrate experts' opinions with attribute values and establish the aggregated decision matrix of attribute values as follows:

$\dot{\tilde{D}} = \left(\dot{\tilde{d}}_{ij}\right)_{m\times n}$

where

$\dot{\tilde{d}}_{ij} = IVIFOWA_w\left(\tilde{d}_{ij}^{(1)}, \tilde{d}_{ij}^{(2)}, \dots, \tilde{d}_{ij}^{(k)}\right); i = 1,2,\dots,m;\ j = 1,2,\dots,n;\ k = 1,2,\dots,t.$

Step II-5: Use a hybrid approach based on GRA and TOPSIS method introduced by Makui et al. [28] for ranking the alternatives.

One of the main advantages of this hybrid approach to ranking the alternatives is to consider both issues of shape similarity and position approximation to the ideal solution. This leads to the authority expansion of the decision makers in choosing the most appropriate alternative according to different circumstances.

5. A Real-World Application

In order to demonstrate how the introduced approach can be applied in practice, it is applied to an Iranian investment company, Omid Investment Management Group Company (OIMGC). With the aim of increasing profits and reducing the risk of investment during the long-term horizon, they sought to select the best mutual fund for creating an effective asset management. Four mutual funds were identified ($y_1, y_2, y_3 \, and \, y_4$) and five experts ($e_1, e_2, e_3, e_4 \, and \, e_5$), whose knowledge and experience were approved by the company that participated in our study. The weight and importance of decision-makers were the same. The attributes considered in the decision process include rate of return (g_1), standard deviation (g_2), treynor ratio (g_3), and turnover rate (g_4). Therefore, a procedure for selection of the most appropriate mutual fund contains the following steps:

Step I-1: Set up the IVIFPR matrices on the attributes based on pairwise comparison.

$$\tilde{G}_1 = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.4,0.7],[0.1,0.2]) & ([0.5,0.6],[0.2,0.3]) & ([0.3,0.5],[0.2,0.4]) \\ ([0.1,0.2],[0.4,0.7]) & ([0.5,0.5],[0.5,0.5]) & ([0.5,0.6],[0.1,0.2]) & ([0.6,0.7],[0.1,0.3]) \\ ([0.2,0.3],[0.5,0.6]) & ([0.1,0.2],[0.5,0.6]) & ([0.5,0.5],[0.5,0.5]) & ([0.3,0.4],[0.5,0.6]) \\ ([0.2,0.4],[0.3,0.5]) & ([0.1,0.3],[0.6,0.7]) & ([0.5,0.6],[0.3,0.4]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix},$$

$$\tilde{G}_2 = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.4,0.6],[0.3,0.4]) & ([0.5,0.7],[0.2,0.3]) & ([0.5,0.7],[0.2,0.3]) \\ ([0.3,0.4],[0.4,0.6]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.6],[0.1,0.3]) & ([0.4,0.5],[0.1,0.2]) \\ ([0.2,0.3],[0.5,0.7]) & ([0.1,0.3],[0.4,0.6]) & ([0.5,0.5],[0.5,0.5]) & ([0.5,0.7],[0.1,0.2]) \\ ([0.2,0.3],[0.5,0.7]) & ([0.1,0.2],[0.4,0.5]) & ([0.1,0.2],[0.5,0.7]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix},$$

$$\tilde{G}_3 = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.7,0.8],[0.1,0.2]) & ([0.6,0.7],[0.1,0.2]) & ([0.6,0.7],[0.2,0.3]) \\ ([0.1,0.2],[0.7,0.8]) & ([0.5,0.5],[0.5,0.5]) & ([0.5,0.7],[0.2,0.3]) & ([0.4,0.6],[0.2,0.3]) \\ ([0.1,0.2],[0.6,0.7]) & ([0.2,0.3],[0.5,0.7]) & ([0.5,0.5],[0.5,0.5]) & ([0.3,0.4],[0.5,0.6]) \\ ([0.2,0.3],[0.6,0.7]) & ([0.2,0.3],[0.4,0.6]) & ([0.5,0.6],[0.3,0.4]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix},$$

$$\tilde{G}_4 = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.5,0.6],[0.3,0.4]) & ([0.3,0.4],[0.5,0.6]) & ([0.7,0.8],[0.1,0.2]) \\ ([0.3,0.4],[0.5,0.6]) & ([0.5,0.5],[0.5,0.5]) & ([0.3,0.4],[0.5,0.6]) & ([0.5,0.6],[0.3,0.4]) \\ ([0.5,0.6],[0.3,0.4]) & ([0.5,0.6],[0.3,0.4]) & ([0.5,0.5],[0.5,0.5]) & ([0.5,0.6],[0.3,0.4]) \\ ([0.1,0.2],[0.7,0.8]) & ([0.3,0.4],[0.5,0.6]) & ([0.3,0.4],[0.5,0.6]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix},$$

$$\tilde{G}_5 = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.3,0.4],[0.4,0.6]) & ([0.5,0.6],[0.3,0.4]) & ([0.4,0.5],[0.3,0.4]) \\ ([0.4,0.6],[0.3,0.4]) & ([0.5,0.5],[0.5,0.5]) & ([0.6,0.7],[0.2,0.3]) & ([0.6,0.7],[0.1,0.3]) \\ ([0.3,0.4],[0.5,0.6]) & ([0.2,0.3],[0.6,0.7]) & ([0.5,0.5],[0.5,0.5]) & ([0.3,0.4],[0.2,0.3]) \\ ([0.3,0.4],[0.4,0.5]) & ([0.1,0.3],[0.6,0.7]) & ([0.2,0.3],[0.3,0.4]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix}.$$

Step I-2: Integrate experts' opinions on preference relation and establish the aggregated preference relation matrix.

$$\begin{aligned} \dot{\tilde{g}}_{12} = & ([0.4^{0.2} \times 0.4^{0.2} \times 0.7^{0.2} \times 0.5^{0.2} \times 0.3^{0.2}, 0.7^{0.2} \times 0.6^{0.2} \times 0.8^{0.2} \times 0.6^{0.2} \times 0.4^{0.2}], [0.1^{0.2} \\ & \times 0.3^{0.2} \times 0.1^{0.2} \times 0.3^{0.2} \times 0.4^{0.2}, 0.2^{0.2} \times 0.4^{0.2} \times 0.2^{0.2} \times 0.4^{0.2} \times 0.6^{0.2}]) \\ & = ([0.4416,0.6044],[0.2048,0.3288]) \end{aligned}$$

$$\dot{\tilde{G}} = \begin{bmatrix} ([0.5000,0.5000],[0.5000,0.5000]) & ([0.4416,0.6044],[0.2048,0.3288]) \\ ([0.2048,0.3288],[0.4416,0.6044]) & ([0.5000,0.5000],[0.5000,0.5000]) \\ ([0.2268,0.3366],[0.4682,0.5885]) & ([0.1821,0.3178],[0.4478,0.5885]) \\ ([0.1888,0.3104],[0.4789,0.6284]) & ([0.1431,0.2930],[0.4919,0.6153]) \end{bmatrix}$$

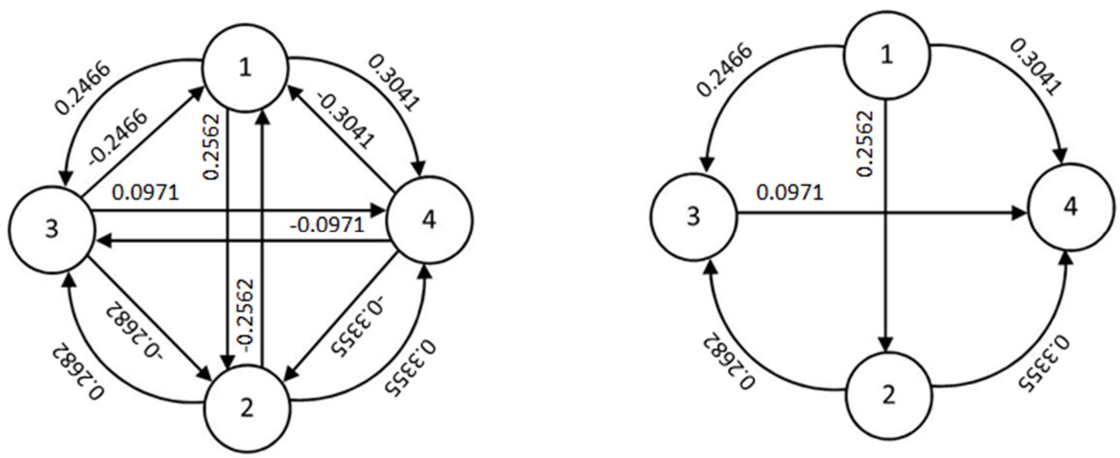
$$\begin{matrix} ([0.4682, 0.5885], [0.2268, 0.3366]) & ([0.4789, 0.6284], [0.1888, 0.3104]) \\ ([0.4478, 0.5885], [0.1821, 0.3178]) & ([0.4919, 0.6153], [0.1431, 0.2930]) \\ ([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.3680, 0.4852], [0.2724, 0.3866]) \\ ([0.2724, 0.3866], [0.3680, 0.4852]) & ([0.5000, 0.5000], [0.5000, 0.5000]) \end{bmatrix}$$

Step I-3: Calculate score function for each element in $\dot{\ddot{G}}$.

$$S_{12} = \frac{1}{2}(0.4416 - 0.2048 + 0.6044 - 0.3288) = 0.2562$$

$S_{11} = 0.0000$,	$S_{12} = 0.2562$,	$S_{13} = 0.2466$,	$S_{14} = 0.3041$,
$S_{21} = -0.2562$,	$S_{22} = 0.0000$,	$S_{23} = 0.2682$,	$S_{24} = 0.3355$,
$S_{31} = -0.2466$,	$S_{32} = -0.2682$,	$S_{33} = 0.0000$,	$S_{34} = 0.0971$,
$S_{41} = -0.3041$,	$S_{42} = -0.3355$,	$S_{43} = -0.0971$,	$S_{44} = 0.0000$.

Step I-4: Draw the directed network and derive the ordered attribute set.



(a) (b)

Fig. 2. Directed network of the group IVIFPR.

For this purpose, the IVIFSs are chosen whose score function meets the condition: $S_{ij} \geq 0$; as shown in Fig. 2(b). Then, the out-degrees of all attributes should be calculated: $D_1^{out} = 3$, $D_2^{out} = 2$, $D_3^{out} = 1$, and $D_4^{out} = 0$; on this basis, we get the ranking of the attributes: $D_1^{out} \geq D_2^{out} \geq D_3^{out} \geq D_4^{out}$. Thus, the best attribute is g_1 and the worst is g_4.

Step I-5: Determine the attribute weights by using the IVIFBWM.

In order to derive the optimal weight vector of the ordered attributes set, two models are constructed as follows:

Model **I-5-I:**

$min\ \varphi$

$S.t.$

$$\left|\frac{\tau_B}{\tau_W} - \mu_{BW}\right| = \left|\frac{\tau_1}{\tau_4} - [0.4789, 0.6284]\right| \leq \varphi,$$
$$\left|\frac{\tau_B}{\tau_2} - \mu_{B2}\right| = \left|\frac{\tau_1}{\tau_2} - [0.4416, 0.6044]\right| \leq \varphi,$$
$$\left|\frac{\tau_B}{\tau_3} - \mu_{B3}\right| = \left|\frac{\tau_1}{\tau_3} - [0.4682, 0.5885]\right| \leq \varphi,$$
$$\left|\frac{\tau_2}{\tau_3} - \mu_{23}\right| = \left|\frac{\tau_2}{\tau_3} - [0.4478, 0.5885]\right| \leq \varphi,$$
$$\left|\frac{\tau_2}{\tau_W} - \mu_{2W}\right| = \left|\frac{\tau_2}{\tau_4} - [0.4919, 0.6153]\right| \leq \varphi,$$

$$\left|\frac{\tau_3}{\tau_W}-\mu_{3W}\right| = \left|\frac{\tau_3}{\tau_4}-[0.3680,0.4852]\right| \le \varphi,$$
$$\tau_1+\tau_2+\tau_3+\tau_4 = 1,$$
$$\varphi > 0, \tau_j \ge 0, \quad j = 1,2,3,4.$$

Model **I-5-II:**

$min\, \psi$

$S.t.$

$$\left|\frac{\sigma_B}{\sigma_W}-v_{BW}\right| = \left|\frac{\sigma_1}{\sigma_4}-[0.1888,0.3104]\right| \le \psi,$$
$$\left|\frac{\sigma_B}{\sigma_2}-v_{B2}\right| = \left|\frac{\sigma_1}{\sigma_2}-[0.2048,0.3288]\right| \le \psi,$$
$$\left|\frac{\sigma_B}{\sigma_3}-v_{B3}\right| = \left|\frac{\sigma_1}{\sigma_3}-[0.2268,0.3366]\right| \le \psi,$$
$$\left|\frac{\sigma_2}{\sigma_3}-v_{23}\right| = \left|\frac{\sigma_2}{\sigma_3}-[0.1821,0.3178]\right| \le \psi,$$
$$\left|\frac{\sigma_2}{\sigma_W}-v_{2W}\right| = \left|\frac{\sigma_2}{\sigma_4}-[0.1431,0.2930]\right| \le \psi,$$
$$\left|\frac{\sigma_3}{\sigma_W}-v_{3W}\right| = \left|\frac{\sigma_3}{\sigma_4}-[0.2724,0.3866]\right| \le \psi,$$

$\sigma_1+\sigma_2+\sigma_3+\sigma_4 = 1,$

$\psi > 0, \sigma_j \ge 0, \quad j = 1,2,3,4.$

Given that the variables in the above models are in the form of interval, Tong's method [27] is applied that can properly interact with the liner programming problems with interval values. Thus, the problem-solving process should be continued as follows:

The ILP problem (I-5-I) is transformed into two sub-problems of the best and the worst which are summarized as follows:

The best sub-problem:

$min\, \varphi^-$

$S.t.$

$$\left|\frac{\tau_B}{\tau_W}-b_{BW}\right| = \left|\frac{\tau_1}{\tau_4}-0.6284\right| \le \varphi^-,$$
$$\left|\frac{\tau_B}{\tau_2}-b_{B2}\right| = \left|\frac{\tau_1}{\tau_2}-0.6044\right| \le \varphi^-,$$
$$\left|\frac{\tau_B}{\tau_3}-b_{B3}\right| = \left|\frac{\tau_1}{\tau_3}-0.5885\right| \le \varphi^-,$$
$$\left|\frac{\tau_2}{\tau_3}-b_{23}\right| = \left|\frac{\tau_2}{\tau_3}-0.5885\right| \le \varphi^-,$$
$$\left|\frac{\tau_2}{\tau_W}-b_{2W}\right| = \left|\frac{\tau_2}{\tau_4}-0.6153\right| \le \varphi^-,$$
$$\left|\frac{\tau_3}{\tau_W}-b_{3W}\right| = \left|\frac{\tau_3}{\tau_4}-0.4852\right| \le \varphi^-,$$

$\tau_1+\tau_2+\tau_3+\tau_4 = 1,$

$\varphi^- > 0, \tau_j \ge 0, \quad j = 1,2,3,4.$

The worst sub-problem:

$min\, \varphi^+$

$S.t.$

$$\left|\frac{\tau_B}{\tau_W}-a_{BW}\right| = \left|\frac{\tau_1}{\tau_4}-0.4789\right| \le \varphi^+,$$
$$\left|\frac{\tau_B}{\tau_2}-a_{B2}\right| = \left|\frac{\tau_1}{\tau_2}-0.4416\right| \le \varphi^+,$$
$$\left|\frac{\tau_B}{\tau_3}-a_{B3}\right| = \left|\frac{\tau_1}{\tau_3}-0.4682\right| \le \varphi^+,$$

$$\left|\frac{\tau_2}{\tau_3} - a_{23}\right| = \left|\frac{\tau_2}{\tau_3} - 0.4478\right| \leq \varphi^+,$$
$$\left|\frac{\tau_2}{\tau_W} - a_{2W}\right| = \left|\frac{\tau_2}{\tau_4} - 0.4919\right| \leq \varphi^+,$$
$$\left|\frac{\tau_3}{\tau_W} - a_{3W}\right| = \left|\frac{\tau_3}{\tau_4} - 0.3680\right| \leq \varphi^+,$$

$\tau_1 + \tau_2 + \tau_3 + \tau_4 = 1,$
$\varphi^+ > 0, \tau_j \geq 0, \; j = 1,2,3,4.$

Consequently, the optimal solutions of the best sub-problem and the worst sub-problem form a box as follows:

$\mathrm{T}^{\pm} = \left(\tau_1^{\pm}, \tau_2^{\pm}, \tau_3^{\pm}, \tau_4^{\pm}\right) = ([0.2620, 0.2630], [0.2580, 0.2580], [0.2410, 0.2410], [0.2380, 0.2390]),$

Similarly, the ILP problem (I-5-II) is transformed into two sub-problems of the best and the worst which are summarized as follows:

The best sub-problem:

$min\, \psi^-$
$S.t.$

$$\left|\frac{\sigma_B}{\sigma_W} - d_{BW}\right| = \left|\frac{\sigma_1}{\sigma_4} - 0.3104\right| \leq \psi^-,$$
$$\left|\frac{\sigma_B}{\sigma_2} - d_{B2}\right| = \left|\frac{\sigma_1}{\sigma_2} - 0.3288\right| \leq \psi^-,$$
$$\left|\frac{\sigma_B}{\sigma_3} - d_{B3}\right| = \left|\frac{\sigma_1}{\sigma_3} - 0.3366\right| \leq \psi^-,$$
$$\left|\frac{\sigma_2}{\sigma_3} - d_{23}\right| = \left|\frac{\sigma_2}{\sigma_3} - 0.3178\right| \leq \psi^-,$$
$$\left|\frac{\sigma_2}{\sigma_W} - d_{2W}\right| = \left|\frac{\sigma_2}{\sigma_4} - 0.2930\right| \leq \psi^-,$$
$$\left|\frac{\sigma_3}{\sigma_W} - d_{3W}\right| = \left|\frac{\sigma_3}{\sigma_4} - 0.3866\right| \leq \psi^-,$$

$\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 1,$
$\psi^- > 0, \sigma_j \geq 0, \; j = 1,2,3,4.$

The worst sub-problem:
$min\, \psi^+$
$S.t.$

$$\left|\frac{\sigma_B}{\sigma_W} - c_{BW}\right| = \left|\frac{\sigma_1}{\sigma_4} - 0.1888\right| \leq \psi^+,$$
$$\left|\frac{\sigma_B}{\sigma_2} - c_{B2}\right| = \left|\frac{\sigma_1}{\sigma_2} - 0.2048\right| \leq \psi^+,$$
$$\left|\frac{\sigma_B}{\sigma_3} - c_{B3}\right| = \left|\frac{\sigma_1}{\sigma_3} - 0.2268\right| \leq \psi^+,$$
$$\left|\frac{\sigma_2}{\sigma_3} - c_{23}\right| = \left|\frac{\sigma_2}{\sigma_3} - 0.1821\right| \leq \psi^+,$$
$$\left|\frac{\sigma_2}{\sigma_W} - c_{2W}\right| = \left|\frac{\sigma_2}{\sigma_4} - 0.1431\right| \leq \psi^+,$$
$$\left|\frac{\sigma_3}{\sigma_W} - c_{3W}\right| = \left|\frac{\sigma_3}{\sigma_4} - 0.2724\right| \leq \psi^+,$$

$\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 1,$
$\psi^+ > 0, \sigma_j \geq 0, \; j = 1,2,3,4.$

Consequently, the optimal solutions of the best sub-problem and the worst sub-problem form a box as follows:

$\Sigma^{\pm} = \left(\sigma_1^{\pm}, \sigma_2^{\pm}, \sigma_3^{\pm}, \sigma_4^{\pm}\right) = ([0.0680, 0.1010], [0.0900, 0.1370], [0.2190, 0.2530], [0.6030, 0.6052]),$

The optimal solutions for Model I-5-I and Model I-5-II are:

$(\tau_1^*, \tau_2^*, \tau_3^*, \tau_4^*) = ([0.2620,0.2630],[0.2580,0.2580],[0.2410,0.2410],[0.2380,0.2390])$, $\varphi^* = [0.1220,0.1510]$,
$(\sigma_1^*, \sigma_2^*, \sigma_3^*, \sigma_4^*) = ([0.0680,0.1010],[0.0900,0.1370],[0.2190,0.2530],[0.6030,0.6052])$ and $\psi^* = [0.0301,0.0310]$. Hence, the optimal weight vector of the attribute set is
$W^* = (\tilde{w}_1^*, \tilde{w}_2^*, \tilde{w}_3^*, \tilde{w}_4^*) = \left((\tau_1^*, \sigma_1^*), (\tau_2^*, \sigma_2^*), (\tau_3^*, \sigma_3^*), (\tau_4^*, \sigma_4^*)\right) =$
$\left(\begin{pmatrix}[0.2620,0.2630],\\ [0.0680,0.1010]\end{pmatrix}, \begin{pmatrix}[0.2580,0.2580],\\ [0.0900,0.1370]\end{pmatrix}, \begin{pmatrix}[0.2410,0.2410],\\ [0.2190,0.2530]\end{pmatrix}, \begin{pmatrix}[0.2380,0.2390],\\ [0.6030,0.6052]\end{pmatrix}\right)$
This means that the most important attribute is g_1 and the least important one is g_4. The numerical results are in accordance with the actual condition.
As $\mu_{BW} = [0.4789,0.6284]$ and $\upsilon_{BW} = [0.1888,0.3104]$, we check Table 1 and get $CI_1 \cong 2.35$ and $CI_2 \cong 0.12$. Consequently, $CR = max\left\{\frac{0.1510}{2.35}, \frac{0.0310}{0.12}\right\} = 0.2583$, which means that the result is reliable because CR is close to 0 (fully consistent) and far away from 1 (least consistent).

Step II-1: Set up the decision matrices of attribute values.

$$\tilde{D}_1 = \begin{bmatrix} ([0.3,0.5],[0.4,0.5]) & ([0.6,0.7],[0.1,0.2]) & ([0.5,0.6],[0.2,0.3]) & ([0.4,0.7],[0.0,0.1]) \\ ([0.6,0.8],[0.1,0.2]) & ([0.6,0.7],[0.2,0.3]) & ([0.6,0.8],[0.1,0.2]) & ([0.5,0.7],[0.1,0.3]) \\ ([0.7,0.8],[0.1,0.2]) & ([0.7,0.8],[0.0,0.1]) & ([0.5,0.7],[0.2,0.3]) & ([0.6,0.8],[0.1,0.2]) \\ ([0.2,0.3],[0.4,0.5]) & ([0.5,0.7],[0.1,0.3]) & ([0.4,0.6],[0.3,0.4]) & ([0.4,0.5],[0.1,0.3]) \end{bmatrix},$$

$$\tilde{D}_2 = \begin{bmatrix} ([0.5,0.6],[0.3,0.4]) & ([0.4,0.6],[0.1,0.2]) & ([0.6,0.7],[0.2,0.3]) & ([0.5,0.6],[0.1,0.2]) \\ ([0.6,0.7],[0.1,0.2]) & ([0.5,0.6],[0.3,0.4]) & ([0.4,0.5],[0.3,0.4]) & ([0.5,0.7],[0.1,0.2]) \\ ([0.6,0.8],[0.1,0.2]) & ([0.6,0.7],[0.1,0.2]) & ([0.5,0.6],[0.3,0.4]) & ([0.7,0.9],[0.0,0.1]) \\ ([0.4,0.6],[0.3,0.4]) & ([0.4,0.5],[0.0,0.1]) & ([0.4,0.5],[0.2,0.4]) & ([0.4,0.6],[0.1,0.2]) \end{bmatrix},$$

$$\tilde{D}_3 = \begin{bmatrix} ([0.5,0.7],[0.2,0.3]) & ([0.5,0.6],[0.1,0.2]) & ([0.5,0.6],[0.2,0.4]) & ([0.4,0.6],[0.1,0.3]) \\ ([0.5,0.6],[0.1,0.2]) & ([0.5,0.7],[0.2,0.3]) & ([0.3,0.6],[0.2,0.4]) & ([0.6,0.8],[0.0,0.1]) \\ ([0.5,0.8],[0.1,0.2]) & ([0.5,0.8],[0.1,0.2]) & ([0.4,0.7],[0.2,0.3]) & ([0.5,0.8],[0.0,0.2]) \\ ([0.4,0.6],[0.1,0.3]) & ([0.4,0.6],[0.0,0.1]) & ([0.3,0.5],[0.2,0.4]) & ([0.4,0.6],[0.2,0.3]) \end{bmatrix},$$

$$\tilde{D}_4 = \begin{bmatrix} ([0.3,0.4],[0.4,0.6]) & ([0.6,0.7],[0.1,0.2]) & ([0.7,0.8],[0.1,0.2]) & ([0.5,0.6],[0.0,0.1]) \\ ([0.5,0.8],[0.1,0.2]) & ([0.6,0.7],[0.2,0.3]) & ([0.4,0.6],[0.1,0.4]) & ([0.5,0.6],[0.1,0.3]) \\ ([0.7,0.8],[0.1,0.2]) & ([0.7,0.8],[0.0,0.1]) & ([0.5,0.7],[0.2,0.3]) & ([0.6,0.7],[0.1,0.2]) \\ ([0.2,0.3],[0.4,0.5]) & ([0.5,0.7],[0.1,0.3]) & ([0.5,0.6],[0.1,0.3]) & ([0.4,0.5],[0.1,0.3]) \end{bmatrix},$$

$$\tilde{D}_5 = \begin{bmatrix} ([0.3,0.4],[0.4,0.6]) & ([0.6,0.7],[0.2,0.3]) & ([0.6,0.7],[0.2,0.3]) & ([0.5,0.7],[0.0,0.1]) \\ ([0.4,0.6],[0.2,0.4]) & ([0.6,0.8],[0.1,0.2]) & ([0.5,0.6],[0.3,0.4]) & ([0.7,0.8],[0.1,0.2]) \\ ([0.5,0.7],[0.1,0.3]) & ([0.7,0.9],[0.0,0.1]) & ([0.5,0.6],[0.2,0.4]) & ([0.6,0.9],[0.0,0.1]) \\ ([0.2,0.3],[0.5,0.6]) & ([0.5,0.7],[0.1,0.3]) & ([0.4,0.6],[0.1,0.2]) & ([0.4,0.6],[0.1,0.3]) \end{bmatrix}.$$

Step II-2: Calculate score function for each element in the decision matrices.

$S_{11}{}^{(1)} = \frac{1}{2}(0.3 - 0.4 + 0.5 - 0.5) = -0.0500$
$S_{11}{}^{(1)} = -0.0500,$
$S_{11}{}^{(2)} = \ \ 0.2000,$
$S_{11}{}^{(3)} = \ \ 0.3500,$
$S_{11}{}^{(4)} = -0.1500,$
$S_{11}{}^{(5)} = -0.1500.$

Step II-3: Reorder the score function results in descending order based on the previous step.
$S_{11}{}^{(3)} = 0.3500 > S_{11}{}^{(2)} = 0.2000 > S_{11}{}^{(1)} = -0.0500 > S_{11}{}^{(4)} = -0.1500 > S_{11}{}^{(5)} = -0.1500$
Then, extract the new permutation of arguments as follows:
$\tilde{d}_{11_{\sigma(1)}} = ([0.5,0.7],[0.2,0.3]),$
$\tilde{d}_{11_{\sigma(2)}} = ([0.5,0.6],[0.3,0.4]),$
$\tilde{d}_{11_{\sigma(3)}} = ([0.3,0.5],[0.4,0.5]),$

$\tilde{d}_{11_{\sigma(4)}} = ([0.3,0.4],[0.4,0.6]),$

$\tilde{d}_{11_{\sigma(5)}} = ([0.3,0.4],[0.4,0.6]).$

Step II-4: Integrate experts' opinions on attribute values and establish the aggregated decision matrix.

$$\dot{\tilde{d}}_{11} = ([1-(1-0.5)^{0.1117}\times(1-0.5)^{0.2365}\times(1-0.3)^{0.3036}\times(1-0.3)^{0.2365}\times(1-0.3)^{0.1117}, 1-(1-0.7)^{0.1117}\times(1-0.6)^{0.2365}\times(1-0.5)^{0.3036}\times(1-0.4)^{0.2365}\times(1-0.4)^{0.1117}],[0.2^{0.1117}\times0.3^{0.2365}\times0.4^{0.3036}\times0.4^{0.2365}\times0.4^{0.1117}, 0.3^{0.1117}\times0.4^{0.2365}\times0.5^{0.3036}\times0.6^{0.2365}\times0.6^{0.1117}]) = ([0.3774,0.5226],[0.3458,0.4774])$$

$$\dot{\tilde{D}} = \begin{bmatrix} ([0.3774,0.5226],[0.3458,0.4774]) & ([0.5521,0.6619],[0.1178,0.2201]) & ([0.5814,0.6831],[0.1851,0.2961]) & ([0.4672,0.6381],[0.0000,0.1332]) \\ ([0.5279,0.7198],[0.1081,0.2161]) & ([0.5677,0.7039],[0.1937,0.2961]) & ([0.4373,0.6205],[0.1859,0.3702]) & ([0.5678,0.7310],[0.0000,0.2132]) \\ ([0.6089,0.7907],[0.1000,0.2093]) & ([0.6600,0.7963],[0.0000,0.1273]) & ([0.4715,0.6684],[0.2093,0.3316]) & ([0.5917,0.8356],[0.0000,0.1571]) \\ ([0.2763,0.4239],[0.3282,0.4572]) & ([0.4607,0.6382],[0.0000,0.1901]) & ([0.4019,0.5677],[0.1777,0.3288]) & ([0.4000,0.5612],[0.1178,0.2867]) \end{bmatrix}$$

Step II-5: Rank the alternatives by using the hybrid approach based on GRA and TOPSIS methods.

With regard to the influence of the third parameter on calculation of distance between the alternatives in the interval-valued intuitionistic fuzzy environment, by utilizing Eq. (3), the uncertainty function for each of the above alternatives has been added to the aggregated interval-valued intuitionistic fuzzy decision matrix as follows:

$$\dot{\tilde{D}} = \begin{bmatrix} ([0.3774,0.5226],[0.3458,0.4774],[0.0000,0.2768]) & ([0.5521,0.6619],[0.1178,0.2201],[0.1180,0.3301]) & ([0.5814,0.6831],[0.1851,0.2961],[0.0208,0.2335]) & ([0.4672,0.6381],[0.0000,0.1332],[0.2287,0.5328]) \\ ([0.5279,0.7198],[0.1081,0.2161],[0.0641,0.3640]) & ([0.5677,0.7039],[0.1937,0.2961],[0.0000,0.2386]) & ([0.4373,0.6205],[0.1859,0.3702],[0.0093,0.3768]) & ([0.5678,0.7310],[0.0000,0.2132],[0.0558,0.4322]) \\ ([0.6089,0.7907],[0.1000,0.2093],[0.0000,0.2911]) & ([0.6600,0.7963],[0.0000,0.1273],[0.0764,0.3400]) & ([0.4715,0.6684],[0.2093,0.3316],[0.0000,0.3192]) & ([0.5917,0.8356],[0.0000,0.1571],[0.0073,0.4083]) \\ ([0.2763,0.4239],[0.3282,0.4572],[0.1189,0.3955]) & ([0.4607,0.6382],[0.0000,0.1901],[0.1717,0.5393]) & ([0.4019,0.5677],[0.1777,0.3288],[0.1035,0.4204]) & ([0.4000,0.5612],[0.1178,0.2867],[0.1521,0.4822]) \end{bmatrix}$$

Then, obtain the Positive Ideal Solution (PIS) and the Negative Ideal Solution (NIS) with interval-valued intuitionistic fuzzy information as follows:

$$A^{+} = [([0.6089,0.7907],[0.1000,0.2093],[0.0000,0.2911]) \quad ([0.6600,0.7963],[0.0000,0.1273],[0.0764,0.3400]) \quad ([0.5814,0.6831],[0.1777,0.2961],[0.0208,0.2409]) \quad ([0.5917,0.8356],[0.0000,0.1332],[0.0312,0.4083])]$$

$$A^{-} = [([0.2763,0.4239],[0.3458,0.4774],[0.0987,0.3779]) \quad ([0.4607,0.6382],[0.1937,0.2961],[0.0657,0.3456]) \quad ([0.4019,0.5677],[0.2093,0.3702],[0.0621,0.3888]) \quad ([0.4000,0.5612],[0.1178,0.2867],[0.1521,0.4822])]$$

To obtain the separation measures, the normalized Hamming distance should be used by utilizing Eq. (11). Therefore, the separation of each alternative from the PIS, A^{+}, is calculated as follows:

$$d_{ij}{}^{+} = \begin{bmatrix} 0.1285 & 0.0631 & 0.0019 & 0.0805 \\ 0.0380 & 0.0906 & 0.0546 & 0.0321 \\ 0.0000 & 0.0000 & 0.0364 & 0.0060 \\ 0.1749 & 0.0894 & 0.0737 & 0.1165 \end{bmatrix}$$

Similarly, the separation of each alternative from the NIS, A^{-} is calculated as follows:

$$d_{ij}^{-}=\begin{bmatrix}0.0500 & 0.0419 & 0.0737 & 0.0678\\ 0.1369 & 0.0432 & 0.0221 & 0.0844\\ 0.1749 & 0.0920 & 0.0426 & 0.1165\\ 0.0095 & 0.0749 & 0.0183 & 0.0000\end{bmatrix}$$

Afterward, calculate the gray relational coefficients of each alternative from the PIS and the NIS as follows:

$$\xi_{ij}^{+}=\frac{\min\limits_{1\le i\le m}\min\limits_{1\le j\le n} d(\tilde{r}_{ij},\tilde{r}_{ij}^{+})+\rho\max\limits_{1\le i\le m}\max\limits_{1\le j\le n} d(\tilde{r}_{ij},\tilde{r}_{ij}^{+})}{d(\tilde{r}_{ij},\tilde{r}_{ij}^{+})+\rho\max\limits_{1\le i\le m}\max\limits_{1\le j\le n} d(\tilde{r}_{ij},\tilde{r}_{ij}^{+})},\qquad i=1,2,\dots,m, j=1,2,\dots,n.$$

$$\xi_{ij}^{+}=\begin{bmatrix}0.4049 & 0.5810 & 0.9793 & 0.5206\\ 0.6972 & 0.4910 & 0.6158 & 0.7313\\ 1.0000 & 1.0000 & 0.7063 & 0.9360\\ 0.3333 & 0.4946 & 0.5425 & 0.4287\end{bmatrix},$$

$$\xi_{ij}^{-}=\frac{\min\limits_{1\le i\le m}\min\limits_{1\le j\le n} d(\tilde{r}_{ij},\tilde{r}_{ij}^{-})+\rho\max\limits_{1\le i\le m}\max\limits_{1\le j\le n} d(\tilde{r}_{ij},\tilde{r}_{ij}^{-})}{d(\tilde{r}_{ij},\tilde{r}_{ij}^{-})+\rho\max\limits_{1\le i\le m}\max\limits_{1\le j\le n} d(\tilde{r}_{ij},\tilde{r}_{ij}^{-})},\qquad i=1,2,\dots,m, j=1,2,\dots,n.$$

$$\xi_{ij}^{-}=\begin{bmatrix}0.6364 & 0.6763 & 0.5425 & 0.5631\\ 0.3898 & 0.6694 & 0.7986 & 0.5088\\ 0.3333 & 0.4872 & 0.6725 & 0.4287\\ 0.9025 & 0.5385 & 0.8273 & 1.0000\end{bmatrix}.$$

Then, calculate the degree of gray relational coefficients of each alternative from the PIS and the NIS as follows:

$$\xi_{i}^{+}=\sum_{j=1}^{n} w_j\xi_{ij}^{+}\ ,\qquad i=1,2,\dots,m.$$

$$\xi_{i}^{-}=\sum_{j=1}^{n} w_j\xi_{ij}^{-}\ ,\qquad i=1,2,\dots,m.$$

$$W^{*}.\xi_{ij}^{+}=\begin{bmatrix}([0.1157,0.1162],[0.3367,0.3952]) & ([0.1592,0.1592],[0.2468,0.3151]) & ([0.2367,0.2367],[0.2260,0.2603]) & ([0.1319,0.1325],[0.7685,0.7699])\\ ([0.1909,0.1917],[0.1535,0.2022]) & ([0.1363,0.1363],[0.3066,0.3768]) & ([0.1562,0.1562],[0.3925,0.4290]) & ([0.1803,0.1811],[0.6908,0.6926])\\ ([0.2620,0.2630],[0.0680,0.1010]) & ([0.2580,0.2580],[0.0900,0.1370]) & ([0.1770,0.1770],[0.3421,0.3788]) & ([0.2246,0.2256],[0.6228,0.6250])\\ ([0.0963,0.0967],[0.4082,0.4657]) & ([0.1372,0.1372],[0.3039,0.3741]) & ([0.1389,0.1389],[0.4387,0.4745]) & ([0.1100,0.1105],[0.8050,0.8063])\end{bmatrix},$$

$$W^{*}.\xi_{ij}^{-}=\begin{bmatrix}([0.1758,0.1765],[0.1807,0.2325]) & ([0.1828,0.1828],[0.1962,0.2607]) & ([0.1389,0.1389],[0.4387,0.4745]) & ([0.1419,0.1426],[0.7521,0.7537])\\ ([0.1117,0.1122],[0.3507,0.4092]) & ([0.1811,0.1811],[0.1995,0.2643]) & ([0.1977,0.1977],[0.2974,0.3337]) & ([0.1292,0.1297],[0.7731,0.7745])\\ ([0.0963,0.0967],[0.4082,0.4657]) & ([0.1353,0.1353],[0.3094,0.3797]) & ([0.1693,0.1693],[0.3601,0.3968]) & ([0.1100,0.1105],[0.8050,0.8063])\\ ([0.2398,0.2407],[0.0884,0.1263]) & ([0.1484,0.1484],[0.2734,0.3429]) & ([0.2040,0.2040],[0.2847,0.3208]) & ([0.2380,0.2390],[0.6030,0.6052])\end{bmatrix}.$$

$$\xi_{1}^{+}=([0.5073,0.5079],[0.0144,0.0250]),$$

$$S_{\xi_1^+}=\frac{1}{2}(0.5073-0.0144+0.5079-0.0250)=0.4879.$$

$S_{\xi_1^+}=0.4879,\quad S_{\xi_2^+}=0.4994,\quad S_{\xi_3^+}=0.6487,\quad S_{\xi_4^+}=0.3475.$

$S_{\xi_1^-}=0.4860,\quad S_{\xi_2^-}=0.4700,\quad S_{\xi_3^-}=0.3759,\quad S_{\xi_4^-}=0.6016.$

Calculate the relative grey relational degree of each alternative from the PIS as follows:

$$S_{\xi_i}=\frac{S_{\xi_i^+}}{S_{\xi_i^+}+S_{\xi_i^-}}\ ,\qquad i=1,2,\dots,m.$$

$S_{\xi_1}=0.5010,\quad S_{\xi_2}=0.5152,\quad S_{\xi_3}=0.6331,\quad S_{\xi_4}=0.3662.$

Finally, rank the alternatives and select the best one(s) in accordance with ξ_i. If any alternative has the highest ξ_i value, then it is the most important alternative.

$$S_\xi\left(\dot{\tilde{d}}_3\right) > S_\xi\left(\dot{\tilde{d}}_2\right) > S_\xi\left(\dot{\tilde{d}}_1\right) > S_\xi\left(\dot{\tilde{d}}_4\right)$$
$$y_3 \succ y_2 \succ y_1 \succ y_4$$

Thus, according to the calculations made in previous stages, the third alternative has the greatest degree of acceptance and is the best one and also the fourth alternative has the least degree of acceptance and is the worst one. In addition, the following figures show that although there is an overall ranking of the alternatives, the decision-maker(s) can observe ranking of the alternatives in any particular attributes and choose any of them depending on different situations.

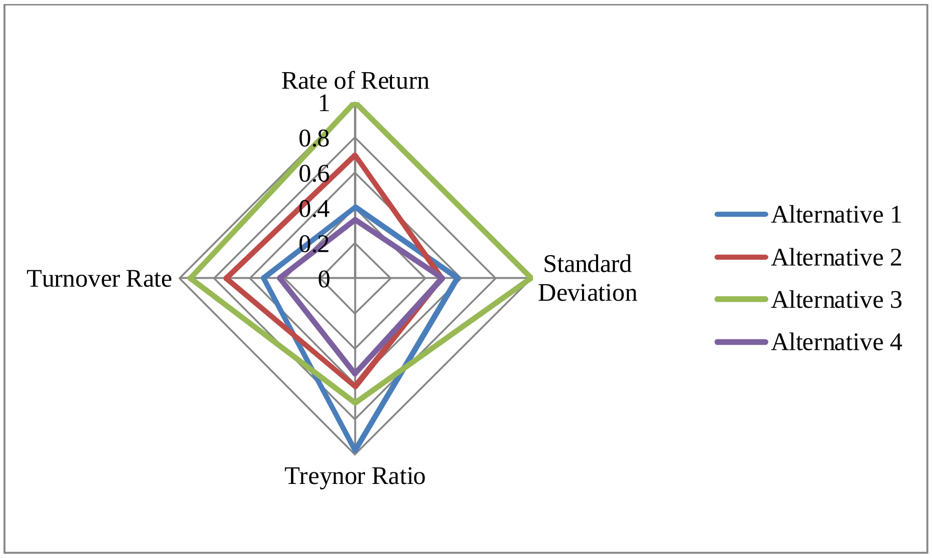


Fig. 3. The shape similarity between the PIS and the alternatives.

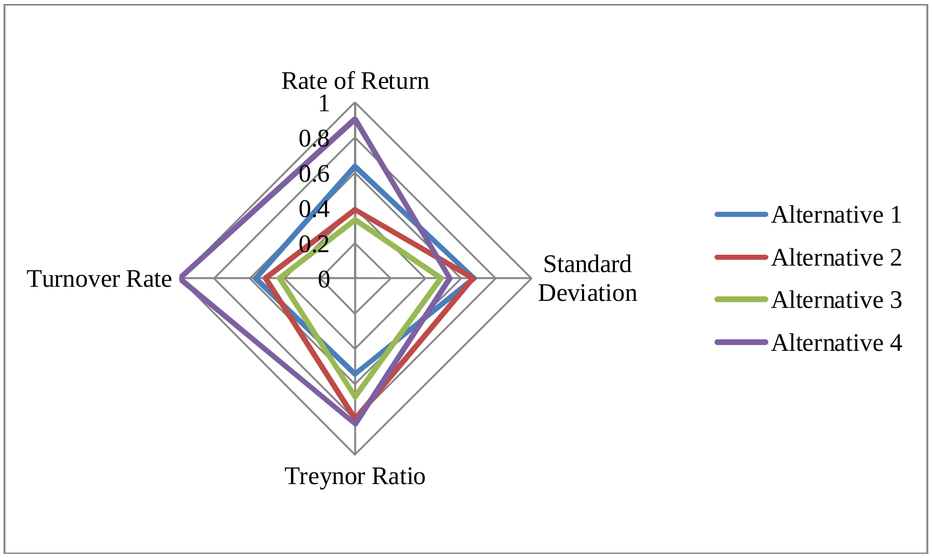


Fig. 4. The shape similarity between the NIS and the alternatives.

6. Conclusion and Future Research

The development and application of tools that could adequately address the uncertainty in the decision-making process and be appropriate for group decision-making is an important area of MCDM. In fact, the use of these tools is a prerequisite for objective decision-making. Therefore, in this paper, first, the traditional

BWM was developed and the IVIFBWM was proposed; then, a novel approach to fuzzy multi-attribute group decision-making was introduced based on the proposed method. Further, to demonstrate the application of the introduced approach, we have used a real-world decision making problem, selecting the best mutual fund for establishing an effective asset management. From the experimental results, it was found that the introduced approach was not only simple in calculation but also convenient in implementation, especially in interval-valued intuitionistic fuzzy environments. Therefore, the introduced approach could meet the objectives mentioned at the beginning of this paper. The introduced approach enjoys several outstanding features that make it a robust and interesting approach:

- The mathematical framework of the approach remains the same regardless of the number of attributes, alternatives, and decision-makers;
- This approach uses a flexible ranking method which, in addition to providing a general ranking of the alternatives, can simultaneously provide other rankings of the alternatives according to the decision-maker's points of view;
- It gives stable solutions regardless of the changes in the measuring scale for qualitative attributes or the changes in the method of formulating quantitative attributes;

Afterward, future research should cover the paradigm to engage a hierarchy of experts in the decision-making process. Another area of interest can be considered as applying this approach to integrating heterogeneous information. Finally, the introduced approach in this paper can be applied in other practical cases to further illustrate its robustness and efficiency.

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9. Compliance with Ethical Standards

This paper does not contain any studies with human participants or animals performed by any of the authors.

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