RESEARCH PAPER

A New Generalization of the Gompertz Makeham Distribution: Theory and Application in Reliability

Gholamhossein Yari[*1], Rezvan Rezaei[2] & Zahra Karimi Ezmareh[3]

ABSTRACT
In this paper, a new five-parameter distribution called Marshall-Olkin Gompertz Makeham Distribution (MOGM) is proposed. This new model can be applied to the analysis of lifetime data, engineering, and actuaries. In addition, several properties of the proposed model such as mode, moment, Reyni entropy, Tsallis entropy, quantile function, and decreasing and unimodal hazard rate function were also investigated. The unknown parameters of MOGM distribution were estimated using Maximum Likelihood Estimation (MLE) and Bayes methods. Then, these methods were compared using Monte Carlo simulation and the best estimator was introduced accordingly. Finally, some other applications of the proposed model were illustrated to show its usefulness and efficiency.

1. Introduction

Gompertz distribution (G), introduced by Gompertz in 1825 ([21] and [10]), is one of the significant distributions in reliability, lifetime data analysis [15], and human mortality studies [22], especially in growth modeling and actuarial tables. Generalization of Gompertz distribution based on the idea given in [3] was proposed by El-Gohary et al. (2013) [8]. This new distribution is known as Generalized Gompertz (GG) distribution and it includes the Exponential (E), Generalized Exponential (GE), and Gompertz (G) distributions. Jafari et al. (2014) [13] proposed a new generalization of Gompertz (G) distribution called Beta-Gompertz (BG) distribution which was the result of the application of the Gompertz distribution to the Beta generator [9]).

Back in 1860, Makeham introduced Gompertz Makham (GM) distribution as a different version of G distribution [16] which was more compatible than G for modeling lifetime data

([4], [19], etc.). Moreover, Jorda [14] studied this family in terms of statistical properties using the Lambert W function. This function proposes a closed-form expression for the quantile function that is one of the remarkable features for GM.

Statistical theories can provide a new distribution by means of some transformations such as methods introduced by Shaw and Buckley (2009) [18]. These transformations have also been applied to introduce a new GM distribution [1], [2], [23].

This study introduces a new extended form of Gompertz Makham distribution (MOGM) by applying the Marshal-Olkin transformation. First, several statistical properties including hazard rate function, Laplace transform of a probability density function, Renyi and Tsallis entropies, residual life, reversed residual life functions, and the density of order statistics are carefully investigated. Then, in Section 2, the maximum likelihood and Bayes methods are presented to estimate the parameters of MOGM. In Section 3, by applying the Monte Carlo simulation, a sample random variable is generated from MOGM. In addition, those two methods mentioned in Section 2 are compared to find out which method is more compatible with the estimation of unknown parameters. Finally, to illustrate how this new distribution functions successfully, a real data set is used for comparing MOGM with some other well-known

---

Corresponding author: *Gholamhossein Yari*
yari@iust.ac.ir

1. *Assistant Professor, Department of Statistics, School of Mathematics, Iran University of Science and Technology, Tehran, Iran.*
2. *Ph.D. student, Department of Statistics, School of Mathematics, Iran University of Science and Technology, Tehran, Iran.*
3. *Ph.D. student, Department of Statistics, School of Mathematics, Iran University of Science and Technology, Tehran, Iran.*

distributions. In the following section, some definitions are presented.

Definition 1.1. *The Cumulative Distribution Function (CDF) of the Gompertz Makeham distribution shown in* $GM(\alpha_1,\alpha_2,\beta,\lambda)$ *is ([16]):*

$$F(x;\alpha_1,\alpha_2,\beta,\lambda)=1-e^{(-\alpha_1 x^\lambda-\frac{\alpha_2}{\beta}(1-e^{\beta x^\lambda}))},x,\alpha_1,\alpha_2,\beta,\lambda>0. \quad (1.1)$$

Definition 1.2. *The Marshal-Olkin method of introducing a parameter into the family of distributions is ([20]):*

$$\bar{G}(x,,p)=\frac{p\bar{F}(x)}{1-\bar{p}\bar{F}(x)},-\infty<x<\infty,0<p<\infty, \quad (1.2)$$

where $\bar{F}(x)=1-F$ *is a survival function and* $\bar{p}=1-p$.

Definition 1.3. *If* X *is a random variable with an absolutely continuous cdf* $F(x)$, *the probability density function (pdf)* $f(x)$, *and support* S_x; *then, Renyi and Tsallis entropies of the random variable* X *are defined by ([6])*

$$H_R(X)=\frac{1}{1-r}\log\int_{S_x}f^r(x)dx, \quad (1.3)$$

$$H_T(X)=-\frac{1}{1-r}(1-\int_{S_x}f^r(x)dx), \quad (1.4)$$

for all $r>0,(r\neq 1)$.

Definition 1.4. *If* $X_{i:n}$ *is the* i^{th} *order statistic of* n *random sample of distribution with* $f(x)$ *and* $F(x)$, *its pdf is calculated as ([17]):*

$$f_{i:n}(x)=\frac{f(x)F^{i-1}(x)(1-F(x))^{n-i}}{\beta(i,n-i+1)},x>0, \quad (1.5)$$

where β *is the beta function [5].*

Definition 1.5. *The residual life is the period from time* t *to the time of failure and defined by the conditional random variable* $R_{(t)}:=X-t|X>t,t>0$; *therefore, for* $R(t)$, *we have:*

$$S_{R_{(t)}}=\frac{1-F(x+t)}{1-F(t)} \quad (1.6)$$

and the reversed residual life is the time elapsed from the failure of a given component with the life of $X\leq t$ *and is defined as the conditional random variable* $\bar{R}(t):=t-X|X\leq t,t\geq 0$; *therefore, for* $R(t)$, *we have:*

$$S_{\bar{R}_{(t)}}=\frac{F(t-x)}{F(t)}, \quad (1.7)$$

where F *is cdf of* X *[2].*

1.1. The proposed model: marshall-olkin gompertz makeham distribution

Now, by substituting Eq. (1.1) into Eq. (1.2), the cdf, generalized pdf, and hazard rate function (h) of Marshall-Olkin Gompertz Makeham presented in this paper with $GM(\alpha_1,\alpha_2,\beta,\lambda)$ are as follows:

$$F(x;\alpha_1,\alpha_2,\beta,\lambda,p)=\frac{1-e^{(-\alpha_1 x^\lambda-\frac{\alpha_2}{\beta}(1-e^{\beta x^\lambda}))}}{1-\bar{p}e^{(-\alpha_1 x^\lambda-\frac{\alpha_2}{\beta}(1-e^{\beta x^\lambda}))}}, \quad (1.8)$$

$$h(x;\alpha_1,\alpha_2,\beta,\lambda,p)=\frac{p(\alpha_1+\alpha_2 e^{\beta x^\lambda})e^{(-\alpha_1 x^\lambda-\frac{\alpha_2}{\beta}(1-e^{\beta x^\lambda}))}}{(1-\bar{p}e^{(-\alpha_1 x^\lambda-\frac{\alpha_2}{\beta}(1-e^{\beta x^\lambda}))})^2}, \quad (1.9)$$

$$h(x;\alpha_1,\alpha_2,\beta,\lambda,p)=\frac{(\alpha_1+\alpha_2 e^{\beta x^\lambda})}{1-\bar{p}e^{(-\alpha_1 x^\lambda-\frac{\alpha_2}{\beta}(1-e^{\beta x^\lambda}))}}, \quad (1.10)$$

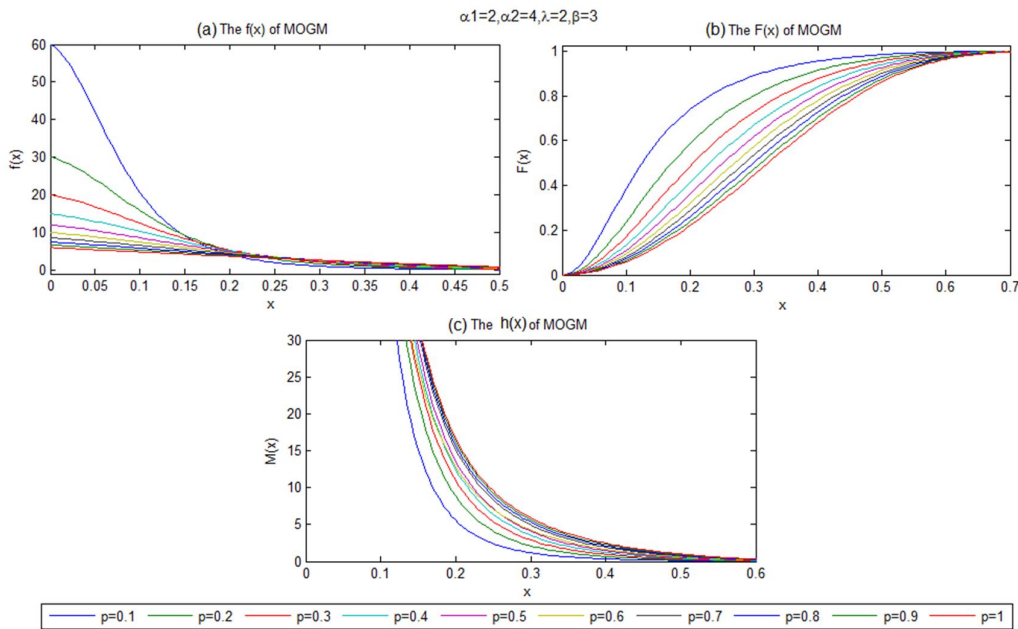**Fig. 1.** $F(x)$, $f(x)$, $h(x)$ **of MOGM distribution**

for $p = 0.1, 0.2,...$ for all $x, \alpha_1, \alpha_2, \beta, \lambda, p > 0$.

Figures 1 (a,b,c) show $f(x)$, $F(x)$, and $h(x)$ functions of the MOGM distribution for some values of p, respectively. As seen in Figure (1), $f(x)$ is the decreasing function and unimodal for

every p, and while p is the increasing function, which looks like a uniform density one. In the following, given that the random variable of X has a distribution with PDF Eq. (1.9), we have:

• The mode is obtained by the following solution:

$$\frac{\partial f(x; \alpha_1, \alpha_2, \beta, p)}{\partial x} = e^{(-\alpha_1 x^\lambda + \frac{\alpha_2}{\beta}(1 - e^{\beta x}))} \{\alpha_2 \beta e^{\beta x} - (\alpha_1 + \alpha_2 e^{\beta x})^2 (1 + 2\overline{p} e^{(-\alpha_1 x - \frac{\alpha_2}{\beta}(1 - e^{\beta x}))})\} = 0$$

with respect to x.

• The Laplace transform of PDF and the central moment:

$$E(e^{-tX}) = \lambda p \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \{(\frac{\binom{-2}{k}\binom{t}{m}(\alpha_2(k+1))^l (m\beta)^j (\alpha_1(k+1))^i (-1)^{2l+i-m}}{\beta^l l! j! i!})$$

$$* (\frac{\alpha_1 \Gamma((j+i+1)\lambda)}{t^{((j+i+1)\lambda)}} + \sum_{n=0}^{\infty} \frac{\alpha_2 \beta^n \Gamma((j+i+1+n)\lambda)}{n! t^{((j+i+1+n)\lambda)}})\}$$

and

$$E(X^r) = \frac{\partial}{\partial t} E(e^{-tX})\Big|_{t=0}$$

$$= \lambda p \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \{(\frac{\binom{-2}{k}\binom{l}{m}(\alpha_2(k+1))^l (m\beta)^j (\alpha_1(k+1))^i (-1)^{2l+i-m+1}}{\beta^l l! j! i!})$$

$$* (\frac{(j+i+1)\lambda \alpha_1 \Gamma((j+i+1)\lambda)}{t^{((j+i+1)\lambda)+1}} + \sum_{n=0}^{\infty} \frac{\alpha_2(j+i+1+n)\lambda \beta^n \Gamma((j+i+1+n)\lambda)}{n! t^{((j+i+1+n)\lambda)+1}})\}.$$

• The Reyni entropy is driven by substituting Eq. (1.9) in Eq. (1.3):

$$H_T(X) = \frac{1}{1-r}\log\{(\lambda p)^r \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{r}\sum_{l=0}^{i}\sum_{h=0}^{\infty}\sum_{m=0}^{h}\binom{h}{m}\binom{r}{j}\binom{i}{l}\binom{-2r}{k}\frac{\alpha_1^i \alpha_2^{r-j+i+h} r^i k^h (p-1)^k}{\lambda i! h! \beta^{i+h}}$$

$$* \frac{\Gamma(1+(1-\frac{1}{\lambda})(r-1))}{[\beta(m+r-j+l)-\alpha_1(r+k)]^{(1-\frac{1}{\lambda})(r-1)+1}}\}.$$

• Te Tsallis entropy is also derived by substituting Eq. (1.9) in Eq. (1.4):

$$H_T(X) = (\lambda p)^r \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{r}\sum_{l=0}^{i}\sum_{h=0}^{\infty}\sum_{m=0}^{h}\binom{h}{m}\binom{r}{j}\binom{i}{l}\binom{-2r}{k}\frac{\alpha_1^i \alpha_2^{r-j+i+h} r^i k^h (p-1)^k x^{r(\lambda-1)}}{i! h! \beta^{i+h}}$$

$$* e^{x^\lambda(\beta(m+r-j+l)-\alpha_1(r+k))}.$$

• The survival function and reversed survival function of the residual life time, $R(t)$ and $\overline{R}(t)$, for the MOGM distribution are obtained by substituting Eq. (1.1) in Eqs. (1.6) and (1.7), respectively:

$$S_{R_{(t)}}(x) = \frac{e^{-\alpha_1 t - \frac{\alpha_2}{\beta}e^{\beta t}}(1-\overline{p}e^{-\alpha_1 x^\lambda - \frac{\alpha_2}{\beta}(1-e^{\beta x^\lambda})})}{1-\overline{p}e^{-\alpha_1(x+t)^\lambda - \frac{\alpha_2}{\beta}(1-e^{\beta(x+t)^\lambda})}},$$

$$S_{\overline{R}_{(t)}}(x) = \frac{e^{\alpha_1 t - \frac{\alpha_2}{\beta}e^{\beta t}}(1-\overline{p}e^{-\alpha_1 x^\lambda - \frac{\alpha_2}{\beta}(1-e^{\beta x^\lambda})})}{1-\overline{p}e^{-\alpha_1(x-t)^\lambda - \frac{\alpha_2}{\beta}(1-e^{\beta(x-t)^\lambda})}}.$$

• The quantile function of the MOGM: the following theorem is presented with respect to this case.
Theorem 1.6. *The quantile function of the MOGM by applying definition of W0 to [14] is*

$$Q(u) = \frac{\alpha_2}{\beta} - \log(\frac{1-u}{1-u\overline{p}}) - \frac{1}{\beta}W_0(\alpha_2 e^{\alpha_2 - \beta\log(\frac{1-u}{1-u\overline{p}})}), \tag{1.11}$$

Where

$$W_0(z) \approx (1+\varepsilon)(\frac{6z}{5\log(\frac{12}{5}(\frac{z}{\log(1+\frac{12}{k}z)}))}) - \varepsilon\log(\frac{2z}{\log(1+2z)}), z \geq 0, \varepsilon = 0.4586887.$$

Proof. By setting $u = \dfrac{1-e^{-\alpha_1 x - \frac{\alpha_2}{\beta}(1-e^{\beta x})}}{1-\overline{p}e^{-\alpha_1 x - \frac{\alpha_2}{\beta}(1-e^{\beta x})}}$, we have:

$$\alpha_1 x + \frac{\alpha_2}{\beta}e^{\beta x} = \frac{\alpha_2}{\beta} - \log(\frac{1-u}{1-u\overline{p}})$$

$$\Rightarrow x = \frac{\alpha_2}{\beta} - \log(\frac{1-u}{1-u\overline{p}}) - \frac{1}{\beta}W_0(\alpha_2 e^{\alpha_2 - \beta\log(\frac{1-u}{1-u\overline{p}})}). \tag{1.12}$$

In the following, the sub-models and the order statistics of MOGM are discussed.

1.2. Sub-models

The MOGM distribution consists of several important sub-models that are widely used in lifetime modeling. These sub-models are:

• When $p=1$, we have the Gompertz Makeham distribution (GM).

• When $p=1$, $\alpha_1 = 0$, we have the Gompertz distribution (G).

• When $p=1$, $\alpha_2 = 0$ and $\lambda = 1$, we have the exponential distribution (E).

• When $p=1$, $\alpha_2 = 0$ and $\alpha_1 = 1$, we have the Weibull $(\lambda,1)$ distribution (Weib).

1.3. The order statistics of MOGM

By substituting Eq. (1.9) in Eq. (1.5), we have:
• The PDF of order statistic of MOGM is given in the following:

$$f_{i:n}(x) = \frac{\lambda(\alpha_1 + \alpha_2 e^{\beta x}) e^{-\alpha_1 x + \frac{\alpha_2}{\beta}(1-e^{\beta x})}}{\beta(i, n-i+1)(1-\overline{p}e^{-\alpha_1 x + \frac{\alpha_2}{\beta}(1-e^{\beta x})})^2} \left(\frac{1-e^{-\alpha_1 x + \frac{\alpha_2}{\beta}(1-e^{\beta x})}}{1-\overline{p}e^{-\alpha_1 x + \frac{\alpha_2}{\beta}(1-e^{\beta x})}}\right)^{i-1} \left(\frac{pe^{-\alpha_1 x + \frac{\alpha_2}{\beta}(1-e^{\beta x})}}{1-\overline{p}e^{-\alpha_1 x + \frac{\alpha_2}{\beta}(1-e^{\beta x})}}\right)^{n-i}. \quad (1.13)$$

Now, by using the expansions of $(1-e^{-\alpha_1 x + \frac{\alpha_2}{\beta}(1-e^{\beta x})})^{i-1}$, $(1-\overline{p}e^{-\alpha_1 x + \frac{\alpha_2}{\beta}(1-e^{\beta x})})^{n-3}$, $(e^{-\alpha_1 x + \frac{\alpha_2}{\beta}(1-e^{\beta x})})^{n+k+j+1-i}$, $(-\alpha_1 x + \frac{\alpha_2}{\beta}(1-e^{\beta x}))^m$, $(1-e^{\beta x})^{m-p}$, and $e^{\beta(m-p-d)x}$, we have

$$f_{i:n}(x) = \sum_{j=0}^{i-1}\sum_{k=0}^{n-3}\sum_{m=0}^{\infty}\sum_{c=0}^{m}\sum_{d=0}^{m-c}\sum_{g=0}^{\infty} \frac{\binom{i-1}{j}\binom{m}{c}\binom{m-c}{d} p^{n+1-i}(-1)^{j+k+m-d}\overline{p}^k \alpha_1^c (\frac{\alpha_2}{\beta})^{m-c} x^{c+g}}{\beta(i, n-i+1)m! g!}. \quad (1.14)$$

• The Laplace transform of pdf and the central moment by using the expansion in the above item is given below:

$$E(e^{-tX}) = \int_0^\infty e^{-tX} f_{i:n}(x)dx$$

$$= \sum_{j=0}^{i-1}\sum_{k=0}^{n-3}\sum_{m=0}^{\infty}\sum_{c=0}^{m}\sum_{d=0}^{m-c}\sum_{g=0}^{\infty} \frac{\binom{i-1}{j}\binom{m}{c}\binom{m-c}{d} p^{n+1-i}(-1)^{j+k+m-d}\overline{p}^k \alpha_1^c (\frac{\alpha_2}{\beta})^{m-c}\Gamma(c+g+1)}{\beta(i, n-i+1)m! g! t^{c+g+1}}.$$

2. Estimation of Parameters

In this section, two methods for estimating the parameters of MOGM distribution are presented. Then, in the next section, the methods are compared using the Monte Carlo simulation [12].

2.1. Maximum likelihood estimator (MLE)

In this subsection, consider a random sample of $X_1, X_2,..., X_n$ in Eq. (1.9). To simplify the calculation, we set $\lambda = 1$.

Now, the parameters are estimated by MLE method. The logarithm of the likelihood function is as follows:

$$L_n = \log L(x, \Theta) = n\log p + \sum_{i=1}^{n}\log(\alpha_1 + \alpha_2 e^{\beta x_i}) - \alpha_1 \sum_{i=1}^{n} x_i + \frac{\alpha_2}{\beta}\sum_{i=1}^{n}(1-e^{\beta x_i}) - 2\sum_{i=1}^{n}\log(A),$$

where $A = 1 - \bar{p}e^{-\alpha_1 x_i + \frac{\alpha_2}{\beta}(1 - e^{\beta x^\lambda})}$ and $C = \frac{a_2}{b^2}(1 - e^{bx_i}) + \frac{a_2}{b}x_i e^{bx_i}$.

The maximum likelihood estimation of Θ denoted by Θ is obtained using a set of equations shown below:

$$\frac{\partial L_n}{\partial \alpha_1} = \sum_{i=1}^{n}[\frac{1}{\alpha_1 + \alpha_2 e^{\beta x_i}} + \frac{2(1-A)}{A}x_i] = 0, \tag{2.1}$$

$$\frac{\partial L_n}{\partial \alpha_2} = \sum_{i=1}^{n}[\frac{e^{\beta x_i}}{\alpha_1 + \alpha_2 e^{\beta x_i}} + \frac{2(1-A)}{\beta A}(1 - e^{\beta x_i})] + \frac{1}{\beta}\sum_{i=1}^{n}(1 - e^{\beta x_i}) = 0, \tag{2.2}$$

$$\frac{\partial L_n}{\partial \alpha_2} = \sum_{i=1}^{n}[\frac{\alpha_2 x_i e^{\beta x_i}}{\alpha_1 + \alpha_2 e^{\beta x_i}} - C(1 + 2\frac{1-A}{A})] = 0, \tag{2.3}$$

$$\frac{\partial L_n}{\partial p} = \frac{n}{p} - \frac{2}{\bar{p}}\sum_{i=1}^{n}\frac{1-A}{A} = 0. \tag{2.4}$$

Of note, in each of the above equations, other parameters were considered as known. Regarded as closed forms, the estimates must be obtained by numerical methods such as Monte Carlo simulation. According to Gupta and Kundu (1999) [11], in the large sample size, Θ has a univariate normal distribution with a mean vector of $\hat{\Theta}$ and covariance matrix of $I^{-1}(\hat{\Theta})$, where:

$$I^{-1}(\hat{\theta}) = \begin{bmatrix} I_{\alpha_1,\alpha_1} & I_{\alpha_1,\alpha_2} & I_{\alpha_1,\beta} & I_{\alpha_1,p} \\ & I_{\alpha_2,\alpha_2} & I_{\alpha_2,\beta} & I_{\alpha_2,p} \\ & & I_{\beta,\beta} & I_{\beta,p} \\ & & & I_{p,p} \end{bmatrix}^{-1} = \begin{bmatrix} \mathrm{var}(\hat{\alpha}_1) & \mathrm{cov}(\hat{\alpha}_1,\hat{\alpha}_2) & \mathrm{cov}(\hat{\alpha}_1,\hat{\beta}) & \mathrm{cov}(\hat{\alpha}_1,\hat{p}) \\ & \mathrm{var}(\hat{\alpha}_2) & \mathrm{cov}(\hat{\alpha}_2,\hat{\beta}) & \mathrm{cov}(\hat{\alpha}_2,\hat{p}) \\ & & \mathrm{var}(\hat{\beta}) & \mathrm{cov}(\hat{\beta},\hat{p}) \\ & & & \mathrm{var}(\hat{p}) \end{bmatrix}$$

and

$$\frac{\partial^2 L_n}{\partial \alpha_1^2} = \sum_{i=1}^{n}[\frac{-1}{(\alpha_1 + \alpha_2 e^{\beta x_i})^2} + \frac{2x_i^2}{A^2} - \frac{2x_i^2}{A}],$$

$$\frac{\partial^2 L_n}{\partial \alpha_2^2} = \sum_{i=1}^{n}[\frac{-e^{2\beta x_i}}{(\alpha_1 + \alpha_2 e^{\beta x_i})^2} + \frac{2(1 - e^{\beta x_i})^2}{\beta^2}\frac{1-A}{A^2}],$$

$$\frac{\partial^2 L_n}{\partial \beta^2} = \sum_{i=1}^{n}[\frac{\alpha_1 \alpha_2 x_i^2 2\alpha_2(1 - e^{\beta x_i})}{(\alpha_1 + \alpha_2 e^{\beta x_i})^2 \beta^3} + \frac{\alpha_2(2 - \beta)x_i e^{\beta x_i}}{A^2} - \frac{2}{A^2}(1 - A)C(1 + \frac{2}{\beta} - \frac{\alpha_2}{\beta C}x_i^2 e^{\beta x_i} + C(1 - A))],$$

$$\frac{\partial^2 L_n}{\partial p^2} = -\frac{n}{p^2} - \frac{2}{\bar{p}^2}\sum_{i=1}^{n}(\frac{1-A}{A})^2,$$

$$\frac{\partial^2 L_n}{\partial \alpha_1 \partial \alpha_2} = \sum_{i=1}^{n}[\frac{e^{\beta x_i}}{(\alpha_1 + \alpha_2 e^{\beta x_i})^2} + \frac{2x_i(1 - e^{\beta x_i})(1-A)}{\beta A^2}],$$

$$\frac{\partial^2 L_n}{\partial \alpha_1 \partial \beta} = \sum_{i=1}^{n}[\frac{-\alpha_2 x_i e^{\beta x_i}}{\alpha_1 + \alpha_2 e^{\beta x_i}} + \frac{2(1-A)}{A^2}C],$$

$$\frac{\partial^2 L_n}{\partial \alpha_1 \partial p} = \sum_{i=1}^{n}2x_i(\frac{1-A}{\bar{p}}),$$

$$\frac{\partial^2 L_n}{\partial \alpha_1 \partial \beta} = \sum_{i=1}^{n}[\frac{-\alpha_2 x_i e^{\beta x_i}}{(\alpha_1 + \alpha_2 e^{\beta x_i})^2} - \frac{C}{\alpha_2} + \frac{1-A}{\bar{p}}(-2\bar{p}x_i e^{\beta x_i} - \bar{p}\beta C(A - (1-A)e^{-\alpha_1 x_i}))],$$

$$\frac{\partial^2 L_n}{\partial \alpha_2 \partial p} = -\frac{2}{\bar{p}} \sum_{i=1}^{n} [(1 - e^{\beta x_i})(\frac{1-A}{A^2})],$$

$$\frac{\partial^2 L_n}{\partial \beta \partial p} = \frac{2\alpha_2}{\beta \bar{p}} \sum_{i=1}^{n} [\frac{1-A}{A^2}(\frac{1-e^{\beta x_i}}{\beta} + x_i e^{\beta x_i})].$$

The approximate $(1-\varepsilon)100\%$ confidence intervals for the parameters $\alpha_1, \alpha_2, \beta, p$ and λ are determined, respectively, in the following:

$$\hat{\alpha}_1 \pm Z_{\frac{\xi}{2}}\sqrt{\frac{Var(\hat{\alpha}_1)}{n}}, \hat{\alpha}_2 \pm Z_{\frac{\xi}{2}}\sqrt{\frac{Var(\hat{\alpha}_2)}{n}}, \hat{\beta} \pm Z_{\frac{\xi}{2}}\sqrt{\frac{Var(\hat{\beta})}{n}}, \hat{p} \pm Z_{\frac{\xi}{2}}\sqrt{\frac{Var(\hat{p})}{n}}, \hat{\lambda} \pm Z_{\frac{\xi}{2}}\sqrt{\frac{Var(\hat{\lambda})}{n}},$$

where $Var(\hat{\alpha}_1), Var(\hat{\alpha}_2), Var(\hat{\beta}), Var(\hat{p})$ and $Var(\hat{\lambda})$ are given by the diagonal elements of $I^{-1}(\hat{\Theta})$ and Z_ξ is the upper 2ξ percentile of the standard normal distribution.

2.2. Bayes estimator (BE)

In estimating the parameters of MOGM distribution by the Bayes method, it was assumed that the previous information of $\alpha_1, \alpha_2, \beta$ and p

was independent of each other, thus $\pi(\alpha_1, \alpha_2, \beta, p) = \pi(\alpha_1)\pi(\alpha_2)\pi(\beta)\pi(p)$. In this method, two prior distributions, i.e., exponential and Weibull, via the Monte Carlo simulation were employed. It is not easy to calculate the denominators of the posterior distributions since they have four integrals; therefore, importance sampling method was employed [24]. In the relevant section, the results are presented.
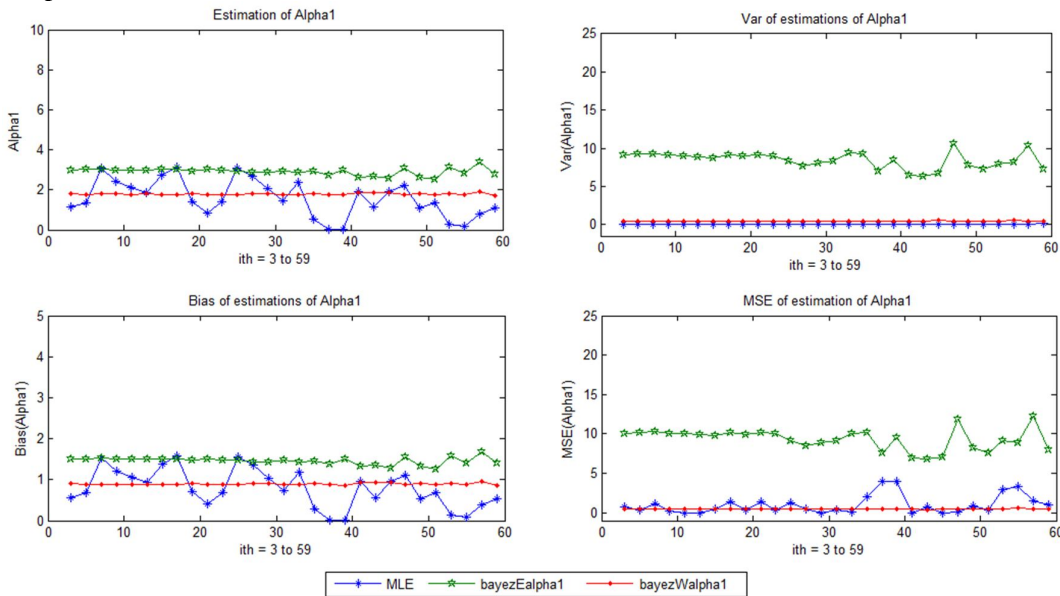

Fig. 2. Comparison of α_1 estimators

3. Simulation Studies

As observed in the results in Section (2), the estimates obtained through MLE (Equations (2.1), (2.2), (2.3), (2.4)) and BE methods do not have a closed form. This is why suitable numerical methods were employed to achieve these estimators. To this end, in MATLAB software, random numbers from the

MOGM distribution with the parameters $\alpha_{1true} = 2$, $\alpha_{2true} = 5$, $\beta_{true} = 3$, and $p_{true} = 0.6$ *were initially generated.* Then, the parameters of this distribution were estimated using MLE and BE methods. This process was repeated for $j = 1000$ times. Through these repetitions, a set of estimates, i.e., α_{1true}, α_{2true}, β_{true}, and p_{true}

was achieved. To compare these two methods, the mean of $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\beta}$ and $\overline{p}$ was used and it

could be calculated as follows:

$$\overline{\alpha}_1 = \frac{1}{j}\sum_{k=1}^{j}\alpha_{1k}, \overline{\alpha}_2 = \frac{1}{j}\sum_{k=1}^{j}\alpha_{2k}, \overline{\beta} = \frac{1}{j}\sum_{k=1}^{j}\beta_k, \text{ and } \overline{p} = \frac{1}{j}\sum_{k=1}^{j}p_k$$

where K index represents the estimate in the K_{th} repetition. It should be noted that in the BE method, two prior distributions, i.e., exponential and Weibull, were employed and for both prior distributions in the denominator of posterior distributions, a four-fold integral, which is not a simple calculation, is required. In order to resolve this problem, the "importance sampling" method

is used. To investigate the effect of sample size on estimates, the above process is repeated for samples of sizes of $n = 3, 5, 7, ..., 59$. To compare these two methods, the criteria including Biased (Bias), Variance (Var), and Mean of Square Error (MSE) are used. The summarized results are shown in Figures (2), (3), (4), and (5).
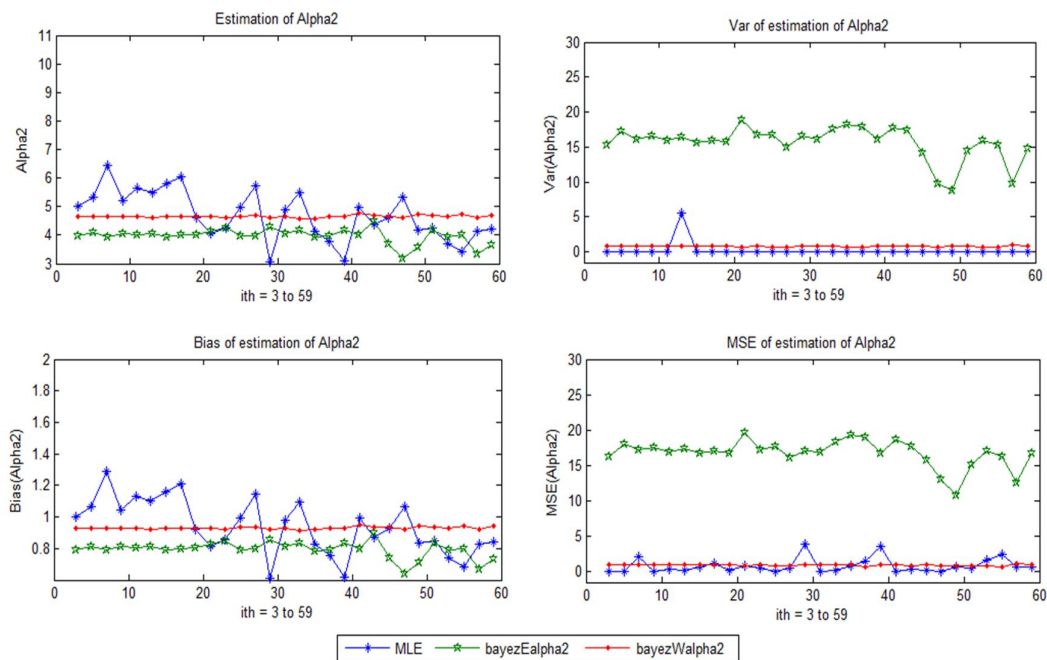

Fig. 3. Comparison of α_2 estimators

In the following, the results of estimating the parameters are analyzed. Figures (2), (3), and (4) show that the BE method, especially with the prior Weibull, provides estimates that are close to real value. In terms of comparative criteria (var, bias, and MSE), the BE method is also appropriate for the prior Weibull. Further, with an increase in the sample size in MLE method, the estimates will be very close to the real values and their fluctuations will decrease; in addition, the MLE method is quite suitable in terms of comparative criteria. According to Figure (5), to estimate p, MLE method is particularly suitable for large sample sizes and other comparative criteria confirm it. The BE method with an

exponential prior can also be considered as the second option to estimate p.

4. Application with A Real Data Set

The MOGM distribution can be a good lifetime model. In order to achieve this purpose, real data is used. Also, to understand the superiority of this model over other known models, indexes AIC, CAIC, and BIC are compared with other models including the exponential, Generalized exponential, Gompertz, Generalized Gompertz, and Beta Gompertz distributions (ED, GE, G, GG, BG) (see [7]). Consider the data obtained from Aarset. The data represent the lifetimes of 50 devices and also, possess a bathtub-shaped failure rate property.
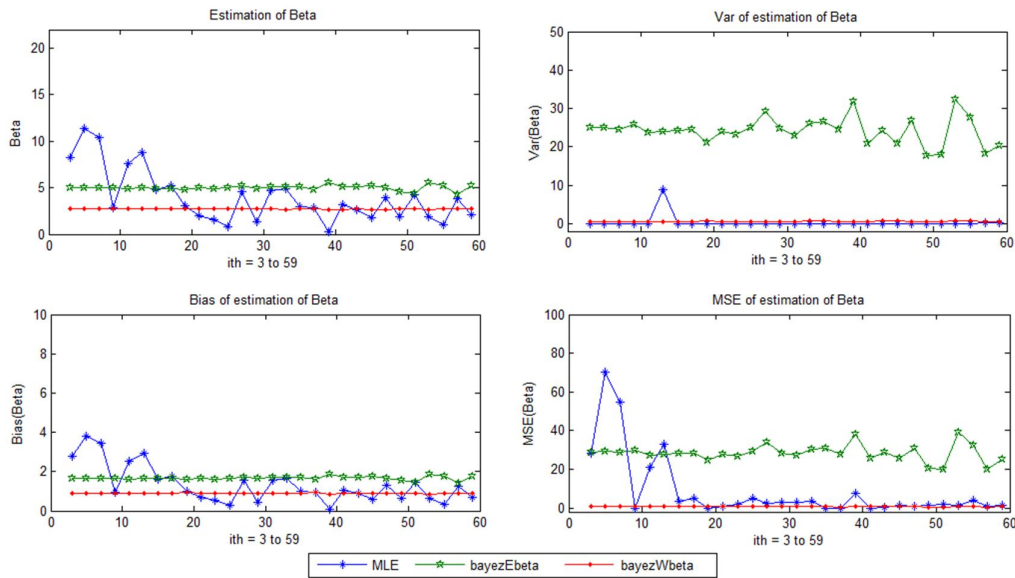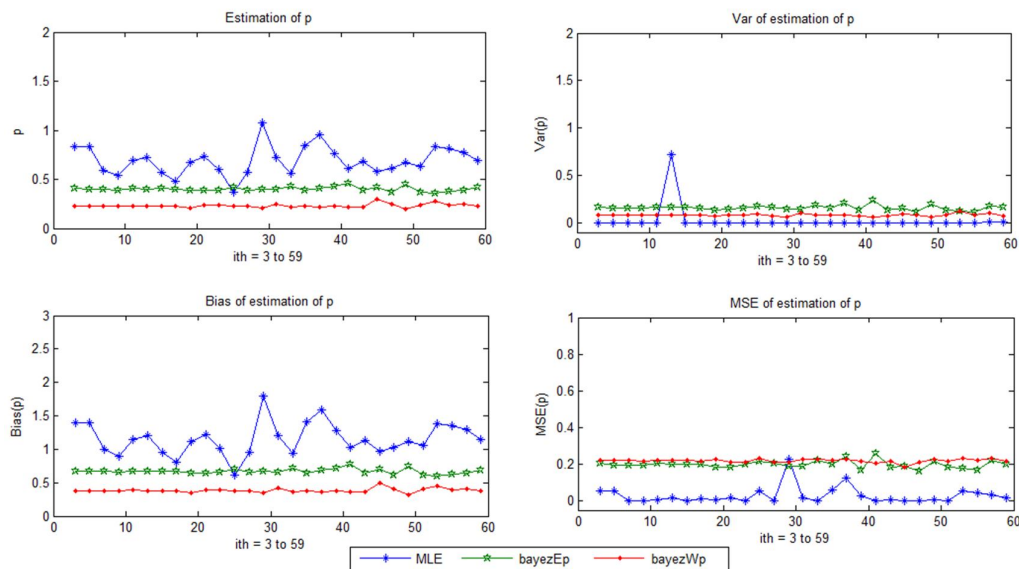
Fig. 4. Comparison of β estimators

Fig. 5. Comparison of *p* estimators

The data include the following values ([7]):
0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

Table (2) lists the values of the maximum likelihood estimation of distribution parameters obtained by different methods. Table (3) includes Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), and corresponding Bayesian Information Criterion (BIC) for the models in Table (2). The formulas of these criteria are summarized in Table (1), where *L* is the log likelihood function, q is the

number of parameters, and *n* is sample size. The AIC, BIC and, CAIC are the measures of the goodness of fit tests for an estimated statistical model. The model with smaller values of AIC, BIC, and CAIC is the preferred model.

The required numerical evaluations are implemented using the *R* software. As seen in Table (3), AIC, BIC, CAIC values of the MOGM distribution are lower than those of other distributions. Therefore, it can be concluded that the MOGM is a good distribution for fitting this dataset. Some other results can also be obtained by examining Figures (6), (7), and (8). These shapes also confirm that MOGM distribution is

more appropriate than other distributions for fitting data.

5. Conclusion

The present study introduced a new distribution based on Gompertz Makeham distribution and Marshall-Olkin transformation and investigated some of its basic statistical and mathematical properties. In addition, unknown parameters were estimated using MLE and BE methods.

According to results of simulations, for α_1, α_2,

and β, the BE method, especially with the prior Weibull for the low sample size and the MLE method for the large sample size is recommended. Further, to estimate *p*, the MLE method for large sample sizes and the BE method with the exponential prior for small sample sizes are appropriate. Moreover, by using a set of real data, the MOGM distribution can be a good lifetime model compared to many known distributions.

Tab. 1. Goodness-of-fit criteria

Goodness-of-fit	Equation
Akaike information criterion	$AIC = -2L + 2q$
Bayesian information criterion	$BIC = -2L + q\log(n)$
Consistent Akaike information criteria	$CAIC = -2L + \frac{2qn}{n-q-1}$

Tab. 2. MLEs for real data.

Distribution	$\hat{\alpha_1}$	$\hat{\alpha_2}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{p}$	$-L$
E	0.0219	–	–	–	–	241.0896
GE	0.0212	0.9012	–	–	–	240.3855
G	–	–	0.00970	0.0203	–	235.3308
GG	–	0.2625	0.00010	0.0828	–	222.2441
BG	0.2158	0.2467	0.00030	0.0882	–	220.6714
OGE-G	0.0400	0.1940	0.000345	0.0780	–	215.9735
MOGM	9×10^{-5}	5×10^{-6}	9×10^{-6}	1.0005	27×10^{-4}	200.065

Tab. 3. Goodness-of-fit statistics corresponding to data set.

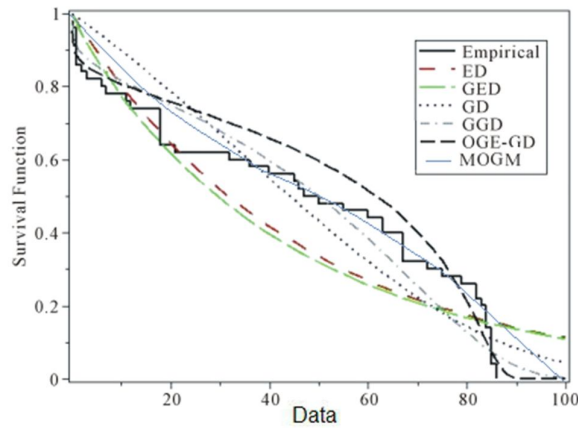| Distribution	AIC	CAIC	BIC
E	484.1792	484.2625	486.0912
GE	484.7710	485.0264	488.5951
G	474.6617	475.1834	482.3977
GG	450.4881	451.0099	456.2242
BG	449.3437	450.2326	456.9918
OGE-G	423.9470	424.8359	447.5951
MOGM	410.1301	411.4937	419.6902

Fig. 6. Graph of survival function of compared distributions
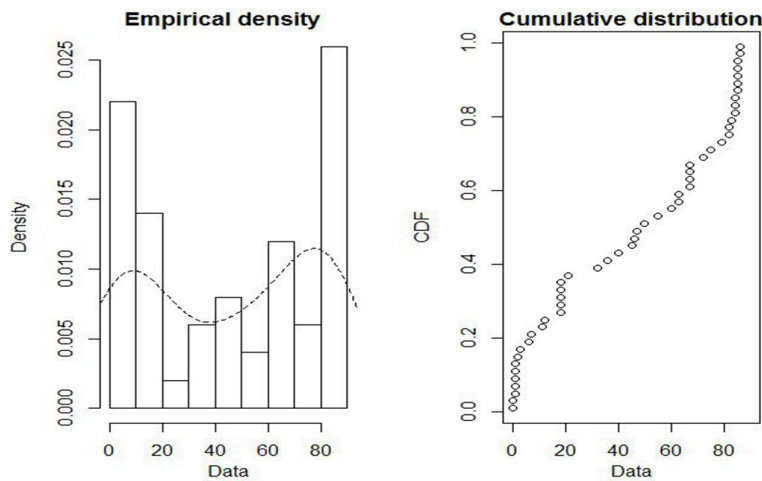

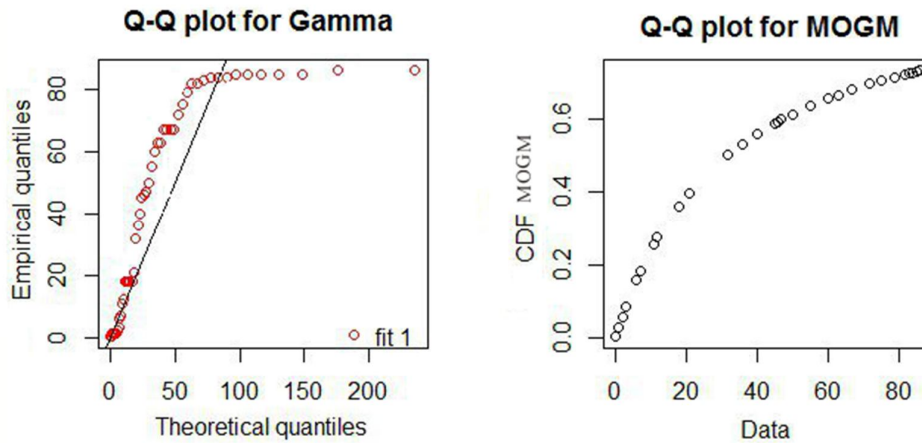**Fig. 7. Graph of emprical density and cdf of MOGM distribution corresponds to data set**

Fig. 8. *Q - Q* **plot for MOGM and Gamma distribution corresponding to the data set**

References

[1] Adebisi Ade, O., Olayode, F., Bamidele, A., *Performance Rating of the Exponentiated Generalized Gompertz Makeham Distribution: An Analytical Approach*, American Journal of Theoretical and Applied Statistics, Vol. 6, No. 5, (2017), pp. 228-235. Doi: 10.11648/j.ajtas.20170605.12.

[2] AbdEl-Bar, A., *An extended Gompertz-Makeham distribution with application to lifetime data*, Communications in Statistics -

Simulation and Computation, 2017, Doi:10.1080/03610918.2017.1348517.

[3] Abu-Zinadah, H.H. and Aloufi, A.S., *Some Characterizations of the Exponentiated Gompertz Distribution*, International Mathematical Forum, Vol. 9, (2014), pp. 1427-1439.

[4] Bailey, R. C., *Limiting forms of the Makeham model and their use for survival analysis of transplant studies*, Biometrics Vol. 34, (1978), pp. 725-725.

[5] Casella, G. and Berger, R.L., Statistical Information, 2nd ed., (2007), Duxbury Press.

[6] Cover, T.M. and Thomas, J.A., Elements of information theory, 2nd ed., Wiley, New york, (2006).

[7] El-Damcese1, M. A., Mustafa, A., El-Desouky, B. S. and Mustafa, M. E., *The Odd Generalized Exponential Gompertz Distribution*, Applied Mathematics, Vol. 6, (2015), pp. 2340-2353.

[8] El-Gohary, A., Alshamrani, A. and Al-Otaibi, A.N., *The Generalized Gompertz Distribution*, Applied Mathematical Modelling, Vol. 37, (2013), pp. 13-24. http://dx.doi.org/10.1016/j.apm.2011.05.017.

[9] Eugene, N., Lee, C. and Famoye, F., *Beta-Normal Distribution and Its Applications*, Communications in Statistics-Theory and Methods, Vol. 31, (2002), pp. 497-512. http://dx.doi.org/10.1081/STA-120003130.

[10] Gavrilov, L., Govrilova, N., The biology of life span: a quantitative approach, Harwood, Chur, (1991).

[11] Gupta R.D., Kundu D., Theory and Methods: Generalized exponential distributions, Vol. 41, No. 2, (1999), pp. 173-188.

[12] J. Lemontea, Artur. And M.Cordeiro, G., *The exponentiated generalized inverse Gaussian distribution*, Statistics and Probability Letters, Vol. 81, (2011), pp. 506-517.

[13] Jafari, A.A., Tahmasebi, S. and Alizadeh, M., *The Beta-Gompertz Distribution*,

Revista Colombiana de Estadstica, Vol. 37, (2014), pp. 141-158. http://dx.doi.org/10.15446/rce.v37n1.44363.

[14] Jodra, P., *A closed form expression for the quantile functions of the Gompertz Makeham distribution*, Mathematics and Computers in Simulation, Vol. 79, No. 10, (2009), pp. 3069-3075. Doi:10.1016/j.matcom.2009.02.002.

[15] Johnson, N. L., Kotz, S. and Balakrishnan, N., Continuous Univariate Distributions, Vol. 2, 2nd ed, John Wiley and Sons, New York, (1995).

[16] Makeham, W. M., *On the Law of Mortality and the Construction of Annuity Tables*, J. Inst. Actuaries and Assurance. Mag. Vol. 8, (1860), pp. 301-310.

[17] Mood, A. M., F. A. Graybill, and D. C. Boes. *Introduction to the Theory of Statistics. McGraw Hill Book Company*, Inc., New York (1963).

[18] Shaw, W. T. and Buckley, I. R., *The alchemy of probability distribution: beyond Gram- Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map*, Technical report, (2009).

[19] Teimouri, M. and Gupta, A. K., *Estimation methods for the Gompertz-Makeham distribution under progressively type-I interval censoring scheme*, Natl. Acad. Sci. Lett, Vol. 35, (2012), pp. 227-235.

[20] W. Marshal, L., *A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families*, Biometrika, Vol. 84, No. 3, (1997), pp. 641-652.

[21] Wetterstrand, WH., *Parametric models for life insurance mortality data: Gompertz's law over time*, Trans Soc Actuar, Vol. 33, (1981), pp. 159-179.

[22] Willemse, W. and Koppelaar, H., *Knowledge elicitation of Gompertz law of mortality*, Scandinavian Actuarial Journal Vol. 2, (2000), pp. 168-179.

[23] Yaghoobzadeh, Sh., *A new generalization of the MarshallOlkin Gompertz distribution*, IntJSyst Assur Eng Manag, Vol. 8, (2017), pp. 1580-1587 DOI: 10.1007/s13198-017-0630-8.

[24] Rubinstein, Y.R. and Kroese, P.D., *simulation and the Monte Carlo method*, A John Wiley and Sons, Inc., Publication, 2nd ed, (2008).

Follow This Article at The Following Site:

Rezaei R, Yari G H, Karimi Ezmareh Z. A new generalization of the Gompertz Makeham distribution: theory and application in reliability. IJIEPR. 2020; 31 (3) :455-467
URL: http://ijiepr.iust.ac.ir/article-1-930-en.html