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NORMAL 6-VALENT CAYLEY GRAPHS OF ABELIAN GROUPS

Mehdi Alaeiyan

***Abstract**: We call a Cayley graph Γ = Cay (G, S) normal for G, if the right regular representation R(G) of G is normal in the full automorphism group of Aut(Γ). In this paper, a classification of all non-normal Cayley graphs of finite abelian group with valency 6 was presented.*

***Keywords**: Cayley graph, normal Cayley graph, automorphism group.*

1. Introduction

Let G be a finite group, and S be a subset of G not containing the identity element 1_G. The Cayley digraph Γ=Cay(G,S) of G relative to S is defined as the graph with vertex set V(Γ) = G and edge set E(Γ) consisting of those ordered pairs (x, y) from G for which $yx^{-1} \in S$. Immediately from the definition we find that, there are three obvious facts: (1) Aut(Γ) contains the right regular representation R(G) of G and so Γ is vertex-transitive.
(2) Γ is connected if and only if G =< S>. (3) Γ is an undirected if and only if $S^{-1}= S$.
A Cayley (di)graph Γ=Cay(G,S) is called normal if the right regular representation R(G) of G is a normal subgroup of the automorphism group of Γ.
The concept of normality of Cayley (di)graphs is known to be important for the study of arc-transitive graphs and half-transitive graphs (see[1,2]). Given a finite group G, a natural problem is to determine all normal or non-normal Cayley (di)graphs of G. This problem is very difficult and is solved only for the cyclic groups of prime order by Alspach [3] and the groups of order twice a prime by Du et al. [4], while some partial answers for other groups to this problem can be found in [5-8]. Wang et al. [8] characterized all normal disconnected Cayley's graphs of finite groups. Therefore the main work to determine the normality of Cayley graphs is to determine the normality of connected Cayley graphs. In [5, 6], all non-normal Cayley graphs of abelian groups with valency at most 5 were classified. The purpose of this paper is the following main theorem.

Theorem 1.1 Let Γ = Cay (G, S) be a connected undirected Cayley graph of a finite abelian group G on S with valency 6. Then Γ is normal except when one of the following cases happens:

(1): $G = Z_2^5 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle$,
$S = \{a, b, c, abc, d, e\}$.

(2): $G = Z_2^3 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$ ($m \geq 3$),
$S = \{a, b, c, abc, d, d^{-1}\}$.

(3): $G = Z_2^2 \times Z_4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$,

$S = \{a, b, ab, c^2, c, c^{-1}\}$.

(4): $G = Z_2^4 \times Z_4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle$,
$S = \{a, b, c, d, e, e^{-1}\}$.

(5): $G = Z_2^3 \times Z_4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$
$S_1 = \{a, b, c, d^2, d, d^{-1}\}$,
$S_2 = \{a, b, ab, c, d, d^{-1}\}$, $S_3 = \{a, b, c, ad^2, d, d^{-1}\}$.

(6): $G = Z_2^2 \times Z_6 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$,
$S = \{a, b, ab, c^3, c, c^{-1}\}$.

(7): $G = Z_2^3 \times Z_6 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$,
$S = \{a, b, c, d^3, d, d^{-1}\}$.

(8): $G = Z_6 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$ ($m \geq 2$),
$S = \{a^3, b^m, a, a^{-1}, b, b^{-1}\}$.

(9): $G = Z_2 \times Z_6 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m \geq 3$),
$S = \{a, b^3, b, b^{-1}, c, c^{-1}\}$.

(10): $G = Z_4 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$ ($m \geq 2$),
$S = \{a, a^{-1}, a^2, b, b^{-1}, b^m\}$.

(11): $G = Z_2 \times Z_4 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m \geq 3$),
$S_1 = \{a, b, b^{-1}, b^2, c, c^{-1}\}$, $S_2 = \{a, b, b^{-1}, ab^2, c, c^{-1}\}$.

(12): $G = Z_2 \times Z_4 \times Z_{2m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ ($m \geq 2$),
$S = \{a, b, b^{-1}, c, c^{-1}, c^m\}$.

(13): $G = Z_2^2 \times Z_4 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$ ($m\geq 3$), $S = \{a, b, c, c^{-1}, d, d^{-1}\}$.

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***M Alaeiyan,** Department of Mathematics, Iran University of Science and Technology, alaeiyan@iust.ac.ir

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(14): G = $Z_2^3 \times Z_m$ = <a> × <b> × <c> × <d> (m≥ 3),
S = {a, b, cd, cd^{-1}, d, d^{-1}}.

(15): G = $Z_2^2 \times Z_m$ = <a> × <b> × <c> (m = 5, 10),
S = {a, b, c, c^{-1}, c^3, c^{-3}}.

(16): G = $Z_2^2 \times Z_{4m}$ = <a> × <b> × <c> (m ≥ 2),
S = {a, b, c, c^{-1}, c^{2m+1}, c^{2m-1}}.

(17): G = $Z_4 \times Z_{2m}$ = <a> × <b> (m≥ 3, m is odd),
S = {a, a^3, b, b^{-1}, b^{m+1},b^{m-1}}.

(18): G = $Z_4^2 \times Z_m$ = <a> × <b> × <c> (m≥ 3),
S = {a, a^3, b, b^3, c, c^{-1}}.

(19): G = $Z_{4m} \times Z_n$ = <a> × <b> (m≥ 2, n≥3),
S = {a, a^{-1}, a^{2m+1}, a^{2m-1}, b, b^{-1}}.

(20): G = $Z_2 \times Z_m \times Z_n$ = <a>×<b>×<c> (m≥ 3, n≥3),
S ={ab, a b^{-1}, b,b^{-1}, c, c^{-1}}.

(21): G = $Z_m \times Z_n$ = <a> × <b> (m = 5, 10, n≥3),
S = {a, a^{-1}, a^3, a^{-3}, b, b^{-1}}.

(22): G = Z_2^4 = <a> × <b> × <c> × <d>,
S = {a, b, ab, c, abc, d}.

(23): G = $Z_2^2 \times Z_4$= <a> × <b> × <c>,
S = {a, b, ac^2, c, c^{-1}, c^2}.

(24): G = $Z_2^3 \times Z_4$= <a> × <b> × <c> × <d>,
S = {a, b, c, d, d^{-1}, abd^2}.

(25): G = $Z_2^2 \times Z_{3m}$ = <a> × <b> × <c> (m≥ 1),
S = {a, b, ac^m, ac^{2m}, c, c^{-1}}.

(26): G = $Z_2 \times Z_{10}$ = <a> × <b>, S= {a, b, b^3, b^5, b^7, b^9}.

(27): G = $Z_2^2 \times Z_{2m}$ = <a> × <b> × <c> (m≥ 2),
S = {ac, ac^{-1}, b, c^m, c, c^{-1}}.

(28): G = $Z_2 \times Z_4 \times Z_{2m}$ = <a> × <b> × <c> (m≥ 2),
S = {a, b^2c^m, b, b^{-1}, c, c^{-1}}.

(29): G = $Z_2 \times Z_{2m}$ = <a> × <b> (m≥ 3),
S = {a, b^m, b, b^{-1} , b^{m+1}, b^{m-1}}.

(30): G = $Z_2^2 \times Z_{2m}$ = <a> × <b> × <c> (m≥ 2),
S = {a, b, ac, ac^{-1}, c, c^{-1}}.

(31): G = $Z_2 \times Z_{6m}$ = <a> × <b> (m ≥3, m is odd),
S = {a, b^2, b^{-2}, b^m, b^{5m}, b^{3m} }.

(32): G = $Z_2^2 \times Z_{6m}$ = <a> × <b> × <c> (m≥ 2),
S = {a, bc^m, bc^{3m}, bc^{5m}, c, c^{-1}}.

(33): G = Z_2^3 = <a> × <b> × <c>, S = {a, b, c, ab, ac, abc}.

(34): G = Z_2^4 = <a> × <b> × <c> × <d>,
S = {a, b, c, d, abc, abd}.

(35): G = $Z_2^2 \times Z_{2m}$ = <a> × <b> × <c> (m≥ 2),
S = {a, b, ac^m, bc^m, c, c^{-1}}.

(36): G = $Z_2^2 \times Z_4$ = <a>×<b>×<c>,
S_1= {a, b, ab, ac^2, c, c^{-1}},
S_2= {a, b, ac^2, abc^2, c, c^{-1}}.

(37): G = $Z_2^3 \times Z_4$= <a> × <b> × <c> × <d>,
S = {a, b, c, $abcd^2$, d, d^{-1}}.

(38): G = $Z_2 \times Z_{6m}$ = <a> × <b> (m≥ 2),
S = {a, b^{3m}, ab^{2m}, ab^{4m}, b, b^{-1}}.

(39): G = $Z_2 \times Z_{4m}$ = <a> × <b> (m≥ 1),
S = {a, ab^m, ab^{2m}, ab^{3m}, b, b^{-1}}.

(40): G = $Z_4 \times Z_{2m}$ = <a> × <b> (m≥ 2),
S = {a, a^{-1}, b^m, a^2b^m, b, b^{-1}}.

(41): G = $Z_2^2 \times Z_{4m}$ = <a> × <b> × <c> (m≥ 1),
S = {a, ac^{2m}, bc^m, bc^{3m}, c, c^{-1}}.

(42): G = $Z_2 \times Z_{10}$ = <a> × <b>,
S = {a, ab^5, b, b^9, b^3, b^7}.

(43): G = $Z_2 \times Z_{2m}$ = <a> × <b>,
S_1= {a, b, b^{-1}, b^m, ab, a b^{-1}}, m≥ 2,
S_2={a, ab^m, b, b^{-1}, ab, a b^{-1}}, m≥ 2,
S_3 = {ab^m, b^m, b, b^{-1}, ab, a b^{-1}}, m≥ 2, S_4 ={a, ab^m, b, b^{-1}, b^{m+1}, b^{m-1}}, m≥ 3, S_5 = {a, b, b^{-1}, b^m, ab^{m+1}, ab^{m-1}}, m≥ 3, S_6 ={a, ab^m, b, b^{-1}, ab^{m+1}, ab^{m-1}}, m≥ 3
S_7={ab^m, b, b^{-1}, b^m, ab^{m+1}, ab^{m-1}}, m≥3

(44): G = $Z_2^2 \times Z_m$ = <a> × <b> × <c>, S_1= {a, b, c, c^{-1}, abc, abc^{-1}}, m≥ 3, S_2={a, b, c, c^{-1}, ac^{k+1}, ac^{k-1}}, m = 2k, k≥ 3, S_3= {a, b, c, c^{-1}, abc^{k+1}, abc^{k-1}}, m =2k, k ≥ 3,
S_4= {a, bc, b c^{-1}, ack, c, c^{-1}}, m = 2k, k ≥ 2,
S_5= {a, bc^{k+1}, bc^{k-1}, c^k, c, c^{-1}}, m = 2k, k ≥ 3,
S_6= {a, bc^{k+1}, bc^{k-1}, ac^k, c, c^{-1}}, m = 2k, k ≥ 3,
S_7= {a, b, c, c^{-1}, ac, ac^{-1}}, m = 2k − 1, k ≥ 2.

(45): G = Z_{4m} = <a> (m≥ 2),
S = {a, a^{-1}, a^m, a^{-m}, a^{2m+1}, a^{2m-1}}.

(46): G = Z_{2m} = <a> (m≥ 4),
S = {a, a^{-1}, a^{m+1}, a^{m-1}, a^k, a^{-k}} (2 ≤k ≤ m − 2),
(m, k) = l, if l > 2 or l = 2 for m = 4i + 2; (k = 2i, with i odd or k = 2i + 2, with i even).

(47): G = $Z_2 \times Z_m$ = <a> × <b> (m≥ 5),
S_1 = {ab, ab^{-1}, b, b^{-1}, b^j , b^{-j}} (2 ≤j < $\frac{m}{2}$), (m, j) = p > 2; m = (t + 1)p,

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$S_2= \{ab, ab\, b^{-1}, b, b\, b^{-1}, ab^j, ab^{-j}\}$, $(2 \leq j < \frac{m}{2})$, $(m, j)= p > 2$; $m = (t + 1)p$.

(48): $G = Z_2 \times Z_8 = \langle a\rangle\times\langle b\rangle$,
$S_1= \{ab, ab^{-1}, b, b^{-1}, b^3, b^{-3}\}$,
$S_2= \{ab, ab^{-1}, b, b^{-1}, ab^3, ab^{-3}\}$.

(49): $G = Z_{2m} \times Z_n = \langle a\rangle \times \langle b\rangle$ $(m\geq 2, n\geq 3)$,
$S = \{a, a^{-1}, a^m b, a^m b^{-1}, b, b^{-1}\}$.

(50): $G = Z_{2m} \times Z_{2n} = \langle a\rangle\times\langle b\rangle$ $(m\geq 3, n\geq 2)$,
$S = \{a, a^{-1}, a^{m+1}b^n, a^{m-1}b^n, b, b^{-1}\}$.

(51): $G = Z_{6m}= \langle a\rangle$ $(m\geq 2)$, $S_1= \{a, a^{-1}, a^3, a^{-3}, a^{3m+1}, a^{3m-1}\}$,
$S_2= \{a, a^{-1}, a^{3m+1}, a^{3m-1}, a^{3m+3}, a^{3m-3}\}$.

(52): $G = Z_m = \langle a\rangle$ $(m = 7, 14)$, $S = \{a, a^{-1}, a^3, a^{-3}, a^5, a^{-5}\}$.

(53): $G = Z_{3m} = \langle a\rangle$ $(m\geq 3)$,
$S = \{a, a^{-1}, a^{m-1}, a^{m+1}, a^{2m-1}, a^{2m+1}\}$.

(54): $G = Z_{16m-4} = \langle a\rangle$ $(m\geq 1)$,
$S = \{a, a^{-1}, a^{4m-2}, a^{12m-2}, a^{8m-3}, a^{8m-1}\}$.

(55): $G = Z_{16m+4} = \langle a\rangle$ $(m\geq 1)$,
$S = \{a, a^{-1}, a^{4m+2}, a^{12m+2}, a^{8m+1}, a^{8m+3}\}$.

(56): $G = Z_3 \times Z_3 = \langle a\rangle \times \langle b\rangle$,
$S = \{a, a^2, b, b^2, a^2b, ab^2\}$.

(57): $G = Z_2\times Z_4 \times Z_4 = \langle a\rangle \times \langle b\rangle\times \langle c\rangle$,
$S = \{a, b, b^{-1}, c, c^{-1}, ab^2 c^2\}$.

2. Primary Analysis

Proposition 2.1 [9, Proposition 1.5] Let $\Gamma = Cay(G, S)$ be a Cayley graph of G over S, and $A = Aut(\Gamma)$. Let A_1 be the stabilizer of the identity element 1 in A.
Then Γ is normal if and only if every element of A_1 is an automorphism of G.

Proposition 2.2 [6, Theorem 1.1] Let G be a finite abelian group and S be a generating subset of $G - 1_G$. Assume S satisfies the condition that, if $s, t, u, v\in S$ with $1 \neq st = uv$, implies $\{s, t\} = \{u, v\}$. Then the Cayley graph Cay (G, S) is normal.

Let X and Y be two graphs. The direct product X×Y is defined as the graph with vertex set $V(X \times Y) = V(X)\times V(Y)$ such that for any two vertices $u = [x_1, y_1]$ and $v = [x_2, y_2]$ in $V(X \times Y)$, $[u, v]$ is an edge in $X \times Y$, whenever $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$ or $y_1 = y_2$ and $[x_1, x_2] \in E(X)$. Two graphs are called relatively prime if they have no nontrivial common direct factor. The lexicographic product X[Y] is defined as the graph vertex set $V(X[Y]) = V(X) \times V(Y)$ such that for any two vertices $u = [x_1, y_1]$ and $v = [x_2, y_2]$ in $V(X[Y])$, $[u, v]$ is an edge in X[Y] whenever $[x_1, x_2] \in E(X)$ or $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$.

Let $V(Y) = \{y_1, y_2, ..., y_n\}$. Then there is a natural embedding nX in X[Y], where for $1 \leq i \leq n$, the ith copy of X is the subgraph induced on the vertex subset $\{(x, y_i)|x \in V(X)\}$ in X[Y]. The deleted lexicographic product $X[Y] - nX$ is the graph obtained by deleting all the edges of (this natural embedding of) nX from X[Y]. Let Γ be a graph and α a permutation V (Γ) and C_n a circuit of length n. The twisted product $\Gamma \times_\alpha C_n$ of Γ by C_n with respect to α is defined by;

$V(\Gamma \times_\alpha C_n) = V(\Gamma) \times V(C_n) = \{(x, i) \mid x \in V(\Gamma), i = 0, 1, ... , n - 1\}$,
$E(\Gamma\times_\alpha C_n) = \{[(x, i), (x, i+1)] \mid x \in V(\Gamma), i = 0, 1, ..., n-2\} \bigcup \{[(x, n-1), (x^\alpha, 0)] \mid x \in V(\Gamma)\}$ [$\{[(x, i), (y, i)] | [x, y] \in E(\Gamma), i = 0, 1, ..., n - 1\}$.

The graph Q_4^d denotes the graph obtained by connecting all long diagonals of 4-cube Q_4, that is, connecting all vertices u and v in Q_4 such that $d(u, v) = 4$. The graph $K_{m,m} \times_c C_n$ is the twisted product of $K_{m,m}$ by C_n such that c is a cycle permutation on each part of the complete bipartite graph $K_{m,m}$. The graph $Q_3 \times_d C_n$ is the twisted product of Q_3 by C_n such that d transposes each pair of elements on long diagonals of Q_3. The graph $C_{2m}^d[2K_1]$ is defined by:

$V(C_{2m}^d[2K_1]) = V(C_{2m}[2K_1])$,

$E(C_{2m}^d[2K_1]) = E(C_{2m}[2K_1]) \bigcup \{[(x_i, y_j), (x_{i+m}, y_j)] \mid i= 0, 1, ..., m - 1, j = 1, 2\}$, where $V(C_{2m}) = \{x_0, x_1, ..., x_{2m-1}\}$ and $V(2K_1) = \{y_1, y_2\}$.

Let $G = G_1\times G_2$ be the direct product of two finite groups G_1 and G_2, let S_1 and S_2 be subsets of G_1 and G_2, respectively, and let $S = S_1 \bigcup S_2$ be the disjoint union of two subsets S_1 and S_2. Then we have,

Lemma 2.3
(1) $Cay(G, S) \cong Cay(G_1, S_1)\times Cay(G_2, S_2)$.
(2) If Cay (G, S) is normal, then $Cay(G_1, S_1)$ is also normal.
(3) If both of $Cay(G_1, S_1)$ and $Cay(G_2, S_2)$ are normal and relatively prime, then Cay (G, S) is normal.

3. Proof of the Main Theorem

In this section, Γ always denotes the Cayley graph Cay(G, S) of an abelian group G on S with valency 6. Let $A = Aut(\Gamma)$. Then A_1 and A_1^* denote the stabilizer of 1 in A and the subgroup of A which fixes $\{1\} \bigcup S$, pointwise, respectively. In order to prove Theorem 1.1 we need several lemmas.

Lemma 3.1 Let $G = Z_{2m} = \langle a\rangle$, $(m\geq 5)$, and $S =\{a^i, a^{-i}, a^{m+i}, a^{m-i}, a, a^{-1}\}$ $2 \leq i < \frac{m}{2}$. Then $\Gamma = Cay(G, S)$ is normal.

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Proof Let $\Gamma_2(1)$ be the subgraph of Γ with vertex set $\{1\}\bigcup S\bigcup S^2$ and edge set $\{[1,s], [s, st] \mid s,t\in S\}$. By observing the subgraph $\Gamma_2(1)$, it is easy to prove that A_1^* fixes S^2 pointwise, which implies that $A_1^* = 1$. Thus A_1 acts faithfully on S. Observing the subgraph $\Gamma_2(1)$ again, A_1, as a permutation group on S, is generated by $(a, a^{-1})(a^{m+i}, a^{m-i})$. So $|A_1| = 2$ and $\Gamma = Cay(G, S)$ is normal.

Lemma 3.2: Let $G = Z_2^2 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, $m = 4k$, $k \geq 2$ and $S = \{a, b, c^k, c^{3k}, c, c^{-1}\}$. Then $\Gamma = Cay(G, S)$ is normal.

Proof Set $G_1 = \langle a,b\rangle$, $G_2 = \langle c\rangle$, $S_1 = \{a, b\}$, $S_2 = \{c^k, c^{3k}, c, c^{-1}\}$. Then $\Gamma_1 = Cay(G_1, S_1) \cong K_2\times K_2$. Note that Γ_1 and $\Gamma_2 = Cay(G_2, S_2)$ are relatively prime. By [5, Theorem 1.1] and [6, Theorem 1.2], Γ_1 and Γ_2 are normal and by Lemma 2.3, $\Gamma = Cay(G, S)$ is normal.

With similar arguments as in Lemmas 3.1 and 3.2, we have the following lemma.

Lemma 3.3 Let G and S be as the following. Then the Cayley graphs $\Gamma = Cay(G, S)$ are normal.

(1): $G = Z_2^4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$,
$S = \{a, b, c, d, ad, abc\}$.

(2): $G = Z_2^2 \times Z_6 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$,
$S = \{a, b, ac^3, c^3, c, c^{-1}\}$.

(3): $G = Z_2^2 \times Z_{2m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m \geq 2)$,
$S = \{a, b, abc^m, c^m, c, c^{-1}\}$.

(4): $G = Z_2 \times Z_{6m} = \langle a\rangle \times \langle b\rangle$ $(m \geq 2)$,
$S = \{a, ab^m, ab^{3m}, ab^{5m}, b, b^{-1}\}$.

(5): $G = Z_2 \times Z_{6m} = \langle a\rangle \times \langle b\rangle$ $(m \geq 2)$,
$S = \{a, b^m, b^{3m}, b^{5m}, b, b^{-1}\}$.

(6): $G = Z_6 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$ $(m \geq 3)$,
$S = \{a, a^{-1}, a^3, a^3b^m, b, b^{-1}\}$.

(7): $G = Z_2 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$,
$S_1 = \{a, ab^2, ab^{-2}, b^m, b, b^{-1}\}$, $(m \geq 4)$,
$S_2 = \{a, ab^{m+2}, ab^{m-2}, ab^m, b, b^{-1}\}$, $(m \geq 5)$,
$S_3 = \{a, b^{m+2}, b^{m-2}, b^m, b, b^{-1}\}$, $(m = 4, m \geq 6)$.

(8): $G = Z_2 \times Z_{4m+2} = \langle a\rangle \times \langle b\rangle$,
$S_1 = \{a, ab^m, ab^{3m+2}, b^{2m+1}, b, b^{-1}\}$,
$(m \geq 2)$, $S_2 = \{a, b, b^{-1}, b^m, b^{3m+2}, b^{2m+1}\}$, $(m \geq 2)$,
$S_3 = \{a, ab^{m+1}, ab^{3m+1}, b^{2m+1}, b, b^{-1}\}$, $(m \geq 2)$,
$S_4 = \{a, b, b^{-1}, b^{m+1}, b^{3m+1}, b^{2m+1}\}$, $(m \geq 3)$.

(9): $G = Z_4 \times Z_{4m+2} = \langle a\rangle \times \langle b\rangle$ $(m \geq 1)$,
$S_1 = \{a^2b^{2m+1}, b^{2m+1}, ab^m, a^3b^{3m+2}, b, b^{-1}\}$,
$S_2 = \{a^2, a^2b^{2m+1}, ab^m, a^3b^{3m+2}, b, b^{-1}\}$.

(10): $G = Z_2^2 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, $(m \geq 4, m \neq 6)$,
$S = \{a, b, c, c^{-1}, ac^2, ac^{-2}\}$.

(11): $G = Z_2 \times Z_{4m} = \langle a\rangle \times \langle b\rangle$ $(m \geq 2)$,
$S_1 = \{a, ab^m, ab^{3m}, b^{2m}, b, b^{-1}\}$,
$S_2 = \{a, b, b^{-1}, b^m, b^{3m}, b^{2m}\}$,
$S_3 = \{a, ab^{2m}, b^m, b^{3m}, b, b^{-1}\}$.

(12): $G = Z_4 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$ $(m \geq 3)$,
$S = \{a^2, a^2b^m, a, a^{-1}, b, b^{-1}\}$.

(13): $G = Z_2^2 \times Z_{4m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m \geq 2)$,
$S_1 = \{a, b, abc^m, abc^{3m}, c, c^{-1}\}$, $S_2 = \{a, b, ac^m, ac^{3m}, c, c^{-1}\}$,
$S_3 = \{a, b, c^m, c^{3m}, c, c^{-1}\}$,
$S_4 = \{a, c^{2m}, bc^m, bc^{3m}, c, c^{-1}\}$.

(14): $G = Z_2^3 \times Z_{4m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$ $(m \geq 2)$,
$S = \{a, b, cd^m, cd^{3m}, d, d^{-1}\}$.

(15): $G = Z_2^2 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m = 7, 9, m \geq 11)$,
$S = \{a, b, c, c^{-1}, c^3, c^{-3}\}$.

(16): $G = Z_2\times Z_4\times Z_{4m+2} = \langle a\rangle\times\langle b\rangle\times\langle c\rangle$ $(m \geq 1)$,
$S = \{a, b^2c^{2m+1}, bc^m, b^3c^{3m+2}, c, c^{-1}\}$.

(17): $G = Z_2^2 \times Z_{4m+2} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m \geq 2)$,
$S = \{a, c^{2m+1}, bc^m, bc^{3m+2}, c, c^{-1}\}$.

(18): $G = Z_2\times Z_4 \times Z_{2m} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m \geq 3)$,
$S = \{a, ac^m, b, b^{-1}, c, c^{-1}\}$.

(19): $G = Z_2 \times Z_{2m} = \langle a\rangle \times \langle b\rangle$ $(m \geq 6)$,
$S_1 = \{a, b^m, b, b^{-1}, b^3, b^{-3}\}$,
$S_2 = \{a, ab^m, b, b^{-1}, b^3, b^{-3}\}$.

(20): $G = Z_{4m} \times Z_n = \langle a\rangle \times \langle b\rangle$ $(m \geq 2, n \geq 3)$,
$S = \{a, a^{-1}, a^m, a^{3m}, b, b^{-1}\}$.

(21): $G = Z_{4m} \times Z_{4n} = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m, n \neq 4)$,
$S = \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$.

(22): $G = Z_4 \times Z_m \times Z_n = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ $(m, n \neq 3)$,
$S = \{a, a^3, b, b^{-1}, c, c^{-1}\}$.

(23): $G = Z_{2m}$ $(m \geq 5)$,
$S = \{a, a^{-1}, a^j, a^{-j}, a^{m+j}, a^{m-j}\}$ $(2 \leq j < \frac{m}{2})$.

(24): $G = Z_m\times Z_n = \langle a\rangle\times\langle b\rangle$ $(m = 7, 9, m \geq 11, n \geq 3)$,
$S = \{a, a^{-1}, a^3, a^{-3}, b, b^{-1}\}$.

(25): $G = Z_{3m-1}\times Z_{3n} = \langle a\rangle\times\langle b\rangle$ $(m \geq 2, n \geq 1)$,
$S = \{a, a^{-1}, b, b^{-1}, a^mb^n, a^{2m-1}b^{2n}\}$.

(26): $G = Z_{3m+1} \times Z_{3n} = \langle a\rangle \times \langle b\rangle$ $(m, n \geq 1)$,
$S = \{a, a^{-1}, b, b^{-1}, a^mb^{2n}, a^{2m+1}b^n\}$.

(27): $G = Z_m \times Z_n = \langle a\rangle \times \langle b\rangle$ $(m \geq 5, n \geq 3)$,
$S = \{a, a^{-1}, b, b^{-1}, a^2b, a^{-2}b^{-1}\}$.

(28): $G = Z_{2m+1} \times Z_n = \langle a\rangle \times \langle b\rangle$ $(m, n \geq 3)$,
$S = \{a, a^{-1}, a^m, a^{m+1}, b, b^{-1}\}$.

(29): $G = Z_{2m+1} \times Z_{2n+1} = \langle a\rangle\times\langle b\rangle$ $(m, n \geq 2)$,
$S = \{a, a^{-1}, b, b^{-1}, a^mb^{n+1}, a^{m+1}b^n\}$.

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(30): $G = Z_2\times Z_{2n+1}\times Z_{2m+1} = <a>\times<b>\times<c>$ $(m, n \geq 1)$, $S = \{ab^mc^{n+1}, ab^{m+1}c^n, b, b^{-1}, c, c^{-1}\}$.

(31): $G = Z_{4m} = <a>$ $(m \geq 2)$, $S = \{a, a^{-1}, a^k, a^{-k}, a^m, a^{-m}\}$, $(1 < k < 2m, k \neq m, 2m-1$.

(32): $G = Z_4\times Z_m = <a>\times<b>$ $(m \geq 3)$,
$S = \{a, a^{-1}, b, b^{-1}, ab^j, a^{-1}b^{-j}\}$, $1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor$,
(When $m \neq 2k$ for every j or $m = 2k$, $j \neq k$).

(33): $G = Z_4 \times Z_{2m} = <a> \times <b>$ $(m \geq 2)$, $S = \{a, a^{-1}, b, b^{-1}, a^2b^j, a^2b^{-j}\}$ $1 \leq j < m$ (for every $j \neq 1, m - 1$).

(34): $G = Z_4 \times Z_{2m-1} = <a> \times <b>$ $(m \geq 2)$,
$S = \{a, a^{-1}, b, b^{-1}, a^2b^j, a^2b^{-j}\}$ $(1 < j < \frac{2m-1}{2})$.

(35): $G = Z_4 \times Z_m = <a> \times <b>$ $(m \geq 5)$,
$S = \{a, a^{-1}, b, b^{-1}, b^j, b^{-j}\}$ $(1 < j < \frac{m}{2})$,
when $m \neq 2k$, 5 or $m = 2k$ $(k \geq 3, k \neq 5)$, $j \neq k - 1$ or $m = 10$, $j \neq 3$.

(36): $G = Z_{2m} = <a>$ $(m \geq 4)$, $S = \{a, a^{-1}, a^j, a^{-j}, a^{m+1}, a^{m-1}\}$ $(2 \leq j \leq m- 2)$, when $(m, j) = 1$ or $(m, j) = 2, m \neq 4i + 2$ $(i \geq 1)$.

(37): $G = Z_2 \times Z_m = <a> \times <b>$ $(m \geq 5, m \neq 8)$,
$S_1= \{ab, ab^{-1}, b, b^{-1}, b^j, b^{-j}\}$,
$S_2= \{ab, ab^{-1}, b, b^{-1}, ab^j, ab^{-j}\}$ $(2 \leq j < \frac{m}{2})$, when $(m, j) = p \leq 2$.

(38): $G = Z_2\times Z_8 = <a>\times<b>$,
$S_1= \{ab, ab^7, b, b^7, b^2, b^6\}$,
$S_2 = \{ab, ab^7, b, b^7, ab^2, ab^6\}$.

(39): $G = Z_m = <a>$ $(m \geq 9, m \neq 14)$,
$S = \{a, a^{-1}, a^3, a^{-3}, a^j, a^{-j}\}$ $j \neq 3, 2 \leq j < \frac{m}{2}$) when $m \neq 6k$, $\forall$ j or $m = 6k$, $j \neq 3k - 1$.

(40): $G = Z_{14} = <a>$, $S = \{a, a^{-1}, a^3, a^{-3}, a^j, a^{-j}\}$ for $j = 2, 4, 6$.

(41): $G = Z_m = <a>$ $(m \geq 7)$,
$S = \{a, a^{-1}, a^{3j}, a^{-3j}, a^j, a^{-j}\}$, $(2 \leq j < \frac{m}{2}$, $3j \not\equiv 0, 1, m - 1, j, m - j, \frac{m}{2} \pmod m$) , when $m \neq 7, 14, 6k$ $(k \geq 2)$ and $m = 7$; $j = 2$ or $m = 14$; $j = 2, 3, 4, 6$ or $m = 6k$; $j \neq 3k - 1$.

(42): $G = Z_m = <a>$ $(m \geq 8, m \neq 14)$,
$S = \{a, a^{-1}, a^{2+j}, a^{-2-j}, a^j, a^{-j}\}$ (if $m = 2k$ then $2 \leq j \leq \frac{m}{2} -3$ and if $m = 2k +1$ then $2 \leq j \leq \frac{m}{2} -1$). When $m \neq 3k$ for every j and when $m = 3k$, for k odd ; $j \neq k - 1$ and for k even ; $j \neq k - 1$, $3\frac{k}{2} - 3$.

(43): $G = Z_{14} = <a>$,
$S = \{a, a^{-1}, a^{2+j}, a^{-2-j}, a^j, a^{-j}\}$ for $j = 2, 4$.

(44): $G = Z_2 \times Z_4 \times Z_{2m} = <a> \times <b> \times <c>$ $(m \geq 3)$, $S = \{a, ab^2c^m, b, b^{-1}, c, c^{-1}\}$.

Now we are in a position to prove Theorem 1.1. Immediately from Lemma 2.3, [5, Theorem 1.1] and [6, Theorem 1.2], we have the Cases (1)-(32) of Theorem 1.1. Assume that Γ is not normal. In view of Proposition 2.2, we have the following assumption: $\exists$ s, t, u, v$\in$S such that $st = ub \neq 1$ but $\{s, t\} \neq \{u, v\}$. (*).
We divide S into four cases:

Case 1: $S = \{a, b, c, d, e, f\}$, where a, b, c, d, e, f are involutions. In this case G is an elementary abelian 2-group and a, b, c, d, e, f are not independent by the assumption (*). Consequently $G = Z_2^3$ or $G = Z_2^4$ or $G = Z_2^5$. If $G = Z_2^3 = <a>\times<b>\times<c>$ by the assumption (*) we can let $S = \{a, b, c, ab, ac, abc\}$. We have $\sigma = (a, abc) \in A_1$, but $\sigma \notin Aut(G, S)$; and by Proposition 2.1 , $\Gamma = Cay(G, S)$ is not normal , the Case (33) of Theorem 1.1. If $G = Z_4^2 = <a> \times <b> \times <c> \times <d>$ by the assumption (*) we see that S is one of the following cases:
(i) $S_1= \{a, b, c, d, abc, abd\}$, (ii) $S_2= \{a, b, c, d, ab, abc\}$,
(iii) $S_3= \{a, b, c, d, abc, abc\}$.

When $S = S_1$, $\sigma = (a, b) \in A_1$, but $\sigma \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (34) of Theorem 1.1. When $S = S_2$, we have the Case (22) of the main theorem. Also when $S = S_3$, Γ is normal by Lemma 3.3. If $G = Z_2^5 = <a> \times <b> \times <c> \times <d> \times <e>$ we can let $S = \{a, b, c, d, e, abc\}$ and hence $\Gamma = Cay(G, S)$ is non-normal, the Case (1) of Theorem 1.1.

Case 2: $S = \{a, b, c, d, e, e^{-1}\}$, where a, b, c, d are involutions but e is not. In this case, $S^2- 1 = \{ab, ac, ad, ae, ae^{-1}, bc, bd, be, be^{-1}, cd, ce, ce^{-1}, de, de^{-1}, e^2, e^{-2}\}$. By the assumption (*) $d = abc$, $o(e) = 4$ or $d = e^3$. Suppose $d = abc$. Then $G = Z_2^2 \times Z_{2m}$,$(m\geq 2)$ or $G = Z_2^3 \times Z_m$, $(m\geq 3)$.

If $G = Z_2^2 \times Z_{2m} = <a>\times<b>\times<c>$, $(m\geq 2)$, we can let
$S = \{a, b, ac^m, bc^m, c, c^{-1}\}$ or
$S = \{a, b, c^m, abc^m, c, c^{-1}\}$.
When $S = \{a, b, ac^m, bc^m, c, c^{-1}\}$,
$\sigma = (ab, abc^m)(abc, abc^{m+1})...(abc^{m-1}, abc^{2m-1}) \in A_1$, but $\sigma \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (35) of the main theorem.
When $S = \{a, b, c^m, abc^m, c, c^{-1}\}$, $\Gamma = Cay(G, S)$ is normal by Lemma 3.3(3). If $G = Z_2^3 \times Z_m = <a> \times <b> \times <c> \times <d>$, $(m\geq 3)$, $S = \{a, b, c, abc, d, d^{-1}\}$, the Case (2) of Theorem 1.1. Suppose $o(e) = 4$. Then G =

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$Z_2^2 \times Z_4$, $Z_2^3 \times Z_4$ or $Z_2^4 \times Z_4$. If G = $Z_2^2 \times Z_4$= <a> × <b> × <c>, we have S is one of the following cases:
S_1= {a, b, ab, ac^2, c, c^{-1}}, S_2= {a, b, ae^2, bc^2, c, c^{-1}},
S_3= {a, b, ac^2, abc^2, c, c^{-1}}.
S_4= {a, b, ab, c^2, c, c^{-1}},
S_5= {a, b, ac^2, c^2, c, c^{-1}},
S_6= {a, b, abc^2, c^2, c, c^{-1}}.

When S = S_1, σ = (ac^2, c)(ac, c^2)(bc, abc^2)(abc, bc^2) ∈ A_1, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (36 − S_1) of Theorem 1.1 . When S = S_2, by Proposition 2.1, Γ = Cay (G, S) is not normal, the Case (35, m = 2) of Theorem 1.1. When S = S_3, σ = (a, c)(ab, bc)(c^2, ac^3)(bc^3, abc^3)∈ A_1, but σ ∉ Aut(G, S); by Proposition 2.4, Γ = Cay(G, S) is not normal the Case (36 − S_2) of Theorem 1.1. When S = S_4, we have the Case (3) of Theorem 1.1. When S = S_5, we have the Case (23) of Theorem 1.1. When S = S_6, Γ is normal by Lemma 3.3 (3, m=2) .If G = $Z_2^3 \times Z_4$= <a> × <b> × <c> × <d>, we have S = {a, b, c, d, d^{-1}, u}, where u = d^2, ab, ad^2, abc, abd^2 or $abcd^2$. When u = d^2, we have the Case (5− S_1) of Theorem 1.1. When u = ab, we have the Case (5 − S_2) of Theorem 1.1. When u = ad^2, we have the Case (5 − S_3) of Theorem 1.1. When u = abc, we have the Case (2) of Theorem 1.1. When u = abd^2, we have the Case (24) of Theorem 1.1. When u = $abcd^2$, σ = ($abcd^2$, d)(bcd^2, ad)(acd^2, bd)(abd^2, cd) (abcd, d^2)(cd^2, abd)(bd^2, acd) and (bcd, ad^2) ∈ A_1, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (37) of Theorem 1.1. If G = Z_2^4 = <a>×<b>×<c>×<d>×<e>, S = {a, b, c, d, e, e^{-1}}, we have the Case (4) of Theorem 1.1. Now suppose d = e^3. Then G = $Z_2^2 \times Z_6$ or G = $Z_2^3 \times Z_6$. If G = $Z_2^2 \times Z_6$= <a> × <b> × <c>, we see that S is one of the following cases: S_1= {a, b, ab, c^3, c, c^{-1}}, S_2= {a, b, ac^3, c^3, c, c^{-1}}, S_3= {a, b, abc^3, c^3, c, c^{-1}}.
When S = S_1, we have the Case (6) of Theorem 1.1. For S_2 and S_3, we have the Cases (2) and (3, m = 3) of Lemma 3.3 respectively. If G = $Z_2^3 \times Z_6$ = <a>×<b>×<c>×<d>, then S = {a, b, c, d^3, d, d^{-1}}, the Case (7) of Theorem 1.1.

Case 3: S = {a, b, c, c^{-1}, d, d^{-1}}, where a, b are involutions but c, d are not. By the assumption (*) and the symmetry of c, c^{-1}, d and d^{-1}, we have five sub cases (I) a = c^3, (II) a = c^2d, (III) o(c) = 4, (IV) c^3 = d and (V) c^2 = d^2. Suppose a = c^3, then G is isomorphic to one of the following: $Z_2 \times Z_{6m}$ (m≥ 2), $Z_2 \times Z_6$, $Z_6 \times Z_{2m}$ (m≥ 2), $Z_2^2 \times Z_{3m}$ (m≥ 1), $Z_2 \times Z_6 \times Z_m$ (m≥ 3). If $Z_2 \times Z_{6m}$ = <a>×<b>, (m≥ 2), we see that S is one of the following cases:
S_1= {a, b^{3m}, ab^{2m}, ab^{4m}, b, b^{-1}}, S_2= {a, ab^{3m}, ab^m, ab^{5m}, b, b^{-1}}, S_3= {a, b^{3m}, b^m, b^{5m}, b, b^{-1}}. When S = S_1, σ = (a, ab^{2m}, ab^{4m})(ab, ab^{2m+1}, ab^{4m+1})...(ab^{2m-1}, ab^{4m-1},ab^{6m-1}) ∈ A_1, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (38) of the main theorem. For the Cases S = S_2 and S = S_3, we have the Cases (4) and (5) of Lemma 3.3. If G = $Z_2 \times Z_6$= <a> × <b>, we see that S is one of the following cases:

S_1 = {a, b^3, ab^2, ab^4, b, b^{-1}}, S_2= {a, b^3, b, b^{-1}, b^2, b^4},
S_3= {a, b^3, b, b^{-1}, ab, ab^{-1}}.
When S = S_1, σ = (a, ab^2, ab^4)(ab, ab^3, ab^5) ∈ A_1, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43 − S_5) of Theorem 1.1. When S = S_2, we have the Case (29, m=3) of Theorem 1.1. When S = S_3, σ = (b^5, ab^5)(b^2, ab^2) ∈ A_1, but σ ∉ Aut(G, S); by Proposition 2.1, Γ= Cay(G, S) is not normal, the Case (43 − S_1) of Theorem 1.1. If G = $Z_6 \times Z_{2m}$ = <a> × <b>, we see that S is one of the following cases:

S_1= {a^3, b^m, a, a^{-1}, b, b^{-1}}, S_2= {a^3, a^3b^m, a, a^{-1}, b, b^{-1}}.
When S =, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (8) of Theorem 1.1.
For S = S_2, when m = 2, σ = (b^2, a^3b)(ab^2, a^4b)(a^2b^2, a^5b)(a^3b^2, b) (a^4b^2, ab)(a^5b^2, a^2b) ∈ A_1, but σ ∉ Aut(G, S); Γ = Cay(G, S) is not normal, the Case (40, m=3) of Theorem 1.1, and when m≥ 3, Γ = Cay(G, S) is normal by Lemma 3.3(6). If G = $Z_2^2 \times Z_{3m}$ =<a>×<b>×<c> (m≥ 1), S = {a, b, ac^m, ac^{2m}, c, c^{-1}}. Then we obtain the Case (25) of Theorem 1.1. If G = $Z_2 \times Z_6 \times Z_m$ = <a>×<b>×<c> (m≥ 3), S = {b^3, a, b, b^{-1}, c, c^{-1}}. Then we obtain the Case (9) of Theorem 1.1. Suppose a = c^2d. Then we have one of the following cases:
(1): G = $Z_2 \times Z_{2m}$ = <a> × <b> (m≥ 3),
S = {a, b^m, b, b^{-1}, ab^{-2}, ab^2}.

(2): G = $Z_2 \times Z_{2m}$ = <a> × <b>,
S_1= {ab^m, a, b, b^{-1}, ab^{m-2}, ab^{m+2}} (m≥ 3),
S_2= {b^m, a, b, b^{-1}, b^{m-2}, b^{m+2}}, m≥ 4,

(3): G = $Z_2 \times Z_{4m+2}$ = <a> × <b>,
S_1 = {a, b, b^{-1}, b^{2m+1}, ab^m, ab^{3m+2}} (m≥ 1),
S_2= {a, b, b^{-1}, b^{2m+1}, b^m, b^{3m+2}}, m≥ 2
S_3= {a, b, b^{-1}, b^{2m+1}, b^{3m+1}, b^{m+1}} (m≥ 1),
S_4= {a, b^{2m+1}, ab^{3m+1}, ab^{m+1}, b, b^{-1}}, m≥ 1,

(4): G = $Z_4 \times Z_{4m+2}$ = <a> × <b>,
S_1= {a^2b^{2m+1}, b^{2m+1}, ab^m, a^3b^{3m+2}, b, b^{-1}}, m≥ 1
S_2= {$^{a2b2m+1}$, a^2, ab^m, a^3b^{3m+2}, b, b^{-1}}, m≥ 1.

(5): G = $Z_2^2 \times Z_m$ = <a> × <b> × <c> (m≥ 3),
S = {a, b, c, c^{-1}, ac^{-2}, ac^2}.

(6): G = $Z_2 \times Z_4 \times Z_{4m+2}$ = <a>×<b>×<c> (m≥ 1),
S = {a, b^2c^{2m+1}, bc^m, $b^{-1}c^{-m}$, c, c^{-1}}.

(7): G = $Z_2^2 \times Z_{4m+2}$ = <a> × <b> × <c> (m≥ 1),
S = {a, c^{2m+1}, bc^m, bc^{-m}, c, c^{-1}}.

In the Case (1), when m = 3, σ = (b^2, b^4) ∈ A_1, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not

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normal, the Case (43− S_5, m = 3) of Theorem 1.1. When m≥ 4, Γ is normal by Lemma 3.3(7− S_1).
In the Case (2), S = S_1 when m = 3, σ = $(b^2, ab^2)(b^5, ab^5) \in A_1$, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43− S_2) of Theorem 1.1.

When m = 4, σ = $(b, b^7)(b^2, b^6)(b^3, b^7) \in A_1$, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case 39 (m = 2) of Theorem 1.1. When m≥ 5, Γ = Cay(G, S) is normal by Lemma 3.3 (7− S_2). In the Case (2), S = S_2, when m = 5, we have the Case (26) of Theorem 1.1. When m≥ 6, Γ is normal by Lemma 3.3 (7− S_3).

In the Case (3), S = S_1, when m = 1, we have the Case (43 − S_1) of Theorem 1.1. When m≥ 2, Γ is normal by Lemma 3.3 (8 − S_1). In the Case (3), S = S_2, Γ is normal by Lemma 3.3 (8 − S_2). In the Case (3), S = S_3, when m = 1, 2, we have the Cases (29,m = 3, 5) of Theorem 1.1 respectively. When m≥ 3, Γ is normal by Lemma 3.3(8 − S_4). In the Case (3), S = S_4, when m= 1, σ = $(ab, ab^5) \in A_1$, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (29,m = 3) of Theorem 1.1. When m≥ 2, Γ = Cay (G, S) is normal by Lemma 3.3(8 − S_3). In the Case (4), Γ = Cay (G, S) is normal by Lemma 3.3(9). In the Case (5), when m = 3, 6, by Proposition 2.1, Γ is not normal, the Case (25, m = 1, 2) of Theorem 1.1. Otherwise Γ is normal by Lemma 3.3(10). In the Case (6), Γ is normal by Lemma 3.3(16). In the Case (7), when m = 1, by Proposition 2.1, Γ is not normal, the Case 27 (m = 1) of Theorem 1.1. When m≥ 2, Γ is normal by Lemma 3.3 (17). Suppose o(c) = 4. Then we have one of the following cases:
(I) G = $Z_2 \times Z_4$ = <a> × <b>, $S_1 = \{a, b^2, b, b^{-1}, ab, ab^{-1}\}$,

(II) G = $Z_2 \times Z_{4m}$ = <a> × <b>, $S_1 = \{a, b^{2m}, ab^m, ab^{3m}, b, b^{-1}\}$, (m≥ 2), $S_2 = \{a, ab^{2m}, ab^m, ab^{3m}, b, b^{-1}\}$, (m≥ 1),
$S_3 = \{a, b^{2m}, b^m, b^{3m}, b, b^{-1}\}$, (m≥ 2),
$S_4 = \{a, ab^{2m}, b^m, b^{3m}, b, b^{-1}\}$, (m≥ 2).

(III) G = $Z_4 \times Z_{2m}$ = <a> × <b> (m≥ 2),
$S_1 = \{a^2, b^m, a, a^{-1}, b, b^{-1}\}$, $S_2 =$
$\{a^2, a^2b^m, a, a^{-1}, b, b^{-}1\}$, $S_3 = \{a^2b^m, b^m, a, a^{-1}, b, b^{-1}\}$.

(IV): G = $Z_2^2 \times Z_4$ = <a>×<b>×<c>,
$S_1 = \{a, b, c, c^{-1}, ac, ac^{-1}\}$, $S_2 = \{a, b, c, c^{-1}, abc, abc^{-1}\}$.
(V): G = $Z_2^2 \times Z_{4m}$ = <a> × <b> × <c> (m≥ 2),
$S_1 = \{a, b, abc^m, abc^{3m}, c, c^{-1}\}$, $S_2 = \{a, b, ac^m, ac^{3m}, c, c^{-1}\}$, $S_3 = \{a, b, c^m, c^{3m}, c, c^{-1}\}$.

(VI): G = $Z_2 \times Z_4 \times Z_m$ = <a> × <b> × <c> (m≥ 3),
$S_1 = \{a, b^2, b, b^{-1}, c, c^{-1}\}$,
$S_2 = \{a, ab^2, b, b^{-1}, c, c^{-1}\}$.

(VII): G = $Z_2 \times Z_4 \times Z_{2m}$ = <a>×<b>×<c> (m≥ 2),
$S_1 = \{a, c^m, b, b^{-1}, c, c^{-1}\}$, $S_2 = \{a, ac^m, b, b^{-1}, c, c^{-1}\}$,
$S_3 = \{a, b^2c^m, b, b^{-1}, c, c^{-1}\}$, $S_4 = \{a, ab^2c^m, b, b^{-1}, c, c^{-1}\}$.

(VIII): G = $Z_2^2 \times Z_{4m}$ = <a> × <b> × <c> (m≥ 1),
$S_1 = \{a, c^{2m}, bc^m, bc^{3m}, c, c^{-1}\}$,
$S_2 = \{a, ac^{2m}, bc^m, bc^{3m}, c, c^1\}$.

(IX): G = $Z_2^2 \times Z_4 \times$ = <a> × <b> × <c> × <d> (m≥ 3),
$S = \{a, b, c, c^{-1}, d, d^{-1}\}$.

(X): G = $Z_2^3 \times Z_{4m}$ = <a> × <b> × <c> × <d> (m≥ 1),
$S = \{a, b, cd^m, cd^{3m}, d, d^{-1}\}$.

In the Case (I), σ = $(ab, b^3) \in A_1$, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43 − S_1) of Theorem 1.1. In the Case (II), S = S_1, Γ = Cay(G, S) is normal by Lemma 3.3(11 − S_1). In the Case (II), S = S_2, σ = $(b, b^{-1})(b^2, b^{-2})...(b^{2m-1}, b^{2m+1})(a, ab^m)...(ab^{2m+1}, ab^{-(m+1)}) \in A_1$, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal , the Case (39) of Theorem 1.1. In the Case (II), S = S_3, and S = S_4, Γ is normal by Lemma 3.3, the Case (11 − S_2, S_3). In the Case (III), when S = S_1, we have the Case (10) of Theorem 1.1. When S = S_2, m = 2, σ = $(a^2b^2, b)(a^3b^2, ab)(ab^2, a^3b)(b^2, a^2b) \in A_1$, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (40, m = 2) of Theorem 1.1. When S = S_2, m≥ 3, Γ = Cay(G, S) is normal by Lemma 3.3(12). When S = S_3, σ = $(a^2, ab^m)(a^2b, ab^{m+1})...(a^2b^{2m-1}, ab^{m+(2m-1)}) \in A_1$ but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal , the Case (40) of Theorem 1.1.

In the Case (IV), when S = S_1, σ = $(c^2, ac^2)(bc^2, abc^2) \in A_1$,but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44− S_2) of Theorem 1.1. When S = S_2, σ = $(ac^2, bc^2) \in A_1$, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44− S_3) of Theorem 1.1. In the Case (V), S = S_1, when m = 1, with an argument similar to the Case (IV − S_2) we obtain the same result. When m≥ 2, Γ is normal by Lemma 3.3 (13− S_1). In the Case (V), S = S_2, when m = 1, with an argument similar to the Case (IV−S_1), we obtain the same result.

When m≥ 2, Γ is normal by Lemma 3.3 (13 − S_2). In the Case (V), S = S_3, Γ is normal by Lemma 3.3(13− S_3). In the Case (VI), we have the Case (11) of Theorem 1.1. In the Case (VII), S = S_1, S = S_3 and S = S_2 (m = 2), we have the Cases (12), (28) and (11 − S_2, m = 4) of Theorem 1.1 respectively. In the Case (VII), S = S_2, m≥ 3, Γ is normal by Lemma 3.3(18). In the Case (VII), S = S_4, for m = 2, σ = $(b^3, c)(ab^3, ac)(abc^2, ab^2c^3)(b^2, bc)(b^3c^3, c^2)(b^2c, b^2c^3)(ab^2, abc)(ab^3c^3, ac^2) \in A_1$, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (57) of Theorem 1.1, and for m≥ 3, Γ is normal by Lemma 3.3(44). In the Case (VIII), S = S_1 when m = 1, we have the Case (21, m = 2) of Theorem 1.1. If m≥ 2, Γ is normal by Lemma 3.3 (13 − S_4). In the Case (VIII), S = S_2, σ = $(ab, abc^{2m})(abc, abc^{2m+1})...(abc^{2m-1}, abc^{4m-1}) \in A_1$, but σ ∉ Aut(G, S); by Proposition 2.1 ,Γ = Cay(G, S) is not normal, the Case (41) of Theorem 1.1. In the Case

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(IX), we have the Case (13) of Theorem 1.1. In the Case (X), m = 1, we have the Case (14) of Theorem 1.1, and for $m\geq 2$, Γ = Cay (G, S) is normal by Lemma 3.3(14). Suppose $c^3 = d$, then G= $Z_2^2 \times Z_{2m}$, ($m\geq 4$) or G = $Z_2^2 \times Z_m$ ($m\geq 5$, $m \neq 6$). If G = $Z_2 \times Z_{2m}$ = <a> × <b> ($m\geq 4$), we can let S to be

S_1= {a, b^m, b, b^{-1}, b^3, b^{-3}} or S_2= {a, ab^m, b, b^{-1}, b^3, b^{-3}}. Let S = S_1, for m = 4, 5 we have the Cases (29), (26) of Theorem1.1 respectively, and for $m\geq 6$, Γ is normal by Lemma 3.3(19 − S_1). Let S = S_2. When m = 4, $\sigma = (ab^2, ab^6) \in A_1$, but $\sigma \notin$ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43− S_4), m = 4) of Theorem 1.1. When m = 5, $\sigma = (b^3, b^7)(ab^3, ab^7)(b^2, b^8)(ab^2, ab^8) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (42) of Theorem 1.1. When $m\geq 6$, Γ = Cay (G, S) is normal by Lemma 3.3(19 − S_2). If G = $Z_2^2 \times Z_m$ = <a> × <b> × <c> ($m\geq 5$, m 6= 6), S = {a, b, c, c^{-1}, c^3, c^{-3}}. When m = 5, 10 and m = 8 we have the Cases (15), and (16) of Theorem 1.1 respectively. When m = 7, 9, $m\geq 11$, Γ = Cay (G, S) is normal by Lemma 3.3(15). Suppose $c^2 = d^2$, then G = $Z_2 \times Z_{2m}$, G = $Z_2^2 \times Z_{2m}$ ($m\geq 3$) G = $Z_2^2 \times Z_{2m-1}$ ($m\geq 2$) or G = $Z_2^2 \times Z_m$ ($m\geq 3$). If G= $Z_2 \times Z_{2m}$ = <a> × <b> we see that S is one of the following cases:

1) S_1= {a, b^m, b, b^{-1}, ab, ab^{-1}}, $m\geq 2$,

2) S_2= {a, ab^m, b, b^{-1}, ab, ab^{-1}}, $m\geq 2$,

3)S_3= {a, b^m, b, b^{-1}, b^{m+1}, b^{m-1}}, $m\geq 3$,

4) S_4= {a, ab^m, b, b^{-1}, b^{m+1}, b^{m-1}}, $m\geq 3$,

5) S_5={a, b^m, b, b^{-1}, ab^{m+1}, ab^{m-1}}, $m\geq 3$,

6) S_6= {a, ab^m, b, b^{-1}, ab^{m+1}, ab^{m-1}}, $m\geq 3$,

7) S_7= {ab^m, b^m, b, b^{-1}, ab, ab^{-1}}, $m\geq 2$,

8) S_8 = {ab^m, b^m, b, b^{-1}, ab^{m+1}, ab^{m-1}}, $m\geq 2$.

In the Case (1), $m\geq 2$, when m = 2i, $\sigma = (b^i, ab^i)(b^{3i}, ab^{3i}) \in A_1$, but $\sigma \notin$ Aut(G, S) and when m = 2i + 1, $\sigma = (b^{i+1}, ab^{i+1})(b^{3i+2}, ab^{3i+2}) \in A_1$, but $\sigma \notin$ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43 − S_1) of Theorem 1.1. In the Case (2), similarly Case (1), Γ = Cay(G, S) is not normal, the Case (43− S_2) of Theorem 1.1. In the Case (3), we have the Case (29) of Theorem 1.1. In the Case (4), when m = 2i, $\sigma = (ab^i, ab^{3i}) \in A_1$, but $\sigma \notin$ Aut(G, S) and when m = 2i + 1, $\sigma = (ab^{i+1}, ab^{3i+2}) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43 − S_4) of Theorem 1.1. In the Case (5), when m = 2i,$\sigma = (b^{3i}, ab^i) \in A_1$, but $\sigma \notin$ Aut(G, S) and when m = 2i+1, $\sigma = (b^{i+1}, ab^{3i+2}) \in A_1$,but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43− S_5) of Theorem 1.1. In the Case (6), when m = 2i, $\sigma = (b^i, ab^{3i})(b^{3i}, ab^i) \in A_1$,but $\sigma \notin$ Aut(G, S) and when m = 2i + 1, $\sigma = (b^{i+1}, ab^{3i+2})(b^{3i+2}, ab^{i+1}) \in A_1$, but $\sigma \notin$ Aut(G, S). Hence by Proposition 2.1, Γ = Cay (G, S) is not normal, the Case (43 − S_6) of Theorem 1.1.

In the Case (7), for m = 2i and m = 2i + 1, $\sigma = (b^{i+1}, ab^{i+1}) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay (G, S) is not normal, the Case (43 − S_3) of Theorem 1.1. In the Case (8), for m = 2i and m = 2i − 1, $\sigma = (b^i, ab^{i+m})(b^{m+i}, ab^i) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43 − S_1) of Theorem 1.1. If G = $Z_2^2 \times Z_{2m}$ = <a> × <b> × <c>, we can let S to be one of the following cases:

(1): S_1= {a, b, c, c^{-1}, ac, ac^{-1}}, $m\geq 2$,
(2): S_2= {a, b, c, c^{-1}, abc, abc^{-1}}, $m\geq 2$,
(3): S_3= {a, b, c, c^{-1}, c^{m+1}, c^{m-1}}, $m\geq 3$,
(4): S_4= {a, b, c, c^{-1}, ac^{m+1}, ac^{m-1}}, $m\geq 2$,
(5): S_5= {a, b, c, c^{-1}, abc^{m+1}, abc^{m-1}}, $m\geq 2$,
(6): S_6= {a, cm, c, c^{-1}, bc, b c^{-1}}, $m\geq 2$,
(7): S_7 = {a, ac^m, c, c^{-1}, bc, bc^{-1}}, $m\geq 2$,
(8): S_8 = {a, c^m, c, c^{-1}, bc^{m+1}, bc^{m-1}}, $m\geq 2$,
(9): S_9 = {a, ac^m, c, c^{-1}, bc^{m+1}, bc^{m-1}}, $m\geq 2$.

In the Case (1), Γ is not normal, the Case (30) of Theorem 1.1. In the Case (2),$\sigma = (ac^{m-1}, bc^{m-1}) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1 , Γ=Cay(G, S) is not normal, the Case (44 − S_1) of Theorem 1.1. In the Case (3), when m = 2i, Γ = Cay (G, S) is not normal, the Case (16) of Theorem 1.1.

When m = 2i+1, Γ =Cay (G, S) is not normal, we have the Case 14 (with m odd) of Theorem 1.1. In the Case (4), when m = 2i, i ≥2, $\sigma = (c^i, ac^{3i})(ac^i, c^{3i})(bc^i, abc^{3i})(abc^i, bc^{3i}) \in A_1$, but $\sigma \notin$ Aut(G, S), and when m =2i+1, $\sigma = (c^{i+1}, ac^{3i+2})(ac^{i+1}, c^{3i+2})(bc^{i+1}, abc^{3i+2})(abc^{i+1}, bc^{3i+2}) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44 − S_2) of Theorem 1.1. In the Case (5), when m = 2i, i ≥2, $\sigma = (c^{3i}, abc^i)(ac^{3i}, bc^i)(bc^{3i}, ac^i)(abc^{3i}, c^i) \in A_1$, but $\sigma \notin$ Aut(G, S) and when m = 2i + 1, $\sigma = (c^{3i+2}, abc^{i+1})(ac^{3i+2}, bc^{i+1})(bc^{3i+2}, ac^{i+1})(abc^{3i+2}, c^{i+1}) \in A_1$, but $\sigma \notin$ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44 − S_3) of Theorem 1.1. In the Case (6), $m\geq 2$, Γ is not normal, we have the Case (27) of Theorem 1.1. In the Case (7), if $m\geq 3$, for m = 2i and m = 2i − 1, σ = (ci, bci)(aci, abci)(ci+m, bci+m) (aci+m, abci+m) $\in A_1$, but $\sigma \notin$ Aut(G, S), and if m = 2, $\sigma = (b, bc^2)(ab, abc^2) \in A_1$, but $\sigma \notin$ Aut(G, S). Then by Proposition 2.1, Γ = Cay (G, S) is not normal , the Case (44 − S_4) of Theorem 1.1. In the Case (8), for m = 2i and m = 2i−1, $\sigma = (c^i, bc^{i+m})(ac^i, abc^{i+m})(c^{i+m}, bc^i)(ac^{i+m}, abc^i) \in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44 − S_5) of Theorem 1.1. In the Case (9), similarly Case (8), Γ = Cay(G, S) is not normal . We have the Case (44 − S_6) of Theorem 1.1.

If G = $Z_2^2 \times Z_{2m-1}$ = <a> × <b> × <c>, ($m\geq 2$), then S is S_1= {a, b, c, c^{-1}, ac, ac^{-1}} or S_2= {a, b, c, c^{-1}, abc, abc^{-1}}. When S = S_1, σ = (cm, acm)(bcm, abcm) $\in A_1$, but $\sigma \notin$ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal , the Case (44− S_7) of the main theorem .

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When $S = S_2$, $\sigma = (ac^{m-1}, bc^{m-1}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (44− S_1) of Theorem 1.1. If $G = Z_2^2 \times Z_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$, we can consider $m \geq 3$, $S = \{a, b, d, d^{-1}, cd, cd^{-1}\}$. In this case for $m = 2i$ and $m = 2i-1$, $(i\geq 2)$ $\sigma = (d^i, cd^i)(ad^i, acd^i)(bd^ibcd^i)(abd^i, abcd^i) \in A_1$, but $\sigma \notin Aut(G, S)$ and by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal the Case (14) of Theorem 1.1.

Case 4: $S = \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$, where the elements of the set S are not involution By the assumption (*), $o(a) = 4$, $a^2 = b^2$, $a^3 = b$ or $c = a^2b$. Suppose $o(a) = 4$, then G is isomorphic to one of the following: Z_{4m} $(m\geq 2)$, $Z_4 \times Z_m$, $Z_{4m} \times Z_n$ $(m\geq 2, n\geq 3)$, $Z_{4m} \times Z_{4n}$ $(m\geq 1, n\geq 1)$, $Z_4 \times Z_m \times Z_n$ $(m, n\geq 3)$. If $G = Z_{4m} = \langle a\rangle$ $(m\geq 2)$, we can let $S = \{a^m, a^{-m}, a, a^{-1}, a^j, a^{-j}\}$, where $1 < j < 2m$, $j \neq m$. When $j = 2m - 1$, $\sigma = (a^m, a^{-m}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (45) of Theorem 1.1. When $j \neq 2m - 1$, $\Gamma = Cay(G, S)$ is normal by Lemma 3.3(31). If $G = Z_4 \times Z_m = \langle a\rangle \times \langle b\rangle$, we can let S to be one of the following cases:

(1): $S_1= \{a, a^3, b, b^{-1}, ab^j, a^3b^{-j}\}$, $m\geq 3$, $1 \leq j \leq [m/2]$,

(2): $S_2= \{a, a^3, b, b^{-1}, a^2b^j, a^2b^{-j}\}$,
$m\geq 2$, $1 \leq j \leq (m/2)$,

(3): $S_3= \{a, a^3, b, b^{-1}, b^j, b^{-j}\}$, $m\geq 5$, $1 < j < (m/2)$.

When $S = S_1$, for $m = 2j$, $\sigma = (a^2, a^2b^j)(a^2b, a^2b^{j+1})...(a^2b^{j-1}, a^2b^{2j-1}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (49) of the main theorem. Otherwise, Γ is normal by Lemma 3.3(32). When $S = S_2$, $j = 1$ for $m = 2k$ and $m = 2k - 1$, $k \geq 2$, $\sigma = (ab^k, a^3b^k) \in A_1$, but $\sigma \notin Aut(G, S)$, and when $j = k - 1, m = 2k$ $(k \geq 3)$, $\sigma = (b^{k-1}, a^2b^{-1})(ab^{k-1}, a^3b^{-1})(a^2b^{k-1}, b^{-1})(a3b^{k-1}, ab^{-1}) \in A_1$, but $\sigma \notin Aut(G, S)$, then these graphs are non-normal and we have the Cases (49, 50) of Theorem 1.1. Otherwise, $\Gamma = Cay(G, S)$ is normal by Lemma 3.3 (33, 34). When $S = S_3$, for $j = k - 1$, $m = 2k$, if k is odd we have the Case (17) of Theorem 1.1 and if k is even we have the Case 19 ($m = 4$) of the main theorem. For $m = 5$; $j = 2$ and $m = 10$; $j = 3$ we have the Case 21($m = 4$) of the main theorem.
Otherwise, $\Gamma = Cay(G, S)$ is normal by Lemma 3.3 (35). If $G = Z_{4m} \times Z_n = \langle a\rangle \times \langle b\rangle$ $(m\geq 2, n\geq 3)$, $S = \{a^m, a^{-m}, a, a^{-1}, b, b^{-1}\}$, then $\Gamma = Cay(G, S)$ is normal by Lemma 3.3(20). If $G = Z_{4m} \times Z_{4n} = \langle a\rangle \times \langle b\rangle$ $(m\geq 1, n\geq 1)$, $S = \{a^mb^n, a^{-m}b^{-n}, a, a^{-1}, b, b^{-1}\}$, then $\Gamma = Cay(G, S)$ is normal by Lemma 3.3(21).If $G = Z_4\times Z_m \times Z_n = \langle a\rangle\times\langle b\rangle\times\langle c\rangle$ $(m, n\geq 3)$, we can consider $S = \{a, a^3, b, b^{-1}, c, c^{-1}\}$. In this case, for $m = 4$, $\Gamma = Cay(G, S)$ is not normal, the Case (18) of Theorem 1.1, and for $m, n \neq 4$, $\Gamma = Cay(G, S)$ is normal by Lemma 3.3(22). Suppose $a^2 = b^2$. Then G is isomorphic to one of the following: Z_{2m}, $Z_2\times Z_m$ $(m\geq 5)$, $Z_{2m} \times Z_{2n+1}$, $Z_{2m} \times Z_{2n}$ $(m\geq 3, n\geq 2)$, $Z_2 \times Z_n$ $(m\geq 3, n\geq 3)$. If $G = Z_{2m} = \langle a\rangle$, we can let S to be $S_1= \{a^j, a^{-j}, a^{m+j}, a^{m-j}, a, a^{-1}\}$, $2 \leq j \leq m/2$, $m\geq 5$, or $S_2= \{a, a^{-1}, a^{m+1}, a^{m-1}, a^j, a^{-j}\}$, $2\leq j \leq m - 2$, $m\geq 4$. When $S = S_1$, $\Gamma = Cay(G, S)$ is normal by Lemma 3.3(23). When $S = S_2$, $(m, j) =2$, for $m = 4i + 2$, $j = 2i$ (with i odd) and $j = 2i + 2$ (with i even), $\sigma =(a^2, a^{2+m/2})(a^6, a^{6+m/2})...(a^{2m-2}, a^{m/2-2}) \in A_1$, but $\sigma \notin Aut(G, S)$, and when $(m, j) = 1 >2$, then $\sigma = (a^2, a^{m+2})(a^{2+1}, a^{m+2+1})...(a^{m+2-1}, a^{2-1}) \in A_1$, but $\sigma \notin Aut(G, S)$, then by Proposition 2.1 these graphs are non-normal, and we have the Case (46) of the main theorem. Otherwise, $\Gamma = Cay(G, S)$ is normal by Lemma 3.3 (36). If $G = Z_2 \times Z_m = \langle a\rangle \times \langle b\rangle$ $m\geq 5$, we can let S to be $S_1= \{b, b^{-1}, ab, ab^{-1}, b^j, b^{-j}\}$,
$2 \geq j > m/2$ or $S_2 = \{b, b^{-1}, ab, ab^{-1}, ab^j, ab^{-j}\}$, $2 \geq j > m/2$. Let $S = S_1$. When $(m, j) = p > 2$; $m = (t + 1)p$, $\sigma = (b, ab)(b^{p+1}, ab^{p+1})...(bt^{p+1}, abt^{p+1}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (47− S_1) of the main theorem.
When $m = 8$, $j = 3$, $\sigma = (b^2, b^6)(ab, ab^7)(ab^3, ab^5) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (48− S_1) of Theorem 1.1. Otherwise, $\Gamma = Cay(G, S)$ is normal by Lemma 3.3(37, 38− S_1). Let $S = S_2$. When $(m, j) = p > 2$; $m = (t + 1)p$, $\sigma = (b, ab)(b^{p+1}, ab^{p+1})\ldots(b^{tp+1}, ab^{tp+1}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (47 − S_2) of Theorem 1.1. When $m = 8$, $j = 3$, $\sigma =(b^2, b^6)(b^3, b^5)(b, b^7) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case(48−S_2) of main theorem. Otherwise, $\Gamma = Cay(G, S)$ is normal by Lemma 3.3(37, 38 − S_2). If $G = Z_{2m} \times Z_n = \langle a\rangle \times \langle b\rangle$, we can let S to be one of the following cases:
(1): $S_1= \{a, a^{-1}, a^{m-1}, a^{m-1}, b, b^{-1}\}$, $m\geq 3$,
(2): $S_2 = \{b, b^{-1}, a^mb, a^mb^{-1}, a, a^{-1}\}$, $m\geq 2$,
(3): $S_3= \{b, b^{-1}, a^{m+1}b^l, a^{m-1}b^l, a, a^{-1}\}$, $n = 2l$, $l \geq 2$.

Let $S = S_l$. When $m = 2i$, $\Gamma = Cay(G, S)$ is not normal, the Case (19) of Theorem 1.1. When $m = 2i + 1$, $\sigma = (a^{m-1}, a^{2m-1})(a^{m-1}b, a^{2m-1}b)\ldots(a^{m-1}b^{n-1}, a^{2m-1}b^{n-1}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.4, $\Gamma = Cay(G, S)$ is not normal, the Case 20 (with m odd) of Theorem 1.1. Let $S = S_2$. When $n = 2j$, $2j - 1$ $(j \geq 2)$, $\sigma = (b^j, a^mb^j)(ab^j, a^{m+1}b^j)\ldots(a^{m-1}b^j, a^{2m-1}b^j) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (49) of Theorem 1.1. When $S = S_3$, $\sigma= (a^{m-1}, a^{-1}b^l)(a^{m-1}b, a^{-1}b^{l+1})\ldots(a^{m-1}b^{2l-1}, a^{-1}b^{l-1}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (50) of Theorem 1.1. If $G = Z_2 \times Z_m \times Z_n = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$, $m\geq 3$, $n\geq 3$, $S = \{b, b^{-1}, ab, ab^{-1}, c, c^{-1}\}$, we have the Case (20) of the main theorem. Suppose $a^3 = b$, then we have one of the following cases :

(1): $G = Z_m = \langle a\rangle$, $m\geq 7$, $S_1= \{a, a^{-1}, a^3, a^{-3}, a^j, a^{-j}\}$, $(j\neq 3, 2 \leq j \leq m/2)$,
$S_2 = \{a^j, a^{-j}, a^{3j}, a^{-3j}, a, a^{-1}\}$, $(2 \leq j \leq m/2, 3j \neq 0, 1, m - 1, j, m - j, m/2 (\bmod m))$.

(2): $G = Z_m \times Z_n = \langle a\rangle \times \langle b\rangle$, $(n\geq 3, m\geq 5, m \neq 6)$,
$S = \{a, a^{-1}, a^3, a^{-3}, b, b^{-1}\}$.

(3): $G = Z_{3m-1}\times Z_{3n} = \langle a\rangle\times\langle b\rangle$, $(m\geq 2, n\geq 1)$,

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$S = \{a^mb^n, a^{2m-1}b^{2n}, a^3, a, a^{-1}, b, b^{-1}\}$.

(4): $G = Z_{3m+1}\times Z_{3n} = \langle a\rangle \times \langle b\rangle$, (m, n≥1), $S = \{a^{2m+1}b^n, a^mb^{2n}, a, a^{-1}, b, b^{-1}\}$.

In the Case (1), when m = 6k, j = 3k−1, k ≥ 2, $\sigma = (a, a^{3k+1})(a^4, a^{3k+4})...(a^{3k-2}, a^{6k-2}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (51) of Theorem 1.1. In this case for S_1, when m= 7, j = 2, $\sigma = (a^2, a^5) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (52) of Theorem 1.1. When m = 8, j = 2, $\sigma = (a^2, a^6) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (45) of the main theorem.

When m = 14; j = 5, $\sigma = (a^2, a^{12})(a^5, a^9) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (52) of Theorem 1.1. Also for S_2, when m = 7; j = 3, $\sigma = (a^3, a^4) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay (G, S) is not normal, the Case (52) of Theorem 1.1. When m = 14; j = 3, $\sigma = (a^2, a^{12})(a^5, a^9) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (52) of Theorem 1.1. Otherwise, Γ = Cay (G, S) is normal by Lemma 3.3(39, 40, 41). In the Case (2), when m = 5, 10 and 8 we have the Cases (21) and (19, m = 2) of Theorem 1.1 respectively. Otherwise, Γ = Cay(G, S) is normal by Lemma 3.3 (24). In the Cases (3) and (4), Γ = Cay (G, S) is normal by Lemma 3.3 (25, 26). Suppose $c = a^2b$. Then we have one of the following cases:

(1): $G = Z_m = \langle a\rangle$ (m≥ 7), $S = \{a, a^{-1}, a^j, a^{-j}, a^{2+j}, a^{-2-j}\}$, if m = 2k, 2 ≤j≤(m/2) − 3 and if m = 2k + 1, 2 ≤ j ≤ (m/2) − 1.

(2): $G = Z_m = \langle a\rangle$ (m≥ 7), $S_1=\{a^j, a^{-j}, a, a^{-1}, a^{2j+1}, a^{-2j-1}\}$, 2 ≤ j ≤ m − 2, j ≠ m/2 and 2j + 1 ≠ m/2, 0, 1, m −1, j, m − j (mod m)

(3): $G = Z_m \times Z_n = \langle a\rangle \times \langle b\rangle$ (m, n≥3), $S = \{a, a^{-1}, b, b^{-1}, a^2b, a^{-2}b^{-1}\}$.

(4): $G = Z_{2m+1} \times Z_n = \langle a\rangle \times \langle b\rangle$ (m≥ 2, n≥3), $S = \{a^m, a^{m+1}, a, a^{-1}, b, b^{-1}\}$.

(5): $G = Z_{2m+1} \times Z_{2n+1} = \langle a\rangle \times \langle b\rangle$ (m, n≥1), $S = \{a^mb^{n+1}, a^mb^n, a, a^{-1}, b, b^{-1}\}$.

(6): $G = Z_2\times Z_{2m+1} \times Z_{2n+1} = \langle a\rangle\times\langle b\rangle\times\langle c\rangle$ (m, n≥1), $S = \{ab^m c^{n+1}, ab^{m+1}c^n, b, b^{-1}, c, c^{-1}\}$.

In the Case (1), if m = 3k, k ≥ 3, j = k −1, $\sigma = (a^k, a^{2k}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (53) of Theorem 1.1. If m = 6k, k ≥3, j = 3k − 3, $\sigma = (a, a^{3k+1})(a^4, a^{3k+4})...(a^{3k-2}, a^{6k-2}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ= Cay(G, S) is not normal, the Case (51 − S_2, m≥ 3) of Theorem 1.1. If m = 7; j = 2, $\sigma = (a^3, a^4) \in A_1$, but $\sigma \notin Aut(G, S)$, and if m = 14, j = 2, $\sigma = (a^2, a^{12})(a^5, a^9) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (52) of the main theorem.

Otherwise, Γ = Cay (G, S) is normal by Lemma 3.3(42, 43). In the Case (2), if m = 7, j = 4, $\sigma = (a^5, a^9) \in A_1$, but $\sigma \notin Aut(G, S)$, and if m = 14, j = 5, $\sigma = (a^2, a^{12})(a^5, a^9) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (52) of Theorem 1.1. If m = 3k, j = k − 1, k ≥ 3, $\sigma = (a^k, a^{2k}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (53) of Theorem 1.1. If m = 4j, j ≥ 2, $\sigma = (a^j, a^{3j}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (45) of Theorem 1.1. If m = 6k, j = 3k+1, k ≥ 3, $\sigma = (a, a^{3k+1})(a^4, a^{3k+4})...(a^{3k-2}, a^{6k-2}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ= Cay(G, S) is not normal, the Case (51- S_1) of Theorem 1.1. If m = 8k + 4, k ≥ 1, for k = 2i − 1, j = 4i− 2, i ≥ 1, $\sigma = (a^2, a^{12i-1})(a^6, a^{12i+3})...(a^{m-2}, a^{12i-5}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (54) of Theorem 1.1, and for k= 2i, j = 12i + 2, i ≥1, $\sigma = (a^2, a^{4i+3})(a^6, a^{4i+7})...(a^{m-2}, a^{4i-1}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ=Cay(G, S) is not normal, the Case (55) of Theorem 1.1. In the Case (3), if m = n = 3, $\sigma = (ab, a^2b^2) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (56) of the main theorem. If m = 4, $\sigma = (ab^2, a^3b^2) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (50) of Theorem 1.1. Otherwise, Γ = Cay(G, S) is normal by Lemma 3.3(27) .

In the Case (4), if m = 2, we have the Case (21) of Theorem 1.1. if m≥ 3, Γ = Cay(G, S) is normal by Lemma 3.3(28). In the Case (5), if m = n = 1, $\sigma = (ab, a^2b^2) \in A_1$, but $\notin Aut(G, S)$, by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (56) of Theorem 1.1. Otherwise, Γ = Cay(G, S) is normal by Lemma 3.3(29). In the Case (6), Γ = Cay(G,S) is normal by Lemma 3.3(30).

4. Conclusion

Let Γ = Cay (G, S) be a connected Cayley graph of a abelian group G on S. In this paper we have shown all non-normal Cayley graph Γ with valency 6.

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