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OPTIMIZING MASS FOR VIBRATION ANALYSIS OF PLANE STRUCTURE

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ABSTRACT

The mass matrix formulation is very important to achieve a high-convergent model in structural dynamics. This study calculates the optimum mass matrix for in-plane free vibrations of the plane problems. In fact, the parameterized mass and stiffness for a rectangular element are formulated by the template approach. By using perturbation theory and sensitivity analysis, the changes of the natural frequencies are obtained as a function of the free parameter variations. Based on the natural frequencies, the objective function is established. Through an optimization process, the optimum values for template-free parameters are determined. Findings are used to calculate the plane problems' natural frequencies. Some structural analyses and comparative studies with the other schemes are performed. Base on the obtained results, the efficiencies and high-convergence properties of the optimal element are demonstrated by numerical examples.

Keywords: eigenvalue; optimum mass; perturbation; plane vibration; sensitivity; template.

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1. INTRODUCTION

In civil, mechanical, and aerospace engineering, plane structures are used extensively. For example, the building shear walls usually are designed for low-frequency dynamic loads alike winds and earthquakes. In the new design and construction building methods, there is an increasing need to design shear walls against high-frequency loads, such as blasts and impacts [1]. Therefore, it is substantial to apply a precise and efficient method for vibration response analysis of such structures in high-frequency vibrations. One of the common shapes for the shear walls is cantilever rectangular plates. For vibration analysis of cantilever

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rectangular plates, several analytical methods, based on the variational approximation [2], Rayleigh-Ritz energy method [3] and superposition approach, have been developed. Nefovska and Petronijevic [1] developed a precise dynamic stiffness method for in-plane vibrations of rectangular plates. Although these methods provide exact solutions, they cannot be applied easily to dynamic updating and damage detection problems. For these plane problems, a simple finite element model with high precision and convergent properties is needed.

One of the very susceptible difficulties that exist in vibration-based finite element models is the need to achieve an accurate mathematical model that reflects the actual structural dynamic behavior. The changes in the measured vibration response due to the slight variations in the stiffness are very small. Therefore, it is impossible to distinguish between insufficient modeling and actual stiffness changes. The way to reduce these errors is to modify the finite element modeling. The modifications can be performed by introducing the appropriate assumptions and parameters in the finite element model [4]. During the process of adjusting certain parameters of the finite element model, optimum values of the parameters are found. Consequently, the finite element results satisfy requirements for actual structural dynamic behavior. Classically, the goal was achieved by a trial-and-error approach. This scheme was a time-consuming process and in some cases did not lead to feasible results. The idea of element parameterization is a suitable approach to construct a finite element model with feasible vibration properties [4, 5].

The parameter selection strategy is very important to define the actual vibration behavior of the structure and should have a good physical explanation. If this idea is applied at the element level, it can lead to the concept of generic elements. In the other words, a generic element model is identified as a parametric form of the element matrices that generates a family of elements with the same configuration [4]. The generic element model must satisfy all of the requirements, including; consistency, stability, parameterization, and observer invariant. For these requirements, the element stiffness matrix should be semi-positive definite and the rigid body modes of the element confined in the null space of the stiffness matrix [4]. The first two conditions: consistency and stability, ensure convergence. Parameterization property allows performance optimization to achieve special purposes [5]. Finite element templates have the features of generic elements and provide the feasibility of constructing the custom elements.

In the dynamic structural analysis and eigenvalue problems, the characteristics of the mass matrix are very important [6, 7]. It is important to notify that the convergence requirements are not satisfied by the traditional lumped and consistent mass models. The diagonal feature of the lumped mass matrices enables solving dynamic problems in a simple manner. However, the rate of convergence in estimating the eigenvalue is low and in some cases, the lumped mass is not a positive definite matrix [7]. The consistent mass underestimates the mass effects and yields higher natural frequencies. On the other hand, the lumped mass matrix is overestimated, and the natural frequencies are in the lower bound of the exact solution [8].

Due to the drawbacks of the lumped and consistent mass, considerable efforts have been focused on obtaining the more accurate mass matrices. Several forms of non-consistent matrices were developed to reduce the discretization errors and enhanced convergence properties [7]. Stavrinidis [9] derived analytical expressions to achieve the more accurate mass matrices for the bar, bending beam, and plane stress elements. Kim [8] proposed the linear combination of the lumped and consistent mass matrices to obtain a more accurate non-consistent mass. Another form of the linear combination was studied by Fried et al. [10] for bar and membrane elements. The optimum non-consistent mass matrix with desired-convergent property can be achieved by an admissible parametric mass matrix [7]. Adjusting the mass matrix parameters to reach the required convergence properties, consequences the optimum mass. Ahmadian and Faroughi [7] used the inverse method and derived a super convergent mass matrix formulation for a plate bending element.

One of the more general approaches for obtaining the parametric mass matrix relies on templates [6]. The template mass features as a parametric form with the applicability of generating a set of mass matrices that satisfy certain convergence requirements. For a given element configuration, a set of mass matrices possesses the requirements such as symmetric, positive definite; invariance, and momentum conservation is obtained [6]. In vibration analysis and wave propagation problems, free parameters of template mass can be optimized for special needs. This idea was first proposed by Fellippa [11] and was applied for Fourier analysis of Euler-Bernoulli and Timoshenko beam elements. Afterward, Guo [12] developed optimal mass matrices using the template approach for membrane triangles with corner drilling freedoms in a wave propagation problem. Badran et. al. [13] studied the template approach for plane stress quadrilateral elements in the wave propagation analysis.

Although finite element templates have many advantages, all of their efficiencies in vibration analysis have not been yet recognized. The dynamic response of the structure is determined by modal characteristics. Hence, the free parameters of the finite element template can be used as the design variables to adjusting the mathematical model for vibration analysis. The sensitivity modal analysis can be a valuable tool to obtain the optimum values of the template-free parameters. Previously, this idea has not been presented in any studies, and the researchers have not utilized the templates as generic elements for vibration analysis of the plane structures.

The main objective of this paper is to optimize admissible parametric stiffness and mass templates for in-plane free vibration analysis of the plane problem. This article introduces a different approach in optimizing the free parameters of the template. The approach is based on perturbation theory and sensitivity analysis. The parameterized mass and stiffness for rectangular elements are used to establish the objective function. The optimum values for free parameters are determined by the quasi-Newton optimization method. In the numerical studies, the obtained mass and stiffness produce a highly accurate eigenvalue and fast rate of convergence. The significant accuracy and convergence rate are observed in high-frequency vibrations that are very important in certain specialized applications.

2. FINITE ELEMENT TEMPLATE

One of the important purposes of finite element studies is the construction of high-performance (HP) elements. The HP elements are defined as simple elements that deliver engineering accuracy with arbitrary coarse meshes [5]. In the late 1960s and early 1970s, to achieve HP elements, some construction techniques such as incompatible shape functions, the patch test, reduced, selective and directional integration were used. Some developments in construction HP elements were made using mixed and hybrid variational principles in the 1980s [5]. New innovative approaches came into existence by the free formulation proposed by Bergan and Nygard [15]. The free formulation creates the fundamental decomposition in the stiffness matrix. The stiffness matrix splits into basic and higher-order matrices, respectively. The consistency and mixability are provided by the basic stiffness, whereas the stability and accuracy requirements are compiled by the higher-order stiffness matrix [16]. This fundamental decomposition had a key role in development of HP elements. This idea accompanied by parameterized variational principles led to the unexpected discovery that is called finite element template [14].

A finite element template is an algebraic form of the element stiffness matrix which contains free parameters. The stiffness template has the fundamental decomposition as the followings [17]:

$$K = K_b(\overline{\alpha_k}) + K_h(\alpha_i) \tag{1}$$

In Equation (1), K is the stiffness matrix of the plane element. Also, K_b and K_h are the basic and higher-order stiffness matrices, respectively. Parameter $\overline{\alpha_k}$ denotes the free parameters of the basic stiffness matrix, and α_i stands for the set of the parameters associated with the higher-order stiffness. A very few free parameter ($\overline{\alpha_k}$) is needed for the basic stiffness of the simple elements. The patch test requirements are satisfied by the basic stiffness part and yields the consistent element. The rank efficiency and accuracy are provided by the parametric form of the higher-order stiffness matrix. Satisfying consistency and stability, ensures convergence. Parameterized form of the stiffness matrix permits performance optimization [14]. By using this method, the elements can be tuned to specific needs. Setting the optimum values of free parameters, the specific element instances are obtained. In the present study, a new parametric form of template is optimized for analysis of the in-plane vibrations of the plane problems.

2.1 Stiffness template

In this study, the rectangular element with 2 degrees of freedom in each node is formulated by template approach for free vibration analysis (Fig. 1). The rectangular element is the simplest element that can be modified to achieve a parameterized template [17]. The element formulation is emanated from a mathematical statement of the convergence requirements proposed by Bergan and Nygard [15]. According to this mathematical statement, the displacement shape functions must be force orthogonal and energy orthogonal. A set of rigid-body and constant strain modes together with a set of linearly independent higher-order modes which is energy orthogonal to the first set are used for element formulation [15].

The nodal displacement and displacement field are described in Eqs. (2) and (3). The displacement field is decomposed in three rigid body, constant strain, and higher-order modes.

$$U = [u_1 \;\; v_1 \;\; u_2 \;\; v_2 \;\; u_3 \;\; v_3 \;\; u_4 \;\; v_4] \tag{2}$$

$$\begin{matrix} u \rightarrow \\ v \rightarrow \end{matrix} \left\langle \begin{matrix} 1 & 0 & -y \\ 0 & 1 & x \end{matrix} \middle| \begin{matrix} x & 0 & y \\ 0 & y & x \end{matrix} \middle| \begin{matrix} xy & 0 \\ 0 & xy \end{matrix} \right\rangle \tag{3}$$

r modes c modes h modes

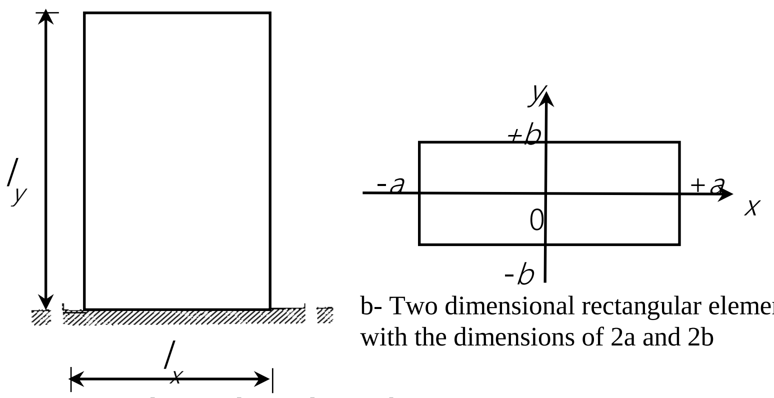
where, U is the vector of nodal displacements. The displacement fields along the x and y-axis are denoted by u and v. This field is described by two sets of polynomials, and their corresponding coefficients by the matrix form presented in Equation (4). N_{rc} and N_h express two complete polynomials correspond to the rc-modes and h-modes. The associated polynomial coefficients represented by q_{rc} and q_h , as follows.

$$\begin{bmatrix} u \\ v \end{bmatrix} = N_{rc} q_{rc} + N_h q_h \tag{4}$$

The kinematic relationship between the generalized modes and the nodal displacements U is easily established by Equation (5). When all displacement modes are linearly independent, the inverse exists and two important matrices H_{rc} and H_h are obtained. The roles of H_{rc} and H_h are essentially that of geometric projectors for basic and higher-order stiffness matrices.

$$U = Gq = G_{rc} q_{rc} + G_h q_h \tag{5}$$

$$q = \begin{bmatrix} q_{rc} \\ q_h \end{bmatrix} = G^{-1} U = \begin{bmatrix} H_{rc} \\ H_h \end{bmatrix} U \tag{6}$$



b- Two dimensional rectangular element with the dimensions of 2a and 2b

a- Rectangular cantilever plate with dimensions of l_x and l_y

Figure 1. The rectangular panel and the element configuration

Consider the rectangular plane stress element shown in Fig. 1. The in-plane strain and stress tensors are e_{ij} and σ_{ij}. The governing plane-stress elasticity equations of the element are described by Equations (7). The operator Δ stands for derivations of the displacement field and tensor S_{ij} stands for the material elasticity coefficients.

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \Delta \begin{bmatrix} u \\ v \end{bmatrix} \quad , \quad \sigma_{ij} = S_{ij} e_{ij} \tag{7}$$

The generalized stiffness matrix associated with the rc-modes is derived from the potential energy principles by the following relation [16]:

$$k_{qrc} = \int_V (\Delta N_{rc})^T S (\Delta N_{rc}) dV \tag{8}$$

The rigid-body modes are not influential in the rc-generalized stiffness matrix [16]. So, the r-modes can be ignored, and associated with the c-modes is obtained as k_{qc} in the next equation.

$$k_{qc} = \int_V \left(\Delta N_{qc}\right)^T S\left(\Delta N_{qc}\right) dV \tag{9}$$

In the same way, the generalized stiffness matrix associated with the h-modes is derived [16]:

$$k_{qh} = \int_V (\Delta N_h)^T S (\Delta N_h) dV \tag{10}$$

The total stiffness matrix of the rectangular element is a combination of basic stiffness (K_b) and higher-order stiffness (K_h). Each part of the stiffness matrix is obtained by the generalized stiffness and geometric projectors. In Equation (11), H_c and H_h are geometric projectors that correspond to the c-modes and h-modes respectively and $\boldsymbol{\alpha}$ is the free parameter of the template stiffness.

$$K = K_b + K_h(\boldsymbol{\alpha}) = H_c^T k_{qc} H_c + \boldsymbol{\alpha} H_h^T k_{qh} H_h \tag{11}$$

Equation (11) is a new parametric form of the rectangular plane element stiffness for performing the vibration analysis and sensitivity optimization. The basic stiffness (K_b) satisfies consistency conditions and is the same for any rectangular elements with specified freedom configuration. The two conditions: consistency and stability are required for convergence ensured by basic and higher-order matrices. The free parameter α provides the possibility to generate finite elements for this specific configuration. Specific elements are obtained by assigning numeric values to α parameter. This freedom of choice can be used to design custom elements for in-plane vibrations of the rectangular cantilever plate. The optimum value for α parameter increases the engineering accuracy, typically 1% in displacements and 10% in strains [5]. Some applications, notably in dynamic updating and damage detection problems require higher precisions in the natural frequencies and vibration mode shapes. In the present study, by performing sensitivity analysis of the frequency equation, the optimum value for α parameter is obtained.

2.2 Mass template

The consistent and diagonally-lumped mass matrices denoted by M_C and M_L, respectively, are the two widely used forms of mass matrices in structural dynamics. However, these models have major drawbacks in the special applications that require higher precisions. These drawbacks can be fulfilled with an approach that relies on the template idea. Availability of free parameters in the template approach, allows the mass matrix to be customized to the special needs in vibration analysis. Different ways can be applied to parameterizing mass matrices. In the study, the simple and effective method in practice is utilized. This method is named Matrix-Weighted Parameterization. In this method, a mass template for element e is a linear combination of *(k+1)* component mass matrices. The components of which $k \geq 1$, are weighted by β_k parameters by the following relation [6]:

$$M^e = M_0^e + \beta_1 M_1^e + \cdots + \beta_k M_k^e \tag{12}$$

where, M_0^e is the baseline mass matrix that should be an acceptable mass matrix if all of the weighted components are zero. The simple and effective mass template can be defined by only one weighted mass component $M_1^e = M_L^e - M_c^e$ according to Equations (13) and (14) [12].

$$M^e = M_0^e + \beta_1 M_1^e = M_c^e + \beta(M_L^e - M_c^e) \tag{13}$$

$$M = M_c + \beta(M_L - M_C) \tag{14}$$

In the last equation, M_C is the consistent mass proposed by Archer [17]. The consistent mass is calculated using the Galerkin formulation that is referred to the variationally consistent. There are several ways to obtain the lumped mass matrix. In the present study, the lumped mass M_L is obtained by HRZ method that was formulated by the contribution of Hinton, Rock and Zienkiewicz [18]. Hinton, Rock and Zienkiewicz [18], constructed the diagonal components of the lump mass matrix by scaling the total translational mass. The scaling factor is derived from the ratio of each diagonal component of the consistent mass with respect to the sum of all the diagonal components of the consistent mass. The HRZ method always produces positive lumped mass.

3.OPTIMIZATION OF FREE PARAMETERS IN THE MASS AND STIFFNESS TEMPLATES

The actual vibrational properties of the structures are essential in many engineering problems, notably damage identification [19, 20]. Detection of damage severity can be considered as an inverse problem that relies on the dynamic updating of a finite element model [20]. It is highly desirable that updating procedures can apply to the simple finite elements with high accuracy and fast convergence properties. Furthermore, the modeling errors should not affect the results [21]. The mass and stiffness templates provide very useful parametric forms to fulfill these requirements. In this section, by using approximation methods based on the perturbation theory, a parametric form of the frequency equation is obtained and utilized for establishing the objective function and the optimum values for mass and stiffness parameters are obtained.

3.1 Optimization method

Eigenvalue perturbation theory is an attractive subject in structural engineering that originated from the work of Rayleigh in the 19th century. There are two main branches of this research, including analytical and numerical perturbation theory [22]. The numerical perturbation theory is applied frequently for eigenvalue optimization problems in vibration analysis [23, 24]. Sensitivity analysis has been carried out for optimization of eigenvalues for free vibrations and appropriate formulations for structural modeling [25]. In structural vibration modal analysis, the eigenvalues and their sensitivities are very important for engineering problems, such as optimization design, model updating, and damage identification [26, 27, 28]. At present, there are two kinds of methods to calculate the sensitivities of eigenvalues: exact methods and approximate methods [29, 30]. The changes of the eigenvalues and eigenvectors can be determined by accurate approximation in eigenvalue perturbation theory [30, 31].

In the present study, with the help of approximation concepts in structural optimization, the objective function is established. The approximation method based on the perturbation theory is applied to achieve the parameterized frequency equations for in-plane vibrations of the plane problems. In such a way, without time-consuming computations, the eigenfrequencies are approximated in terms of the changes of the design variables. This type of problem is structural optimization with respect to the modal characteristics, including eigenvalues and eigenvectors.

The modal characteristics of a structure with n degrees of freedom in the initial state are determined by the eigenvalue equation:

$$K_0\phi_{0i} - \omega_{0i}^2 M_0 \phi_{0i} = 0 \tag{15}$$

where, $K_0(n\times n)$ and $M_0(n\times n)$ are stiffness and mass matrices of the structure, ω_{0i} and ϕ_{0i} are the ith eigenvalue and mode shape of the structure. The changes of eigenvalues and eigenvectors of the structure can be determined by the perturbation method. The perturbed eigenvalue equation can be written as follows:

$$(K_0 + \delta K)(\phi_{0i} + \delta\phi_i) - (\omega_{0i}^2 + \delta\omega_i^2)(M_0 + \delta M)(\phi_{0i} + \delta\phi_i) = 0 \tag{16}$$

Here, δK and δM are corresponding perturbations in the stiffness and mass matrices. Eigenvalue and eigenvector perturbations denoted by $\delta\omega_i^2$ and $\delta\phi_i$, respectively. If the changes of structural parameters are slight, the second and higher-order terms can be neglected and the first order perturbed equation is obtained:

$$\delta K\phi_{0i} = \omega_{0i}^2 \delta M\phi_{0i} + \delta\omega_i^2 M_0\phi_{0i} \tag{17}$$

By Multiplying both sides of Equation (17) by ϕ_{0i}^T, the perturbed eigenfrequency can be defined as a function of δK and δM:

$$\delta\omega_i^2 = \frac{\phi_{0i}^T \delta K\phi_{0i} - \omega_{0i}^2 \phi_{0i}^T \delta M\phi_{0i}}{\phi_{0i}^T M_0 \phi_{0i}} \tag{18}$$

According to the Equations (11) and (14) presented in section 2 of this paper, the changes of stiffness and mass matrices depend on the changes of two free parameters α and β respectively. So, the perturbed stiffness and mass are defined by $\delta\alpha$ and $\delta\beta$:

$$\delta K = \delta\alpha K_h \quad , \quad \delta M = \delta\beta(M_L - M_C) \tag{19}$$

Equation (18) and (19) yields new equation for the changes of eigenfrequencies as a function of $\delta\alpha$ and $\delta\beta$:

$$\delta\omega_i^2 = \frac{\phi_{0i}^T(\delta\alpha K_h)\phi_{0i} - \omega_{0i}^2 \phi_{0i}^T\big(\delta\beta(M_L - M_C)\big)\phi_{0i}}{\phi_{0i}^T M_0 \phi_{0i}} \tag{20}$$

$$\omega_i^2 = \omega_{0i}^2 + \delta\omega_i^2 \quad , \quad \omega_i = \sqrt{\omega_{0i}^2 + \delta\omega_i^2} \tag{21}$$

By using Equations (20) and (21), the natural frequencies due to the changes of template-free parameters can be determined. It is worth emphasizing that this scheme is a very efficient computational method. Because, there is no need to solve the eigenvalue problem for finding these changes.

3.2 Proposed optimization algorithm for finding optimum values of free parameters

When a general template is configured, among the numerous mass and stiffness matrices that can be generated, the best ones should be found. The best elements depend on the optimum values of the free parameters. In previous studies, these optimum values are determined at the local level by using a conventional error analysis in the bending tests of a patch of elements. It was usually followed by several heuristic optimization constraints. In the study, an optimization procedure is developed to obtain the optimum values for free parameters of the mass and stiffness templates. The suggested technique is based on the sensitivity of the frequency equation with respect to the free parameters α and β of the template (Eq. (20)). The objective function is established by minimizing the natural frequency of the finite element model ω and its desired value ω_d (Eq. (22)). In Equation (22), two parameters $\Delta\alpha$ and $\Delta\beta$ are design variables of the objective function:

$$Minimize\ \Delta = \left(\frac{\omega - \omega_d}{\omega_d}\right)^2 \tag{22}$$

The desired value for the natural frequencies (ω_d) can be assumed as experimental data or the exact solutions of the benchmarks. The authors' procedure is outlined by a flowchart in Fig. 2. As observed in Fig. 2, the initial values for α and β are chosen as 1 and 0 respectively. By these initial values, the eigenvalue equation of the structure is solved. Afterward, the frequency equation based on the sensitivity analysis is established by the eigenvalues and eigenvectors. The objective function in Eq. (22) is set upped and solved using the quasi-Newton optimization method. The quasi-Newton optimization method gives the optimum values for $\Delta\alpha$ and $\Delta\beta$. The free parameters are updated by $\alpha=\alpha_0-\Delta\alpha$ and $\beta=\beta_0+\Delta\beta$. When the updated values of α and β convinced the requirement, the calculation process is ended, and the new optimum mass and stiffness matrices are generated.

The accurate element stiffness and mass matrices are determined during numerical studies. Several numerical examples with different shapes and element aspect ratios are considered. The proposed optimization method is applied to achieve the element mass model that satisfies the accepted criteria from a parametric family of admissible mass matrices. In this optimization process, special attention is paid to reducing the errors in each vibration mode. Hence, a pair of stiffness and mass model is proposed with minimum errors and the highest convergence rate. The stiffness and mass models are obtained by optimum values of α and β, 0.5 and 0.35 respectively.

The advantages of using the proposed mass and stiffness model are simple and fast convergence, especially in the high-frequency range of vibration modes. The number of unknowns is significantly decreased with respect to the other methods, without any loss of accuracy. The efficiencies in estimating the eigenvalues are presented in the numerical case studies.

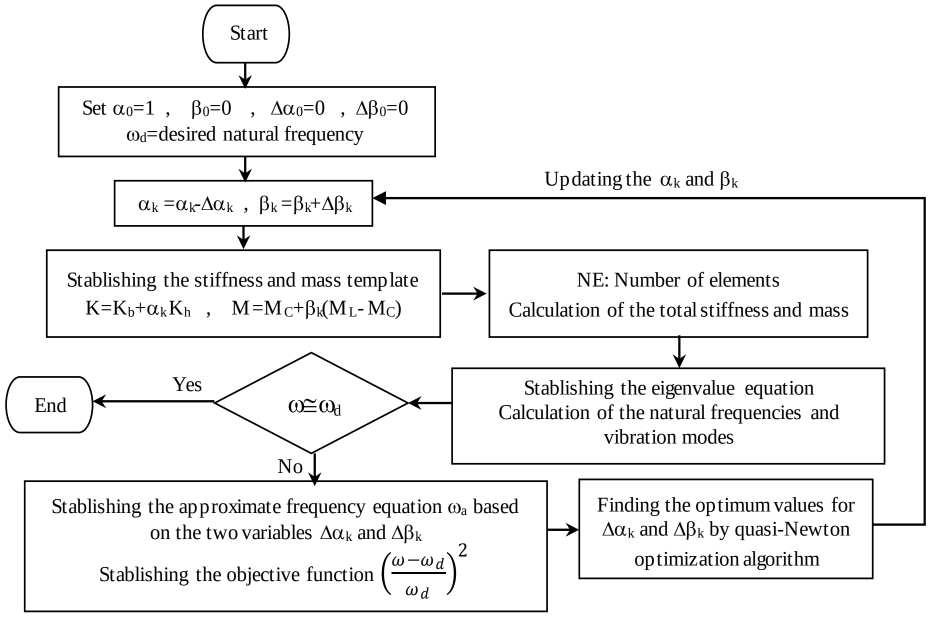


Figure 2. The proposed flowchart to find the optimum values for the free parameters of the mass and stiffness templates

4. NUMERICAL STUDY

In this section, the relative efficiencies of the proposed method over the usually associated methods are demonstrated through the three numerical examples. The exact solutions of these examples are derived from the previous analytical studies and the numerical studies that were carried out by ANSYS commercial software using the fine meshes of PLANE 42 elements. The optimum values of the free parameters of the mass and stiffness templates are obtained by the proposed optimization procedure (Sec. 3 and Fig. (2)).

4.1 Square cantilever plate ($l_x/l_y=1$)

In this numerical example, a square cantilever plate that is vibrating in its own plane, is analyzed. The side lengths of the square plate are $l_x=l_y=10$. The plate thickness is one unit, and Poisson's ratio is 0.3. This plate was analyzed by Gupta [32] and Stavrinidis et. al. [9] by the finite-element method (Fig. 3). The exact analytical dimensionless natural frequencies of the square plate are obtained by Seok et al. [2] that are presented in Table 1.

Table1: Exact solutions for dimensionless natural frequencies of the cantilever square plate

	First Vibration mode	Second Vibration mode	Third Vibration mode
Parametric exact solution (Seok et al., [2])	$(0.3370) \times \frac{\pi}{l_x} \times \sqrt{\frac{E}{\rho(2(1+\upsilon))}}$	$(0.8102) \times \frac{\pi}{l_x} \times \sqrt{\frac{E}{\rho(2(1+\upsilon))}}$	$(0.9093) \times \frac{\pi}{l_x} \times \sqrt{\frac{E}{\rho(2(1+\upsilon))}}$
Exact solution for ν=0.3 , 2b=10	$(0.06566) \times \sqrt{\frac{E}{\rho}}$	$(0.15785) \times \sqrt{\frac{E}{\rho}}$	$(0.17716) \times \sqrt{\frac{E}{\rho}}$

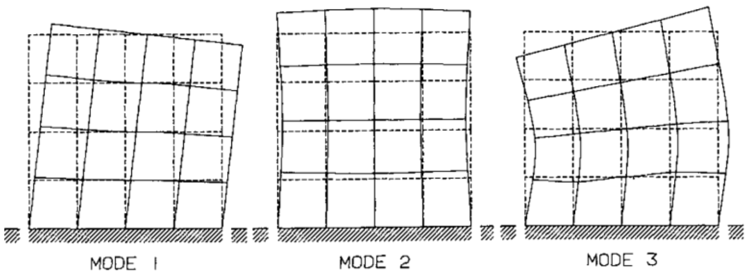


Figure 3. The first three vibration modes of the square plate (l_x=10, l_y=10)

The cantilever square plate is discretized by a coarse mesh of the template elements and the natural frequencies and associated modes are calculated and used for establishing the approximate frequency equation. The optimum values for the free parameters of the template are determined by the proposed optimization procedure described in Sec. 3 of the paper. By implementing this algorithm, the optimum values α=0.5 and β=0.35 are chosen for the analysis.

By using Equation (20), the sensitivities of the first natural frequencies of the square plate with respect to the changes of the free parameter in the mass template, β are calculated. The results are depicted in Fig. 4. As observed in Fig. 4, the rate of the sensitivity depends on the size of the mesh. For two types of very coarse mesh, the values β close to 0.35, lead to accurate frequencies.

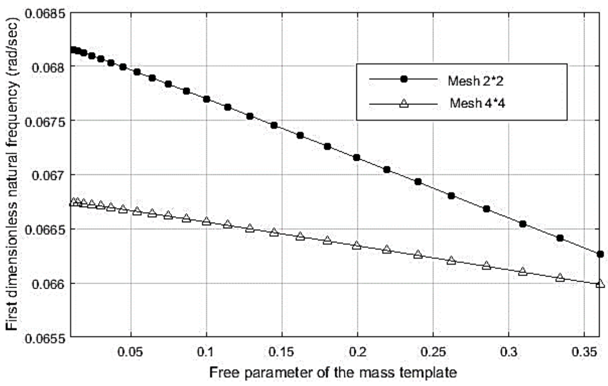


Figure 4. Sensitivity study for the first dimensionless natural frequency of the square plate (l_x=10, l_y=10)

In order to convergence studies, the cantilever plate is discretized by an increasing number of the proposed elements, and the convergence of the method is compared to those obtained by ANSYS software. According to the Figs. 5-7, compromising results are found. The proposed finite element mass and stiffness converge very fast in the small number of degrees of freedom. The accurate results are obtained by ANSYS for 882 degrees of freedom. Whereas, the same accurate results are obtained by the optimum mass and stiffness in 50 degrees of freedom. The difference between the usual finite element modeling and the proposed method is more significant in the high-frequency vibration modes (Fig. 7). As shown in Fig. 7, the precise natural frequencies are obtained with respect to ANSYS results. Furthermore, the proposed mass and stiffness pair is more accurate than the mass matrix formulation proposed by Stavrinidis et. al. [9].

To reach a more general conclusion, the square plate is also analyzed by the optimum stiffness with the free parameter α=0.5 and four different forms of the mass matrices, including: lumped, consistent, linear combination, and template mass. For each form of the mass matrices, the averages of the errors of the three first natural frequencies are calculated. Fig. 8 shows the effect of the mass matrix formulation on the convergence. Based on the findings, the proposed template mass with the free parameter β=0.35 yield minimum errors during the convergence study.

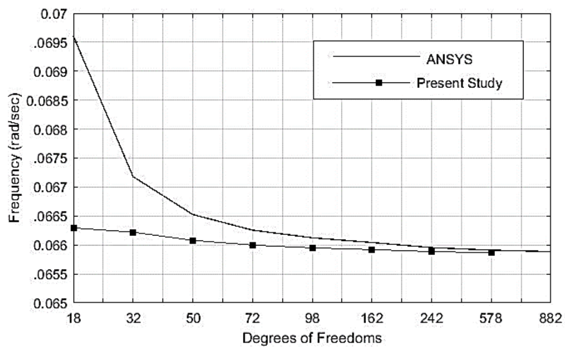


Figure 5. Convergence study for the first dimensionless natural frequency of the square plate (l_x=10, l_y=10)

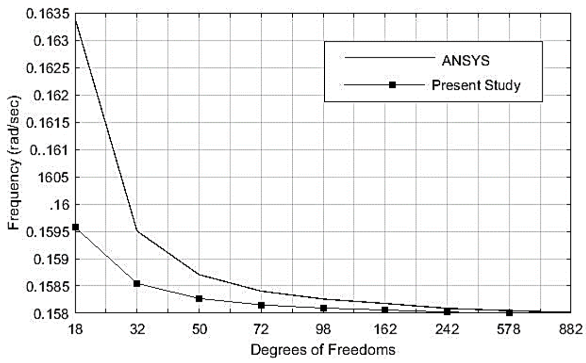


Figure 6. Convergence study for the second dimensionless natural frequency of the square plate (l_x=10, l_y=10)

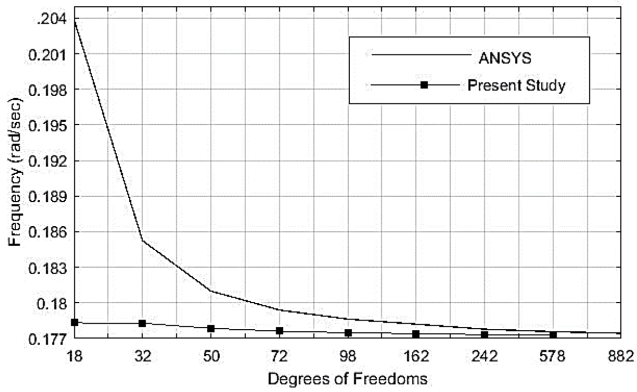


Figure 7. Convergence study for the third dimensionless natural frequency of the square plate (l_x=10, l_y=10)

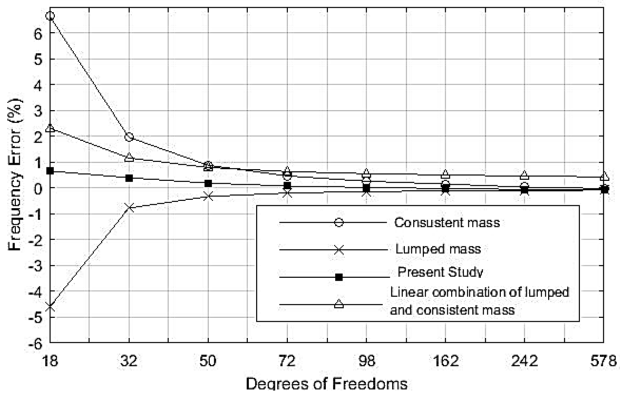


Figure 8. Average frequency errors for three first vibration modes of the square plate (l_x=10, l_y=10) with optimum stiffness template and four different mass matrices

4.2 Rectangular cantilever plate (l_x/l_y=2/5)

In the case of rectangular cantilever plates similar to the concrete shear walls of the buildings, this numerical study is undertaken and discussed (Fig. 9). The dimensions of the rectangular plate are l_x=4, l_y=10, and Poisson's ratio is 0.2. The free in-plane vibration of this cantilever plate was analyzed by Seok et al. [2]. They used a variational approximation procedure to obtain the exact solutions [2]. In the present study, this rectangular plate is analyzed utilizing the optimum mass and stiffness templates.

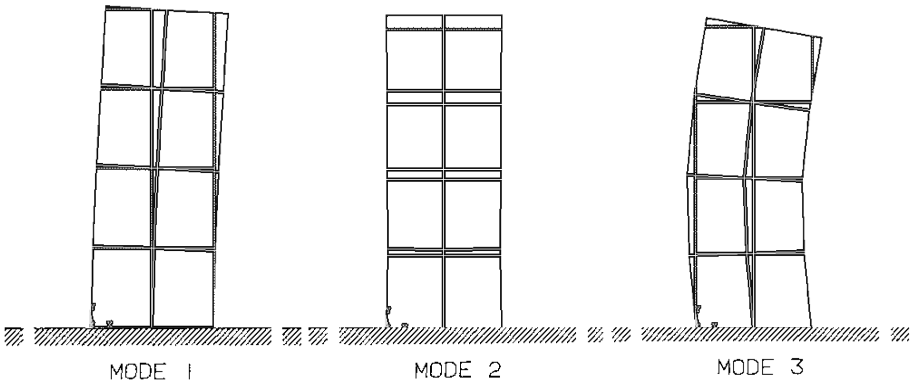


Figure 9. The first three vibration modes of the rectangular plate (l_x=4, l_y=10)

According to Sec. (3), the optimum values for the free parameters of the mass and stiffness templates are determined by perturbation theory and sensitivity analysis. The stiffness and mass templates with α=0.5 and β=0.35 are selected for the finite element analysis. The results for mesh 2×4 and mesh 4×8 of the present study are summarized in Table (2). According to the obtained results in Table (2), by using the template approach, the model of mesh 4×8 with 90 degrees of freedoms yields accurate results closed to the exact solution. To achieve these accurate results, it is needed to utilize the finite-element model with about 900 degrees of freedom in ANSYS software (Figs. 10-12). The results indicate that the model is precise and fast convergent.

Table 2: Dimensionless natural frequencies of the cantilever rectangular plate $\lambda = \sqrt{\frac{E}{\rho}}$, $l_x/l_y=2/5$

	First Vibration mode	Second Vibration mode	Third Vibration mode
Exact solution for ν=0.2, l_x =4	$(0.03669) \times \lambda$	$(0.15740) \times \lambda$	$(0.15810) \times \lambda$
mesh (2×4) –present study α=0.5 , β=0.35	$(0.03640) \times \lambda$	$(0.1578) \times \lambda$	$(0.1651) \times \lambda$
mesh (4×8) –present study α=0.5 , , β=0.35	$(0.0366) \times \lambda$	$(0.1575) \times \lambda$	$(0.1598) \times \lambda$

The convergence study is carried out, and the results are compared to the numerical results of ANSYS software (Figs. 10-12). To study the influence of the mass matrix formulation, the mass matrices derived by consistent, lumped, template and combination methods are utilized with the optimum stiffness template (α=0.5) in the finite-element model. For these mass models, the average errors of the first three vibration modes are calculated and presented in Fig. 13. According to Fig. 13, template mass gives more accurate results.

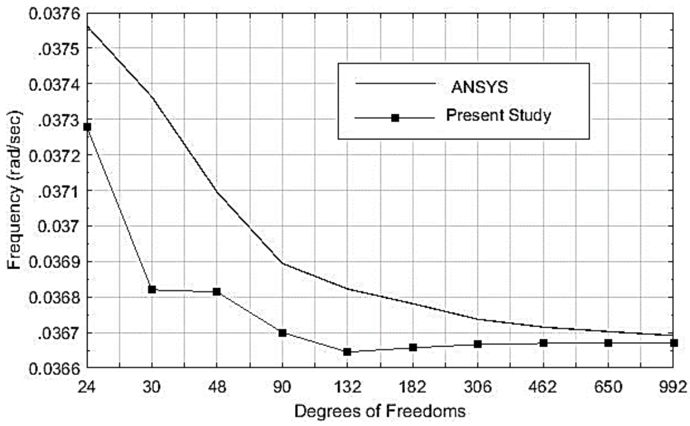


Figure 10. Convergence study for the first dimensionless natural frequency of the rectangular plate (l_x=4, l_y=10)

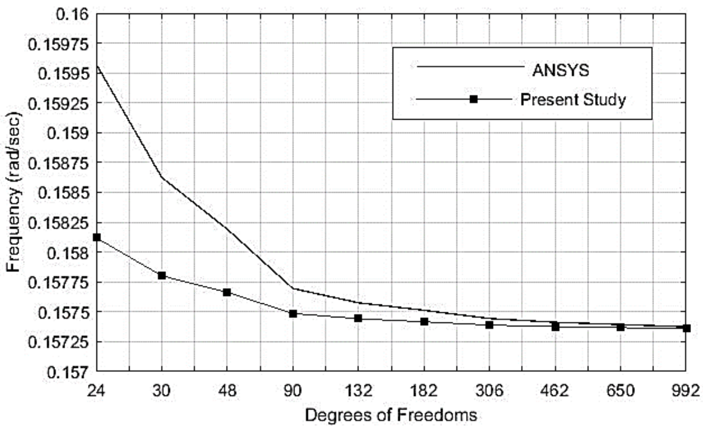


Figure 11. Convergence study for the second dimensionless natural frequency of the rectangular plate (l_x=4, l_y=10)

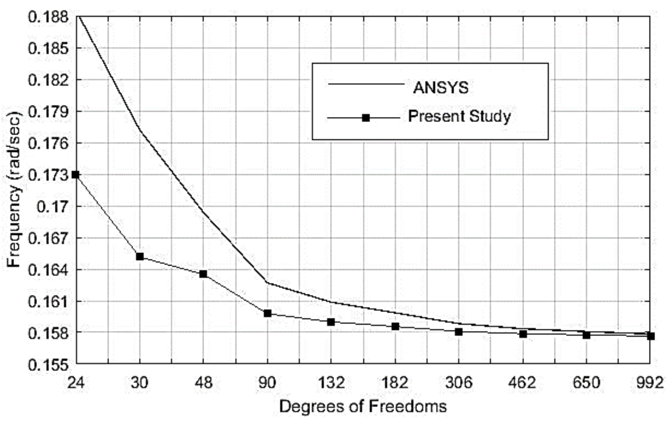


Figure 12. Convergence study for the third dimensionless natural frequency of the rectangular plate (l_x=4, l_y=10)

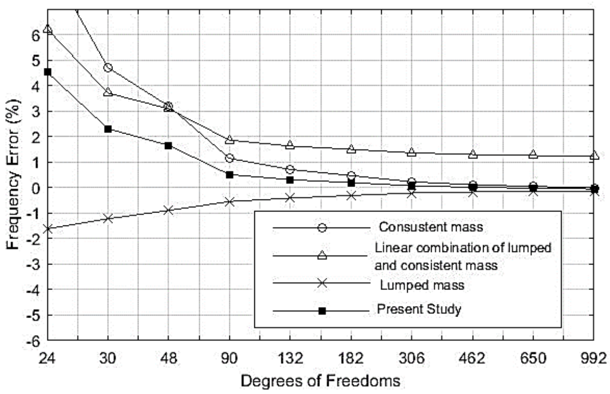


Figure 13. Average frequency errors for three first vibration modes of the rectangular plate (l_x=4, l_y=10) with optimum stiffness template and four different mass matrices

4.3 Rectangular cantilever plate (l_x/l_y=3/2)

In this numerical example, the vibration response of a short height cantilever rectangular plate is studied by the proposed method. The rectangular plate with the height of four units and the width of six units is analyzed. It is similar to the rectangular plate employed by Seok et al. [2]. To find the exact eigenfrequencies, Seok et al. [2] utilized a variational approximation procedure.

Table 2: Dimensionless natural frequencies of the cantilever rectangular plate *lx/l_y=3/2*

	First Vibration mode	Second Vibration mode	Third Vibration mode
Exact solution for ν=0.2, l_x =6	$(0.19534) \times \sqrt{\frac{E}{\rho}}$	$(0.39360) \times \sqrt{\frac{E}{\rho}}$	$(0.43478) \times \sqrt{\frac{E}{\rho}}$

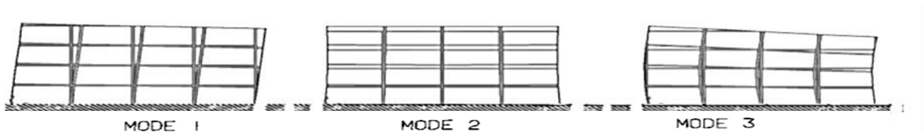


Figure 14. The first three vibration modes of the rectangular plate (l_x=6, l_y=4)

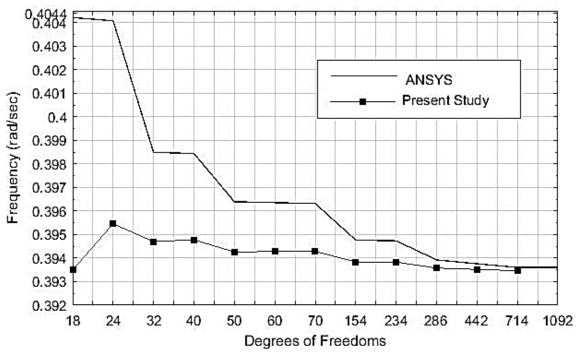


Figure 15. Convergence study for the first dimensionless natural frequency of the rectangular plate (l_x=6, l_y=4)

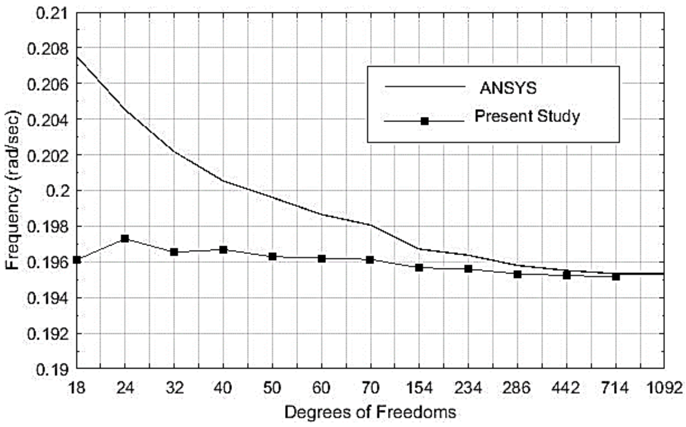


Figure 16. Convergence study for the second dimensionless natural frequency of the rectangular plate (l_x=6, l_y=4)

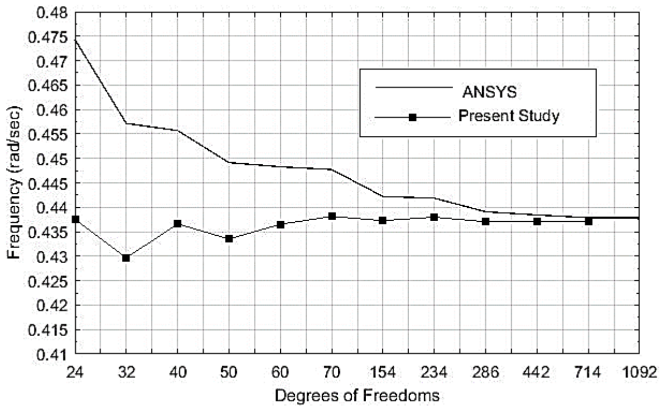


Figure 17. Convergence study for the third dimensionless natural frequency of the rectangular plate (l_x=6, l_y=4)

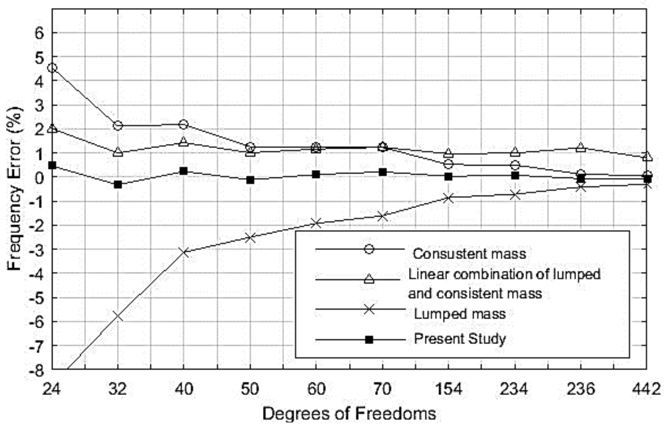


Figure 18. Average frequency errors for three first vibration modes of the rectangular plate (l_x=6, l_y=4) with optimum stiffness template and four different mass matrices

4.4 Results and discussion

In this study, in-plane vibrations of the plane problems with different dimensions, were analyzed by the proposed mass and stiffness matrices. Because of the fast convergence properties and high accuracy of the suggested scheme, all the obtained numerical results were better than ANSYS software. Moreover, the authors' technique led to the better responses than the other conventional finite element formulations, such as, assumed shape functions and variational methods. The conventional methods are the basis for construing lumped and consistent mass formulations. These ways cannot analyze efficiently all the engineering problems [5, 6]. The general approach relies on the template has the virtue of generating custom elements for engineering applications, such as, plane vibrations.

According to the findings, even when using only small numbers of degrees of freedoms, the great distances between the results of ANSYS and the present study in the prepared figures confirm firmly the efficiencies and accuracy of the suggested parametric form of the stiffness template. No correction is needed for calculated coefficients, because they led to the precise and fast convergent. The achieved responses by using only small numbers of degrees of freedoms demonstrated clearly the efficiencies and accuracy of the authors' scheme.

It is worth mentioning that the proposed method has the limitations in the high aspect ratios of the mesh. This behavior is similar and common to the other finite element solutions.

5. CONCLUDING REMARKS

The proposed element mass and stiffness matrices were optimized in this study. Based on the perturbation theory and sensitivity analysis, the Authors' technique utilized an optimization method for a finite element template. The optimum values for the free parameters of the mass and stiffness template were found to obtain precise natural frequencies of the plane problems. By performing several numerical and comparison studies, the efficiencies of the optimum elements were demonstrated, and the following conclusions were drawn:

- A new pair of mass and stiffness matrices for in-plane vibration analysis of the plane problem was obtained based on the template approach. Two vital optimum values of $\alpha=0.5$ and $\beta=0.35$ were found for the optimum stiffness and mass respectively.
- The benefits of the proposed element mass and stiffness matrices are the highest convergence rate and precision in the limited number of degrees of freedom. These properties are valuable in the updating process of some actual experimental applications with a limited number of sensors and degrees of freedom.
- The precision of the proposed pair of mass and stiffness matrices became more significant in the higher-frequency vibration modes. Due to the increasing interest to use the higher frequencies of in-plane vibrations in some engineering designs such as acoustics, blast loads, and ship hull, it is a very useful property.

- The new mass and stiffness matrices require no complicated and time-consuming computations and can be implemented easily into finite element codes.

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